Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica

Benjamin Cahen

Global Parametrization of Scalar Holomorphic Coadjoint Orbits of a Quasi-Hermitian Lie Group

Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 52 (2013), No. 1, 35--48

Persistent URL: `http://dml.cz/dmlcz/143389`

Terms of use:

© Palacký University Olomouc, Faculty of Science, 2013

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://project.dml.cz`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **52**, 1 (2013) 35–48

Global Parametrization of Scalar Holomorphic Coadjoint Orbits of a Quasi-Hermitian Lie Group

Benjamin CAHEN

Université de Metz, UFR-MIM, Département de mathématiques
LMMAS, ISGMP-Bât. A
Ile du Saulcy 57045, Metz cedex 01, France
e-mail: cahen@univ-metz.fr

(Received January 8, 2013)

Abstract

Let G be a quasi-Hermitian Lie group with Lie algebra $\mathfrak{g}$ and K be a compactly embedded subgroup of G. Let ξ_0 be a regular element of $\mathfrak{g}^*$ which is fixed by K. We give an explicit G-equivariant diffeomorphism from a complex domain onto the coadjoint orbit $\mathcal{O}(\xi_0)$ of ξ_0. This generalizes a result of [B. Cahen, *Berezin quantization and holomorphic representations*, Rend. Sem. Mat. Univ. Padova, to appear] concerning the case where $\mathcal{O}(\xi_0)$ is associated with a unitary irreducible representation of G which is holomorphically induced from a unitary character of K. In particular, we consider the case $G = SU(p,q)$ and the case where G is the Jacobi group.

Key words: quasi-Hermitian Lie group, coadjoint orbit, stereographic projection, Berezin quantization, unitary holomorphic representation, unitary group, Jacobi group

2000 Mathematics Subject Classification: 32M10, 22E15, 22E10, 22E45, 32M05, 32M15, 81S10

1 Introduction

Let us first consider the following situation. Let $G = SU(1,1)$ and K be the torus of G consisting of matrices of the form $\mathrm{Diag}(e^{i\theta}, e^{-i\theta})$ where $\theta \in \mathbb{R}$. The Lie algebra $\mathfrak{g}$ of G has basis

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Let (u_1^*, u_2^*, u_3^*) be the dual basis of $\mathfrak{g}^*$. For $r > 0$, let $\xi_0 = r u_3^*$. Then the orbit $\mathcal{O}(\xi_0)$ of ξ_0 for the coadjoint action of G is the upper sheet $x_3 > 0$ of the two-sheet hyperboloid $\{\xi = x_1 u_1^* + x_2 u_2^* + x_3 u_3^* \colon -x_1^2 - x_2^2 + x_3^2 = r^2\}$. Since the stabilizer of ξ_0 for the coadjoint action of G is K, we have $\mathcal{O}(\xi_0) \simeq G/K$. On the other hand, G/K is diffeomorphic to the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. Then, by composition, we get a global chart $\psi \colon \mathbb{D} \to \mathcal{O}(\xi_0)$. Explicitly, we have

$$\psi(z) := r \left(\frac{z + \bar{z}}{1 - z\bar{z}} u_1^* + \frac{z - \bar{z}}{i(1 - z\bar{z})} u_2^* + \frac{1 + z\bar{z}}{1 - z\bar{z}} u_3^* \right).$$

Note that ψ intertwines the natural action on G on $\mathbb{D}$ (by fractional linear transforms) and the coadjoint action of G on $\mathcal{O}(\xi_0)$. Note also that ψ^{-1} is an analog of the stereographic projection from the two-sphere $\mathbb{S}^2$ onto $\mathbb{C} \cup (\infty)$. Moreover, if we take $r = n/2$ where n is an integer ≥ 2 then $\mathcal{O}(\xi_0)$ is associated with a holomorphic discrete series representation π_n of G by the Kirillov–Kostant method of orbits [26], [27]. In that case, the differential $d\pi_n$ of π_n is related to ψ by the Berezin calculus S, that is, we have $S(d\pi_n(X))(z) = i\langle(\psi(z), X)$ for each $X \in \mathfrak{g}$ and each $z \in \mathbb{D}$ [12].

The goal of the present note is to extend the above considerations to a large setting. To this aim, we consider a quasi-Hermitian Lie group G and a compactly embedded subgroup $K \subset G$. In [20], we considered a unitary representation π of G which is holomorphically induced from a unitary character of K and we proved that the dequantization of $d\pi$ by means of the Berezin calculus provides an explicit diffeomorphism from a complex domain onto the coadjoint orbit of G associated with π (see also [16] and [18]). Here we show that, more generally, such a diffeomorphism can also be constructed for the coadjoint orbit $\mathcal{O}(\xi_0) := \mathrm{Ad}^*(G)\,\xi_0$ of an element $\xi_0 \in \mathfrak{g}^*$ which is fixed by K and assumed to be regular (in a sense defined below). We call such an orbit $\mathcal{O}(\xi_0)$ a *scalar* orbit.

Note that similar parametrizations for coadjoint orbits of compact Lie groups can be found in [30] and [8]. For unitary groups, explicit expressions for generalized stereographic projections are given in [30].

Parametrizations of coadjoint orbits have many applications in deformation theory, harmonic analysis and mathematical physics. Let us mention some of them:

1. Construction of covariant star-products on coadjoint orbits [1], [11], [22];
2. Construction of some quantization maps, as adapted Weyl correspondences and Stratonovich-Weyl correspondences [13], [19];
3. Geometric quantization of coadjoint orbits [3], [21];
4. Contractions and restrictions of unitary irreducible representations associated with integral coadjoint orbits [15], [17], [23], [2], [14].

This note is organized as follows. Section 2 is devoted to generalities about quasi-Hermitian Lie groups. In Section 3 and Section 4, we review some results from [20]. In Section 5, we give a G-equivariant parametrization of a scalar coadjoint orbit of a quasi-Hermitian Lie group G. In Section 6, we consider the case of the unitary group $SU(p,q)$ and, in Section 7, the case of the (generalized) Jacobi group.

2 Generalities

The material of this section and of the first part of Section 3 is taken from the excellent book of K.-H. Neeb, [28], Chapter VIII and Chapter XII (see also [29], Chapter II and, for the Hermitian case, [25], Chapter VIII).

