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Control Systems on the Orthogonal Group SO(4)

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Abstract. We classify the left-invariant control affine systems evolving on the orthogonal group $\mathsf{SO}(4)$. The equivalence relation under consideration is detached feedback equivalence. Each possible number of inputs is considered; both the homogeneous and inhomogeneous systems are covered. A complete list of class representatives is identified and controllability of each representative system is determined.

1 Introduction

A control system is given by a dynamical polysystem together with a class of "admissible inputs" (also called controls). More precisely, a (smooth) *control system* Σ on M consists of a family $\mathcal{X} = (\Xi_u)_{u\in U}$ of smooth vector fields on the state space M and an input class $\mathcal{U}$. M is a smooth (real, finite-dimensional) manifold, and an element of $\mathcal{U}$ is a U-valued map (defined on some interval of $\mathbb{R}$) which is (Lebesgue) measurable or piecewise constant, or of some regularity type between these two possibilities. The input set U is usually equipped with a separable metric space structure. For the purposes of this paper, we shall assume that $U = \mathbb{R}^\ell$. In classical notation, a control system Σ on M is written as

$$\Sigma: \quad \dot{x} = \Xi(x, u), \qquad x \in \mathsf{M}, \ u \in U .$$

Here $\Xi\colon \mathsf{M} \times U \to T\mathsf{M}, (x, u) \mapsto \Xi(x, u) = \Xi_u(x) \in T_x\mathsf{M}$ is the map describing the dynamics (i.e., the vector fields) of the system. We assume that Ξ is a smooth map. Standard references for nonlinear control systems are [16], [24]. When the state space is a (real, finite-dimensional) Lie group G and the dynamics $\Xi_u = \Xi(\cdot, u)$ are left invariant, the control system is termed as *left-invariant*. Such control systems have been studied by a number of authors over the past few decades (see, e.g., [3], [19], [20], [26], [28]).

A *trajectory* of Σ (corresponding to an admissible input $u(\cdot) \in \mathcal{U}$) is an absolutely continuous curve γ in M such that $\dot{\gamma}(t) = \Xi_{u(t)}(\gamma(t))$ for almost all t.

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Carathéodory's existence and uniqueness theorem guarantees the local existence and global uniqueness of trajectories. The initial condition (initial state) is just a starting point for the trajectory; different admissible inputs provide, generally speaking, different trajectories starting from a fixed state. All these trajectories fill the set attainable from the given initial state. To characterize such sets is the first natural problem in control theory: the *controllability problem*. As soon as the possibility to attain a certain state is established, we try to do it in the best possible way. This is the *optimal control problem* (see, e.g., [3], [19]).

The most natural equivalence relation for control systems is equivalence up to coordinate changes in the state space. This is called *state space equivalence*. We say that two control systems Σ and $\widetilde{\Sigma}$ are state space equivalent if there exists a diffeomorphism ϕ between the state spaces which transforms the dynamics Ξ_u to $\widetilde{\Xi}_u$. State space equivalence is well understood ([17]). It establishes a one-to-one correspondence between the trajectories of the equivalent systems (corresponding to the same admissible inputs). This equivalence relation is very strong; any general classification appears to be very difficult if not impossible. However, some reasonable classification in low dimensions is possible (see [2], [11]).

Another important equivalence relation for control systems is that of *feedback equivalence*. Applying feedback transformations means that we also modify the controls (which remain unchanged for state space equivalence) in a way that is state dependent. (Feedback control may be used to achieve desired dynamical properties of the system, like stabilizability.) We say that two control systems Σ and $\widetilde{\Sigma}$ are feedback equivalent if there exists a diffeomorphism $\widetilde{x} = \phi(x)$ between the state spaces and an invertible transformation $\widetilde{u} = \varphi(x, u)$ of controls such that the diffeomorphism $\Phi(x, u) = (\phi(x), \varphi(x, u))$ brings Σ into $\widetilde{\Sigma}$. Feedback equivalent systems have geometrically the same set of trajectories which are parametrized differently by admissible inputs. Feedback equivalence has been extensively studied in the last few decades (see [25] and the references therein). Many problems concerning feedback equivalence are studied and solved for control affine systems (i.e., control systems with dynamics affine in controls) and then extended to the general case (for details, see [17], [25]).