Let $\mathfrak{g}$ be a real quasi-Hermitian Lie algebra [28, p. 241]. We assume that $\mathfrak{g}$ is not compact. Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$ and let $Z = X + iY \to Z^* = -X + iY$ be the corresponding involution. We fix a compactly embedded Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$, [28, p. 241] and we denote by $\mathfrak{h}^c$ the corresponding Cartan subalgebra of $\mathfrak{g}^c$. We write $\Delta := \Delta(\mathfrak{g}^c, \mathfrak{h}^c)$ for the set of roots of $\mathfrak{g}^c$ relative to $\mathfrak{h}^c$ and $\mathfrak{g}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ for the root space decomposition of $\mathfrak{g}^c$. Note that $\alpha(\mathfrak{h}) \subset i\mathbb{R}$ for each $\alpha \in \Delta$ [28, p. 233]. We write Δ_k, respectively Δ_p, for the set of compact, respectively non-compact, roots [28, p. 233–235]. Note that one has $\mathfrak{k}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta_k} \mathfrak{g}_\alpha$ [28, p. 235]. We fix a positive adapted system Δ^+ [28, p. 236] and we set $\Delta_p^+ := \Delta^+ \cap \Delta_p$ and $\Delta_k^+ := \Delta^+ \cap \Delta_k$, see [28, p. 241].

Let G^c be a simply connected complex Lie group with Lie algebra $\mathfrak{g}^c$ and $G \subset G^c$, respectively, $K \subset G^c$, the analytic subgroup corresponding to $\mathfrak{g}$, respectively, $\mathfrak{k}$. We also set $K^c = \exp(\mathfrak{k}^c) \subset G^c$ as in [28, p. 506].

Let $\mathfrak{p}^+ = \sum_{\alpha\in\Delta_p^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \sum_{\alpha\in\Delta_p^+} \mathfrak{g}_{-\alpha}$. Let P^+ and P^- be the analytic subgroups of G^c with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$. Then G is a group of the Harish-Chandra type [28, p. 507], that is, the following properties are satisfied:

1. $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$ is a direct sum of vector spaces, $(\mathfrak{p}^+)^* = \mathfrak{p}^-$ and $[\mathfrak{k}^+, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$;
2. The multiplication map $P^+K^cP^- \to G^c$, $(z,k,y) \to zky$ is a biholomorphic diffeomorphism onto its open image;
3. $G \subset P^+K^cP^-$ and $G \cap K^cP^- = K$.

Moreover, there exists an open connected subset $\mathcal{D} \subset \mathfrak{p}^+$ such that $GK^cP^- = \exp(\mathcal{D})K^cP^-$ [28, p. 497]. We denote by $\zeta\colon P^+K^cP^- \to P^+$, $\kappa\colon P^+K^cP^- \to K^c$ and $\eta\colon P^+K^cP^- \to P^-$ the projections onto P^+-, K^c- and P^--components. For $Z \in \mathfrak{p}^+$ and $g \in G^c$ with $g \exp Z \in P^+K^cP^-$, we define the element $g\cdot Z$ of $\mathfrak{p}^+$ by $g \cdot Z := \log \zeta(g \exp Z)$. Note that we have $\mathcal{D} = G \cdot 0$.

We also denote by $g \to g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X \to X^*$. We denote by $p_{\mathfrak{p}^+}$ the projection of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$ associated with the direct decomposition $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$.

3 Holomorphic representations

In this section, we consider the case of a coadjoint orbit associated with a scalar holomorphic discrete series representation of G.

We fix a unitary character χ of K. We also denote by χ the extension of χ to K^c. We set $K_\chi(Z,W) = \chi(\kappa(\exp W^* \exp Z))^{-1}$ for $Z, W \in \mathcal{D}$ and $J_\chi(g,Z) = \chi(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$. Let $\mathcal{H}_\chi$ be the Hilbert space of holomorphic functions on $\mathcal{D}$ such that

$$\|f\|_\chi^2 := \int_{\mathcal{D}} |f(Z)|^2 \, K_\chi(Z,Z)^{-1} d\mu(Z) < +\infty$$

Here μ denotes the G-invariant measure on $\mathcal{D}$, that is,

$$d\mu(Z) := \chi_0(\kappa(\exp Z^* \exp Z)) \, d\mu_L(Z)$$

where χ_0 is the character on K^c defined by $\chi_0(k) = \mathrm{Det}_{\mathfrak{p}^+}(\mathrm{Ad}\, k)$ and $d\mu_L(Z)$ is a Lebesgue measure on $\mathcal{D}$ [28, p. 538].

In this section, we assume that $\mathcal{H}_\chi \neq (0)$. Then $\mathcal{H}_\chi$ contains the polynomials [28, p. 546] and the formula

$$\pi_\chi(g) f(Z) = J_\chi(g^{-1}, Z)^{-1} \, f(g^{-1} \cdot Z)$$

defines a unitary representation of G on $\mathcal{H}_\chi$ which is a highest weight representation with highest weight $\lambda := d\chi|_{\mathfrak{h}^c}$ [28, p. 540].

We introduce the constant c_χ defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} K_\chi(Z,Z)^{-1} \, d\mu(Z).$$

and we set $e_Z(W) := c_\chi K_\chi(W,Z)$. Then we have the reproducing property $f(Z) = \langle f, e_Z \rangle_\chi$ for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$ [28, p. 540]. Here $\langle \cdot, \cdot \rangle_\chi$ denotes the inner product on $\mathcal{H}_\chi$.

The Berezin calculus on $\mathcal{D}$ is then defined as follows [4], [5], [21]. Consider an operator (not necessarily bounded) A on $\mathcal{H}_\chi$ whose domain contains e_Z for each $Z \in \mathcal{D}$. Then the Berezin symbol of A is the function $S_\chi(A)$ defined on $\mathcal{D}$ by

$$S_\chi(A)(Z) := \frac{\langle A\, e_Z, \, e_Z \rangle_\chi}{\langle e_Z, \, e_Z \rangle_\chi}.$$

It is known that each operator is determined by its Berezin symbol and that if an operator A has adjoint A^* then we have $S_\chi(A^*) = \overline{S_\chi(A)}$ [4], [21]. The Berezin calculus is G-equivariant with respect to π_χ, that is, we have the following property: for each operator A on $\mathcal{H}_\chi$ whose domain contains the coherent states e_Z for each $Z \in \mathcal{D}$ and each $g \in G$, the domain of $\pi_\chi(g^{-1}) A \pi_\chi(g)$ also contains e_Z for each $Z \in \mathcal{D}$ and we have

$$S_\chi(\pi_\chi(g)^{-1} A \pi_\chi(g))(Z) = S_\chi(A)(g \cdot Z) \tag{3.1}$$

for each $g \in G$ and $Z \in \mathcal{D}$.

Now, we consider the linear form ξ on $\mathfrak{g}^c$ defined by $\xi = -id\chi$ on $\mathfrak{k}^c$ and $\xi = 0$ on $\mathfrak{p}^\pm$. Then we have $\xi(\mathfrak{g}) \subset \mathbb{R}$ and the restriction ξ_0 of ξ to $\mathfrak{g}$ is an element of $\mathfrak{g}^*$. Let $\mathcal{O}(\xi_0)$ be the orbit of ξ_0 in $\mathfrak{g}^*$ for the coadjoint action of G. In [20], we proved the following proposition (see also [17]).

Proposition 3.1

1. *For each $X \in \mathfrak{g}^c$ and each $Z \in \mathcal{D}$, we have*

$$S(d\pi_\chi(X))(Z) = i\langle \psi(Z), X\rangle$$

where $\psi(Z) := \mathrm{Ad}^*\big(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z)\big)\,\xi_0$.

2. *For each $g \in G$ and each $Z \in \mathcal{D}$, we have $\psi(g \cdot Z) = \mathrm{Ad}^*(g)\,\psi(Z)$.*
3. *The map ψ is a diffeomorphism from $\mathcal{D}$ onto $\mathcal{O}(\xi_0)$.*

Note that (2) immediately follows from the G-equivariance of the Berezin calculus. In the following section, we extend (2) and (3) to scalar coadjoint orbits.

4 Parametrization of scalar coadjoint orbits

If $\xi_0 \in \mathfrak{g}^*$ is associated with a unitary character of K as in Section 3 then we have $\mathrm{Ad}^*(k)\xi_0 = \xi_0$ for each $k \in K$ and, by Lemma 3.1 of [20], the Hermitian form $(Z, W) \to \langle \xi_0, [Z, W^*]\rangle$ is not isotropic. This leads us to consider the elements $\xi_0 \in \mathfrak{g}^*$ which are fixed by K and regular in the sense that the Hermitian form $(Z, W) \to \langle \xi_0, [Z, W^*]\rangle$ is not isotropic. Such elements ξ_0 are called *scalar* and we say that the coadjoint orbit $\mathcal{O}(\xi_0)$ of a scalar element ξ_0 is a *scalar* orbit.

Lemma 4.1 *Let $\xi_0 \in \mathfrak{g}^*$ fixed by K. Let us also denote by ξ_0 the linear extension of ξ_0 to $\mathfrak{g}^c$.*

1. *We have $\xi_0|_{\mathfrak{p}^\pm} \equiv 0$;*
2. *Let $E_1, E_2, \ldots, E_m$ be a basis of $\mathfrak{p}^+$ such that $E_j \in \mathfrak{g}_{\alpha_j}$ where $\alpha_j \in \Delta_p^+$ for $j = 1, 2, \ldots, m$. Then ξ_0 is regular hence scalar if and only if we have $i\langle \xi_0, [E_j^*, E_j]\rangle > 0$ for each $j = 1, 2, \ldots, m$ or $i\langle \xi_0, [E_j^*, E_j]\rangle < 0$ for each $j = 1, 2, \ldots, m$.*

Proof (1) If $\xi_0 \in \mathfrak{g}^*$ is fixed by K then one has $\mathrm{ad}^*\, U\, \xi_0 = 0$ for each $U \in \mathfrak{k}$ or, equivalently, $\langle \xi_0, [U, X]\rangle = 0$ for each $U \in \mathfrak{k}$ and $X \in \mathfrak{g}$. Then, taking $X = E_j$ where $j = 1, 2, \ldots, m$ and $U \in \mathfrak{g}_{\alpha_j}$ such that $\alpha_j(U) \neq 0$ we get $\langle \xi_0, E_j\rangle = 0$ for each $j = 1, 2, \ldots, m$ hence the result.

(2) Let $Z = \sum_{j=1}^m z_j E_j \in \mathfrak{p}^+$. Then, by using (1), we get

$$\langle \xi_0, [Z^*, Z]\rangle = \sum_{j=1}^m \langle \xi_0, [E_j^*, E_j]\rangle |z_j|^2$$

where $i[E_j^*, E_j] \in \mathfrak{h}$ for each j [28], p. 233. The result then follows. □

In the rest of this section, we fix a scalar element $\xi_0 \in \mathfrak{g}^*$. For $Z \in \mathcal{D}$, we set

$$\psi(Z) := \mathrm{Ad}^*\big(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z)\big)\,\xi_0.$$

Proposition 4.2 *For each $g \in G$ and each $Z \in \mathcal{D}$, we have*

$$\psi(g \cdot Z) = \mathrm{Ad}^*(g)\,\psi(Z).$$

Proof Let $g \in G$ and $Z \in \mathcal{D}$. We write $g \exp Z = zky$ where $z \in P^+$, $k \in K^c$ and $y \in P^-$. Then, since $g^* = g^{-1}$, we have $\exp Z^* \exp Z = y^* k^* z^* zky$. This implies that

$$\zeta(\exp Z^* \exp Z) = y^* k^* \zeta(z^* z) {k^*}^{-1}.$$

Thus, noting that $z = \exp(g \cdot Z)$, we get

$$\begin{aligned} &\exp(-(g\cdot Z)^*)\,\zeta(\exp(g\cdot Z)^* \exp(g \cdot Z)) = {z^*}^{-1}\zeta(z^* z) \\ &= g\exp(-Z^*) y^* k^* \zeta(z^* z) = g \exp(-Z^*)\,\zeta(\exp Z^* \exp Z) k^*. \end{aligned}$$

Hence we obtain $\psi(g \cdot Z) = \mathrm{Ad}^*(g)\,\psi(Z)$. □

Corollary 4.3 *The stabilizer of ξ_0 for the coadjoint action of G is K.*

Proof First, we prove that for $Z \in \mathcal{D}$ the equality $\psi(Z) = \xi_0$ implies that $Z = 0$. Assume that $\psi(Z) = \xi_0$. Then we have

$$\mathrm{Ad}^*\big(\zeta(\exp Z^* \exp Z)\big)\,\xi_0 = \mathrm{Ad}^*(\exp Z)\,\xi_0$$

or, equivalently,

$$\langle \xi_0, \mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1})X\rangle = \langle \xi_0, \mathrm{Ad}(\exp(-Z^*))X\rangle.$$

for each $X \in \mathfrak{g}^c$. Thus, taking $X = Z$ and using (1) of Lemma 4.1, we get $\langle \xi_0, [Z^*, Z]\rangle = 0$ hence $Z = 0$.

Now, consider $g \in G$ such that $\mathrm{Ad}^*(g)\xi_0 = \xi_0$. Then, by Proposition 4.2, we have $\psi(g \cdot 0) = \xi_0$ and, by the assertion already proved, we get $g \cdot 0 = \xi_0$. Hence we obtain $g \in K^c P^- \cap G = K$. □

Proposition 4.4 *The map ψ is a diffeomorphism from $\mathcal{D}$ onto $\mathcal{O}(\xi_0)$.*

Proof Let $Z \in \mathcal{D}$. There exists $g \in G$ such that $g \cdot 0 = Z$. Then, by Proposition 4.2, we have $\psi(Z) = \mathrm{Ad}^*(g)\xi_0$. This shows that ψ has values in $\mathcal{O}(\xi_0)$ and that ψ is surjective. Now, suppose that $\psi(Z) = \psi(Z')$ for some $Z, Z' \in \mathcal{D}$. Let $g, g' \in G$ such that $g \cdot 0 = Z$ and $g' \cdot 0 = Z'$. Then, by Proposition 4.2, we have $\mathrm{Ad}^*(g)\xi_0 = \mathrm{Ad}^*(g')\xi_0$. Thus, by Corollary 4.3, we get $g^{-1}g' \in K$ hence $Z = g \cdot 0 = g' \cdot 0 = Z'$. This proves that ψ is injective hence bijective.

Now, we show that ψ is regular. Using Proposition 4.2, we have just to verify that ψ is regular at $Z = 0$. By differentiating the multiplication map from $P^+ \times K^c \times P^-$ onto $P^+K^cP^-$, we easily see that, for each $g \in G$ such that $g = zky$ with $z \in P^+$, $k \in K^c$ and $y \in P^-$ and each $X \in \mathfrak{g}^c$, we have

$$d\zeta_g(X^+(g)) = (\mathrm{Ad}(z)\,p_{\mathfrak{p}^+}(\mathrm{Ad}(z^{-1})\,X))^+(z).$$

Here, we have denoted by Y^+ the right-invariant vector field generated by Y. From this, it follows that, for each $Y \in \mathfrak{p}^+$ and each $X \in \mathfrak{g}^c$, we have

$$\langle (d\psi)_0(Y), X\rangle = \langle \xi_0, [X, Y - Y^*]\rangle. \tag{4.1}$$

Now, assume that $(d\psi)_0(Y) = 0$ for some $Y \in \mathfrak{p}^+$. By taking $X = Y$ in (4.1) we get $\langle \xi_0, [Y, Y^*]\rangle = 0$ hence $Y = 0$. □

Now, we construct a section of the action of G on $\mathcal{D}$, that is, a map $Z \to g_Z$ from $\mathcal{D}$ to G such that $g_Z \cdot 0 = Z$ for each $Z \in \mathcal{D}$ and we show that ψ can be recovered by using this section. Note that such sections are useful in practice, in particular to determine explicitly $\mathcal{D}$, see, for instance [28, p. 501].

Proposition 4.5 *Let $Z \in \mathcal{D}$. There exists an element k_Z in K^c such that $k_Z^* = k_Z$ and $k_Z^2 = \kappa(\exp Z^* \exp Z)^{-1}$. Each $g \in G$ such that $g \cdot 0 = Z$ is then of the form $g = \exp(-Z^*)\,\zeta(\exp Z^* \exp Z)k_Z^{-1}h$ where $h \in K$. Consequently, the map $Z \to g_Z := \exp(-Z^*)\,\zeta(\exp Z^* \exp Z)k_Z^{-1}$ is a section for the action of G on $\mathcal{D}$. In particular, by using the equality $\psi(Z) = \mathrm{Ad}^*(g_Z)\xi_0$, we recover the expression of ψ given above.*

Proof Let $Z \in \mathcal{D}$ and $g \in G$ such that $g \cdot 0 = Z$. Then we can write $g = (\exp Z)ky$ where $k \in K^c$ and $y \in P^-$. Thus we have

$$g^*g = y^*k^*(\exp Z^* \exp Z)ky = e.$$

Consequently, passing to the K^c-component, we get $k^*\kappa(\exp Z^* \exp Z)k = e$. Now, using the polar decomposition $K^c = \exp(i\mathfrak{k})K$ [28, p. 506], we can write $k = k_Z h$ where $k_Z \in \exp(i\mathfrak{k})$ and $h \in K$. Hence we obtain $k_Z^2 = \kappa(\exp Z^* \exp Z)^{-1}$. Moreover, passing similarly to the P^--component, we get $k^{-1}\eta(\exp Z^* \exp Z)ky = e$ hence $ky = \eta(\exp Z^* \exp Z)^{-1}k$. This gives

$$\begin{aligned} g &= \exp Z\eta(\exp Z^* \exp Z)^{-1}k \\ &= \exp(-Z^*)(\exp Z^* \exp Z)\eta(\exp Z^* \exp Z)^{-1}k_Z h \\ &= \exp(-Z^*)\,\zeta(\exp Z^* \exp Z)k_Z^{-1}h. \end{aligned}$$

This shows the second assertion of the proposition. Finally, writing

$$\begin{aligned} \psi(Z) = \mathrm{Ad}^*(g_Z)\xi_0 &= \mathrm{Ad}^*(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z)k_Z^{-1})\xi_0 \\ &= \mathrm{Ad}^*(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z))\xi_0, \end{aligned}$$

we recover the expression of ψ. □

5 Example 1: the unitary group $SU(p, q)$

In this section, we take $G = SU(p, q)$ and $K = S(U(p) \times U(q))$. Recall that K consists of the matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad A \in U(p),\ D \in U(q), \quad \mathrm{Det}(A)\,\mathrm{Det}(D) = 1.$$

For $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{g}^c$ we have $X^* = \left(\begin{smallmatrix} -A^\star & C^\star \\ B^\star & -D^\star \end{smallmatrix}\right)$ where $\star$ denotes conjugate-transposition.

Let $\mathfrak{h}$ be the abelian subalgebra of $\mathfrak{k}$ consisting of the matrices

$$\begin{pmatrix} iaI_p & 0 \\ 0 & ibI_q \end{pmatrix}, \qquad a, b \in \mathbb{R}, \quad pa + bq = 0.$$

Then $\mathfrak{h}^c$ consists of all matrices $X = \mathrm{Diag}(x_1, x_2, \ldots, x_{p+q})$, $x_k \in \mathbb{C}$, such that $\sum_{k=1}^{p+q} x_k = 0$. The set of roots of $\mathfrak{h}^c$ on $\mathfrak{g}^c$ is $\lambda_i - \lambda_j$ for $1 \leq i \neq j \leq p+q$ where $\lambda_i(X) = x_i$ for $X \in \mathfrak{h}^c$ as above. The set of compact roots is $\lambda_i - \lambda_j$ for $1 \leq i \neq j \leq p$ and $p+1 \leq i \neq j \leq p+q$. We take the set of positive roots Δ^+ to be $\lambda_i - \lambda_j$ for $1 \leq i < j \leq p+q$. Then we have

$$P^+ = \left\{ \begin{pmatrix} I_p & Z \\ 0 & I_q \end{pmatrix} : Z \in M_{pq}(\mathbb{C}) \right\}, \qquad P^- = \left\{ \begin{pmatrix} I_p & 0 \\ Y & I_q \end{pmatrix} : Y \in M_{qp}(\mathbb{C}) \right\}.$$

In the rest of this section, we identify $\mathfrak{p}^+$ to $M_{pq}(\mathbb{C})$ by means of the map $Z \to \left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right)$.

The $P^+K^cP^-$-decomposition of a matrix $g \in G^c$ is given by

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_p & BD^{-1} \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_p & 0 \\ D^{-1}C & I_q \end{pmatrix}. \tag{5.1}$$

Note that a matrix $g \in G^c$ have such a decomposition if and only if $\mathrm{Det}(D) \neq 0$. In particular we verify that $G \subset P^+K^cP^-$. Moreover, the action of G^c on $\mathcal{D}$ is then given by

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Note that $g \cdot 0 = BD^{-1} = Z$ satisfies $I_p - ZZ^\star > 0$ [28]. From this we see that

$$\mathcal{D} = \{Z \in M_{pq}(\mathbb{C}) \colon I_p - ZZ^\star > 0\}.$$

The Killing form β on $\mathfrak{g}^c$ is defined by $\beta(X, Y) := 2(p+q)\,\mathrm{Tr}(XY)$ [31, p. 295]. We identify G-equivariantly $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of β. We easily verify that the set of all elements of $\mathfrak{g}$ fixed by K is $\mathfrak{h}$. Each $\xi_0 \in \mathfrak{h}$ can be written as

$$\xi_0 = i\lambda \begin{pmatrix} -qI_p & 0 \\ 0 & pI_q \end{pmatrix}$$

where $\lambda \in \mathbb{R}$. Then we have $\langle \xi_0, [Z^*, Z] \rangle = -2i\lambda(p+q)^2\,\mathrm{Tr}(ZZ^\star)$ for each $Z \in \mathcal{D}$. This shows that ξ_0 is regular if and only if $\lambda \neq 0$. In that case, we can compute the section $Z \to g_Z$ hence $\psi(Z)$ as follows. For $Z \in \mathcal{D}$, we have

$$\exp Z^* \exp Z = \begin{pmatrix} I_p & Z \\ -Z^\star & I_q - Z^\star Z \end{pmatrix}.$$

Then, by (5.1), we get

$$\kappa(\exp Z^* \exp Z) = \begin{pmatrix} (I_p - ZZ^\star)^{-1} & 0 \\ 0 & I_q - Z^\star Z \end{pmatrix},$$

$$\zeta(\exp Z^* \exp Z) = \begin{pmatrix} I_p & Z(I_q - Z^\star Z)^{-1} \\ 0 & I_q \end{pmatrix}$$

and we can take

$$k_Z = \begin{pmatrix} (I_p - ZZ^\star)^{1/2} & 0 \\ 0 & (I_q - Z^\star Z)^{-1/2} \end{pmatrix}.$$

Thus we have

$$g_Z = \exp(-Z^*)\zeta(\exp Z^* \exp Z)k_Z^{-1} = \begin{pmatrix} (I_p - ZZ^\star)^{-1/2} & Z(I_q - Z^\star Z)^{-1/2} \\ Z^\star(I_p - ZZ^\star)^{-1/2} & (I_q - Z^\star Z)^{-1/2} \end{pmatrix}.$$

Hence we obtain

$$\psi(Z) = i\lambda \begin{pmatrix} (I_p - ZZ^\star)^{-1}(-pZZ^\star - qI_p) & (p+q)Z(I_q - Z^\star Z)^{-1} \\ -(p+q)(I_q - Z^\star Z)^{-1}Z^\star & (pI_q + qZ^\star Z)(I_q - Z^\star Z)^{-1} \end{pmatrix}.$$

6 Example 2: the Jacobi group

The Jacobi group is the semi-direct product of the $(2n+1)$-dimensional real Heisenberg group by the symplectic group $Sp(n, \mathbb{R})$. This group plays an important role in different areas of Mathematics and Physics, see [10] and [6]. In particular, the Jacobi group appears as an important example of non-reductive Lie group of Harish-Chandra type [29], [28] and its holomorphic unitary representations were studied in [28], [9], [10], [6] and [7].

Consider the symplectic form ω on $\mathbb{C}^{2n} \times \mathbb{C}^{2n}$ defined by

$$\omega((z,w),(z',w')) = \frac{i}{2}\sum_{k=1}^{n}(z_k w'_k - z'_k w_k).$$

for $z, w, z', w' \in \mathbb{C}^n$. The $(2n+1)$-dimensional real Heisenberg group is

$$H := \{((z,\bar{z}),c) \colon z \in \mathbb{C}^n, c \in \mathbb{R}\}$$

endowed with the multiplication

$$((z,\bar{z}),c)\cdot((z',\bar{z}'),c') = ((z+z',\bar{z}+\bar{z}'),c+c'+\tfrac{1}{2}\omega((z,\bar{z}),(z',\bar{z}'))). \quad (6.1)$$

Then the complexification H^c of H is

$$H^c := \{((z,w),c) \colon z, w \in \mathbb{C}^n, c \in \mathbb{C}\}$$

and the multiplication of H^c is obtained by replacing $(z,\bar{z})$ by (z,w) and $(z',\bar{z}')$ by (z',w') in (6.1). We denote by $\mathfrak{h}$ and $\mathfrak{h}^c$ the Lie algebras of H and H^c.

Now consider the group $S := Sp(n,\mathbb{C}) \cap SU(n,n) \simeq Sp(n,\mathbb{R})$ [28, p. 501], [24, p. 175]. Then S consists of all matrices

$$h = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \quad P, Q \in M_n(\mathbb{C}), \quad PP^\star - QQ^\star = I_n, \quad PQ^t = QP^t$$

and $S^c = Sp(n,\mathbb{C})$.

The group S acts on H by $h \cdot ((z,\bar{z}), c) = h(z,\bar{z}) = Pz + Q\bar{z}$ where the elements of $\mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}^n$ are considered as column vectors. Then we can form the semi-direct product $G := H \rtimes S$ called the Jacobi group. The elements of G can be written as $((z,\bar{z}), c, h)$ where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $h \in S$. The multiplication of G is thus given by

$$((z,\bar{z}), c, h) \cdot ((z',\bar{z}'), c', h') = ((z,\bar{z}) + h(z',\bar{z}'), c + c' + \tfrac{1}{2}\omega((z,\bar{z}), h(z',\bar{z}')), hh').$$

The complexification G^c of G is then the semi-direct product $G^c = H^c \rtimes Sp(n,\mathbb{C})$ and the multiplication of G^c is obtained by replacing $\bar{z}$ and $\bar{z}'$ by w and w' in the preceding formula. We denote by $\mathfrak{s}$, $\mathfrak{s}^c$, $\mathfrak{g}$ and $\mathfrak{g}^c$ the Lie algebras of S, S^c, G and G^c. The Lie bracket of $\mathfrak{g}^c$ is given by

$$[((z,w), c, A), ((z',w'), c', A')] = (A(z',w') - A'(z,w), \omega((z,w),(z',w')), [A, A']).$$

We easily verify that

$$\text{if } X = \Big((z,w), c, \big(\begin{smallmatrix} A & B \\ C & -A^t \end{smallmatrix}\big)\Big) \in \mathfrak{g}^c \text{ then } X^* = \Big((-\bar{w}, -\bar{z}), -\bar{c}, \big(\begin{smallmatrix} \bar{A}^t & -\bar{C} \\ -\bar{B} & -\bar{A} \end{smallmatrix}\big)\Big).$$

We take K to be the subgroup of G consisting of all elements $\big((0,0), c, \big(\begin{smallmatrix} P & 0 \\ 0 & \bar{P} \end{smallmatrix}\big)\big)$ where $c \in \mathbb{R}$ and $P \in U(n)$. Then the Lie algebra $\mathfrak{k}$ of K is a maximal compactly embedded subalgebra of $\mathfrak{g}$ and the subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ consisting of elements of the form $((0,0), c, A)$ where A is diagonal is a compactly embedded Cartan subalgebra of $\mathfrak{g}$ [28, p. 250]. Choosing an adapted positive system of non-compact positive roots relative to $\mathfrak{t}$ as in [28, p. 249], we get

$$\mathfrak{p}^+ = \left\{ a(z,Z) := \left((z,0), 0, \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}\right) : z \in \mathbb{C}^n, Z \in M_n(\mathbb{C}), Z^t = Z \right\}$$

and

$$\mathfrak{p}^- = \left\{ \left((0,w), 0, \begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}\right) : w \in \mathbb{C}^n, W \in M_n(\mathbb{C}), W^t = W \right\}.$$

Then we obtain

$$P^+ = \left\{ \left((z,0), 0, \begin{pmatrix} I_n & Z \\ 0 & I_n \end{pmatrix}\right) : z \in \mathbb{C}^n, Z \in M_n(\mathbb{C}), Z^t = Z \right\}$$

and

$$P^- = \left\{ \left((0,w), 0, \begin{pmatrix} I_n & 0 \\ W & I_n \end{pmatrix}\right) : w \in \mathbb{C}^n, W \in M_n(\mathbb{C}), W^t = W \right\}.$$

Thus we easily verify that $g = \big((z_0,w_0),c_0,\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)\big) \in G^c$ has a $P^+K^cP^-$-decomposition

$$g=\left((z,0),0,\begin{pmatrix} I_n & Z\\ 0 & I_n\end{pmatrix}\right)\cdot\left((0,0),c,\begin{pmatrix} P & 0\\ 0 & (P^t)^{-1}\end{pmatrix}\right)\cdot\left((0,w),0,\begin{pmatrix} I_n & 0\\ W & I_n\end{pmatrix}\right)$$

if and only if $\mathrm{Det}(D)\neq 0$ and, in this case, we have $z=z_0-BD^{-1}w_0$, $Z=BD^{-1}$, $w=D^{-1}w_0$, $W=D^{-1}C$, $P=A-BD^{-1}C=(D^t)^{-1}$ and $c=c_0-(1/4)i(z_0-BD^{-1}w_0)^tw_0$. From this, we deduce that the action of $g=\big((z_0,w_0),c_0,\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)\big)\in G^c$ on $a(z,Z)\in\mathfrak{p}^+$ is given by $g\cdot a(z,Z)=a(z',Z')$ where $Z'=(AZ+B)(CZ+D)^{-1}$ and

$$z'=z_0+Az-(AZ+B)(CZ+D)^{-1}(w_0+Cz).$$

This implies that

$$\mathcal{D}=G\cdot 0=\{a(z,Z)\in\mathfrak{p}^+ : I_n-Z\bar{Z}>0\}.$$

Now we aim to compute the coadjoint action of G^c. This can be done as follows. First, we compute the adjoint action of G^c. Let $g=(v_0,c_0,h_0)\in G^c$ where $v_0\in\mathbb{C}^{2n}$, $c_0\in\mathbb{C}$ and $h_0\in S^c=Sp(n,\mathbb{C})$ and $X=(w,c,U)\in\mathfrak{g}^c$ where $w\in\mathbb{C}^{2n}$, $c\in\mathbb{C}$ and $U\in\mathfrak{s}^c$. We set $\exp(tX)=(w(t),c(t),\exp(tU))$. Then, since the derivatives of $w(t)$ and $c(t)$ at $t=0$ are w and c, we find that

$$\begin{aligned}\mathrm{Ad}(g)X &= \frac{d}{dt}(g\exp(tX)g^{-1})|_{t=0}\\ &= \big(h_0w-(\mathrm{Ad}(h_0)U)v_0, c+\omega(v_0,h_0w)-\tfrac{1}{2}\omega(v_0,(\mathrm{Ad}(h_0)U)v_0),\mathrm{Ad}(h_0)U\big).\end{aligned}$$

On the other hand, let us denote by $\xi=(u,d,\varphi)$, where $u\in\mathbb{C}^{2n}$, $d\in\mathbb{C}$ and $\varphi\in(\mathfrak{s}^c)^*$, the element of $(\mathfrak{g}^c)^*$ defined by

$$\langle\xi,(w,c,U)\rangle=\omega(u,w)+dc+\langle\varphi,U\rangle.$$

Moreover, for $u,v\in\mathbb{C}^{2n}$, we denote by $v\times u$ the element of $(\mathfrak{s}^c)^*$ defined by $\langle v\times u,U\rangle:=\omega(u,Uv)$ for $U\in\mathfrak{s}^c$.

Let $\xi=(u,d,\varphi)\in(\mathfrak{g}^c)^*$ and $g=(v_0,c_0,h_0)\in G^c$. Then, by using the relation $\langle\mathrm{Ad}^*(g)\xi,X\rangle=\langle\xi,\mathrm{Ad}(g^{-1})X\rangle$ for $X\in\mathfrak{g}^c$, we obtain

$$\mathrm{Ad}^*(g)\xi=\big(h_0u-dv_0,d,\mathrm{Ad}^*(h_0)\varphi+v_0\times(h_0u-\tfrac{d}{2}v_0)\big)$$

By restriction, we also get the formula for the coadjoint action of G. Now, we are in position to determine the scalar elements of $(\mathfrak{g}^c)^*$.

Proposition 6.1

1. *The elements ξ_0 of $\mathfrak{g}^*$ fixed by K are the elements of the form $(0,d,\varphi_\lambda)$ where $d,\lambda\in\mathbb{R}$ and $\varphi_\lambda\in\mathfrak{s}^*$ is defined by $\langle\varphi_\lambda,\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)\rangle=i\lambda\,\mathrm{Tr}(A)$.*
2. *Let $\xi_0=(0,d,\varphi_\lambda)$ as above. Then ξ_0 is regular hence scalar if and only if $\lambda d\neq 0$.*

Proof (1) Let $\xi_0 = ((u_0, \bar{u}_0), d, \varphi) \in \mathfrak{g}^*$ where $u_0 \in \mathbb{C}^n$, $d \in \mathbb{R}$ and $\varphi \in \mathfrak{s}^*$. Assume that ξ_0 is fixed by K. Then for each $k = \big((u_0, \bar{u}_0), c_0, \left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P}\end{smallmatrix}\right)\big) \in K$ with $u_0 \in \mathbb{C}^n$, $c_0 \in \mathbb{R}$ and $P \in U(n)$, we have

$$\mathrm{Ad}^*(k)\xi_0 = \big((Pu_0, \bar{P}\bar{u}_0), d, \mathrm{Ad}^*\left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P}\end{smallmatrix}\right)\varphi\big) = ((u_0, \bar{u}_0), d, \varphi).$$

This gives $Pu_0 = u_0$ for each $P \in U(n)$ hence $u_0 = 0$ and $\mathrm{Ad}^*(k_0)\varphi = \varphi$ for each k_0 in the subgroup K_0 of S consisting of the matrices of the form $\left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P}\end{smallmatrix}\right)$ where $P \in U(n)$. Then, denoting by $\mathfrak{k}_0$ the Lie algebra of K_0, we have $\langle \varphi, [U, X]\rangle = 0$ for each $U \in \mathfrak{k}_0$ and each $X \in \mathfrak{s}$. This implies that φ is zero on $[\mathfrak{k}_0, \mathfrak{k}_0]$ and also on the elements of $\mathfrak{s}$ of the form $\left(\begin{smallmatrix} 0 & Q \\ \bar{Q} & 0\end{smallmatrix}\right)$. Then φ is completely determined by its value on the element $\left(\begin{smallmatrix} iI_n & 0 \\ 0 & -iI_n\end{smallmatrix}\right)$ which generates the center of $\mathfrak{k}_0$, hence the result.

(2) Let ξ_0 as above. Then we have $\langle \xi_0, [a(z,Z)^*, a(z,Z)]\rangle = d|z|^2 + i\lambda\,\mathrm{Tr}(Z\bar{Z})$. The result follows. $\square$

In the rest of this section, we fix a scalar element $\xi_0 = (0, d, \varphi_\lambda)$ of $\mathfrak{g}^*$ as above and we compute $\psi(a(z,Z))$ for $a(z,Z) \in \mathcal{D}$. In order to make the expression of $\psi(a(z,Z))$ more explicit, we introduce the following notation. For $\varphi \in \mathfrak{s}^*$, let $\theta(\varphi)$ the unique element of $\mathfrak{s}$ such that $\langle \varphi, X\rangle = \mathrm{Tr}(\theta(\varphi)X)$ for each $X \in \mathfrak{s}$. In particular, one has $\theta(\varphi_\lambda) = \frac{\lambda}{2}\left(\begin{smallmatrix} iI_n & 0 \\ 0 & -iI_n\end{smallmatrix}\right)$. Moreover, for $u = (x, \bar{x}) \in \mathbb{C}^{2n}$ and $u = (y, \bar{y}) \in \mathbb{C}^{2n}$ we have

$$\theta(v \times u) = \frac{1}{2}\begin{pmatrix} -iy\bar{x}^t & iyx^t \\ -i\bar{y}\bar{x}^t & i\bar{y}x^t\end{pmatrix}.$$

Note also that θ intertwines Ad^* and Ad.

Proposition 6.2 *The map $\psi\colon \mathcal{D} \to \mathcal{O}(\xi_0)$ is given by*

$$\psi(a(y,Z)) = \big(-d(y_1, \bar{y}_1), d, \varphi(y,Z)\big)$$

where $y_1 = (I_n - Z\bar{Z})^{-1}(y + Z\bar{y})$ and

$$\varphi(y,Z) := \mathrm{Ad}^*\begin{pmatrix} (I_n - Z\bar{Z})^{-1/2} & (I_n - Z\bar{Z})^{-1/2}Z \\ (I_n - \bar{Z}Z)^{-1/2}\bar{Z} & (I_n - \bar{Z}Z)^{-1/2}\end{pmatrix}\varphi_\lambda - \frac{d}{2}(y_1, \bar{y}_1) \times (y_1, \bar{y}_1).$$

Moreover, we have

$$\theta(\varphi(y,Z)) = -\frac{d}{4}\begin{pmatrix} -iy_1\bar{y}_1^t & iy_1y_1^t \\ -i\bar{y}_1\bar{y}_1^t & i\bar{y}_1y_1^t\end{pmatrix} + \frac{\lambda}{2}i$$

$$\times\begin{pmatrix} (I_n + Z\bar{Z})(I_n - Z\bar{Z})^{-1/2}(I_n - \bar{Z}Z)^{-1/2} & -2Z(I_n - \bar{Z}Z)^{-1/2}(I_n - Z\bar{Z})^{-1/2} \\ 2\bar{Z}(I_n - Z\bar{Z})^{-1/2}(I_n - \bar{Z}Z)^{-1/2} & -(I_n + \bar{Z}Z)(I_n - \bar{Z}Z)^{-1/2}(I_n - Z\bar{Z})^{-1/2}\end{pmatrix}.$$

Proof For $(y, Z) \in \mathbb{C}^n \times M_n(\mathbb{C})$ such that $a(y,Z) \in \mathcal{D}$ we set

$$g(y,Z) := \left((y_1, \bar{y}_1), 0, \begin{pmatrix} (I_n - Z\bar{Z})^{-1/2} & (I_n - Z\bar{Z})^{-1/2}Z \\ (I_n - \bar{Z}Z)^{-1/2}\bar{Z} & (I_n - \bar{Z}Z)^{-1/2}\end{pmatrix}\right) \in G$$

where $y_1 = (I_n - Z\bar{Z})^{-1}(y + Z\bar{y})$. Then the map $a(y, Z) \to g(y, Z)$ is a section for the action of G on $\mathcal{D}$ and we have $\psi(a(y, Z)) = \mathrm{Ad}^*(g(y, Z))\xi_0$ (in fact, we use here this section since the expression of the section given by Proposition 4.5 is too complicated in this case). Thus, by using the formula for the coadjoint action of G and the above considerations on θ, we easily obtain the desired result. $\square$

References

[1] Arnal, D.. Cortet, J.-C.: *Nilpotent Fourier transform and applications.* Lett. Math. Phys. **9** (1985), 25–34.

[2] Bar-Moshe, D.: *A method for weight multiplicity computation based on Berezin quantization.* SIGMA **5** (2009), 091, 1–12.

[3] Bar-Moshe, D., Marinov, M. S.: *Realization of compact Lie algebras in Kähler manifolds.* J. Phys. A: Math. Gen. **27** (1994), 6287–6298.

[4] Berezin, F. A.: *Quantization.* Math. USSR Izv. **8**, 5 (1974), 1109–1165.

[5] Berezin, F. A.: *Quantization in complex symmetric domains.* Math. USSR Izv. **9**, 2 (1975), 341–379.

[6] Berceanu, S.: *A holomorphic representation of the Jacobi algebra.* Rev. Math. Phys. **18** (2006), 163–199.

[7] Berceanu, S., Gheorghe, A.: *On the geometry of Siegel–Jacobi domains.* Int. J. Geom. Methods Mod. Phys. **8** (2011), 1783–1798.

[8] Bernatska, J., Holod, P.: *Geometry and topology of coadjoint orbits of semisimple Lie groups.* Mladenov I. M., de León M. (eds) Proceedings of the 9th international conference on 'Geometry, Integrability and Quantization', June 8–13, 2007, Varna, *Bulgarian Academy of Sciences*, Sofia, 2008, 146–166.

[9] Berndt, R., Böcherer, S.: *Jacobi forms and discrete series representations of the Jacobi group.* Math. Z. **204** (1990), 13–44.

[10] Berndt, R., Schmidt, R.: Elements of the Representation Theory of the Jacobi Group. Progress in Mathematics **163**, *Birkhäuser Verlag*, Basel, 1988.

[11] Cahen, B.: *Deformation program for principal series representations.* Lett. Math. Phys. **36** (1996), 65–75.

[12] Cahen, B.: *Contraction de $SU(1,1)$ vers le groupe de Heisenberg.* In: Mathematical works, Part XV, Séminaire de Mathématique *Université du Luxembourg*, Luxembourg, (2004), 19–43.

[13] Cahen, B.: *Weyl quantization for semidirect products.* Diff. Geom. Appl. **25** (2007), 177–190.

[14] Cahen, B.: *Multiplicities of compact Lie group representations via Berezin quantization.* Mat. Vesnik **60** (2008), 295–309.

[15] Cahen, B.: *Contraction of compact semisimple Lie groups via Berezin quantization.* Illinois J. Math. **53**, 1 (2009), 265–288.

[16] Cahen, B.: *Berezin quantization on generalized flag manifolds.* Math. Scand. **105** (2009), 66–84.

[17] Cahen, B.: *Contraction of discrete series via Berezin quantization.* J. Lie Theory **19** (2009), 291–310.

[18] Cahen, B.: *Berezin quantization for discrete series.* Beiträge Algebra Geom. **51** (2010), 301–311.

[19] Cahen, B.: *Stratonovich-Weyl correspondence for discrete series representations.* Arch. Math. (Brno) **47** (2011), 41–58.

[20] Cahen, B.: *Berezin quantization and holomorphic representations.* Rend. Sem. Mat. Univ. Padova, to appear.

[21] Cahen, M., Gutt, S., Rawnsley, J.: *Quantization on Kähler manifolds I, Geometric interpretation of Berezin quantization.* J. Geom. Phys. **7** (1990), 45–62.

[22] Cahen, M., Gutt, S., Rawnsley, J.: *Quantization on Kähler manifolds III.* Lett. Math. Phys. **30** (1994), 291–305.

[23] Cotton, P., Dooley, A. H.: *Contraction of an adapted functional calculus.* J. Lie Theory **7** (1997), 147–164.

[24] Folland, B.: Harmonic Analysis in Phase Space., *Princeton Univ. Press*, Princeton, 1989.

[25] Helgason, S.: Differential Geometry, Lie Groups and Symmetric Spaces. Graduate Studies in Mathematics **34**, *American Mathematical Society*, Providence, Rhode Island, 2001.

[26] Kirillov, A. A.: Lectures on the Orbit Method, Graduate Studies in Mathematics. **64**, *American Mathematical Society*, Providence, Rhode Island, 2004.

[27] Kostant, B.: *Quantization and unitary representations.* In: Modern Analysis and Applications, Lecture Notes in Mathematics **170**, *Springer-Verlag*, Berlin, Heidelberg, New York, (1970), 87–207.

[28] Neeb, K-H.: Holomorphy and Convexity in Lie Theory. de Gruyter Expositions in Mathematics **28**, *Walter de Gruyter*, Berlin, New York, 2000.

[29] Satake, I: Algebraic Structures of Symmetric Domains. *Iwanami Sho-ten, Tokyo and Princeton Univ. Press*, Princeton, NJ, 1971.

[30] Skrypnik, T. V.: *Coadjoint orbits of compact Lie groups and generalized stereographic projection.* Ukr. Math. J. **51** (1999), 1939–1944.

[31] Varadarajan, V. S.: Lie groups, Lie Algebras and Their Representations. Graduate Texts in Mathematics **102**, *Springer-Verlag*, Berlin, 1984.