In the context of *left-invariant* control systems, feedback equivalence is specialized by requiring that the feedback transformations are independent of the state variable. Such transformations are precisely those that are compatible with the Lie group structure. This is called *detached feedback equivalence*. It turns out that two (full-rank) left-invariant control systems are detached feedback equivalent if and only if there exists a Lie group isomorphism between the state spaces, relating their dynamics. Several classes of left-invariant control affine systems have recently been classified (cf. [7], [9]).

In this paper we consider left-invariant control affine systems, evolving on the (six-dimensional) orthogonal group $\mathsf{SO}(4)$. These systems have the form

$$\Sigma : \quad \dot{g} = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \qquad g \in \mathsf{SO}(4), \; u \in \mathbb{R}^\ell$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{so}(4)$. (The elements $B_1, \ldots, B_\ell$ are assumed to be linearly independent.) The aim is to classify, under detached feedback equivalence, all such systems; a list of class representatives will be produced. In addition, we identify precisely those systems which are controllable. The homogeneous systems are considered first. The single-input, two-input, and three-input systems are classified by exploiting the singular value decomposition. The classification of the four-input and five-input systems follow as corollaries. For the inhomogeneous systems, the classification is based, in each case, on its homogeneous counterpart.

We conclude the paper with a few remarks. Moreover, we refer briefly to other works on $\mathsf{SO}(4)$ (and its Lie algebra) dealing with some interesting variational problems as well as integrable Hamiltonian systems (and their applications).

A tabulation of the classification in matrix form is appended.

2 Invariant control systems

An (ℓ-input) left-invariant control affine system Σ on G is a control system of the form

$$\Sigma: \quad \dot{g} = g\,\Xi(\mathbf{1}, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G}\,,\ u \in \mathbb{R}^\ell.$$

Here G is a (real, finite-dimensional) connected matrix Lie group with Lie algebra $\mathfrak{g}$. The *parametrization map* $\Xi(\mathbf{1}, \cdot)\colon \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine map (i.e., $B_1, \dots, B_\ell$ are linearly independent). Note that the dynamics $\Xi_u = \Xi(\cdot, u)$ are invariant under left translations, i.e., $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$. Such a system is denoted by $\Sigma = (\mathsf{G}, \Xi)$ (cf. [6]). Σ is completely determined by the specification of its state space G and its parametrization map $\Xi(\mathbf{1}, \cdot)$. Hence, for a fixed G, we shall specify Σ by simply writing

$$\Sigma: \quad A + u_1 B_1 + \cdots + u_\ell B_\ell\,.$$

The *trace* $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot) = A + \Gamma^0 = A + \langle B_1, \dots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$. A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Σ has *full rank* if the Lie algebra generated by its trace coincides with $\mathfrak{g}$.

The admissible inputs are piecewise-continuous maps $u(\cdot)\colon [0, T] \to \mathbb{R}^\ell$. A *trajectory* for an admissible input $u(\cdot)$ is an absolutely continuous curve $g(\cdot)\colon [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. A system Σ is said to be controllable if, given any pair of points $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot)$ such that $g(0) = g_0$ and $g(T) = g_1$. If Σ is controllable, then it has full rank. Moreover, if Σ is homogeneous or if G is compact, then the full-rank condition implies controllability. For more details on invariant control systems see, e.g., [19], [20], [26].

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ be two systems on G. We say that Σ and Σ' are (locally) *detached feedback equivalent* if there exist open neighbourhoods N and N' of (the unit element) $\mathbf{1}$ and a (local) diffeomorphism $\Phi = \phi \times \varphi\colon N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^\ell$ such that $\phi(\mathbf{1}) = \mathbf{1}$ and $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. (Here $T_g\phi$ denotes the tangent map of ϕ at g.)

Proposition 1 ([12]). *Two full-rank systems Σ and Σ' are detached feedback equivalent if and only if there exists a Lie algebra automorphism $\psi\colon \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \Gamma = \Gamma'$.*

Proof. (Sketch) Suppose Σ and Σ' are detached feedback equivalent. Then

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'\big(\mathbf{1}, \varphi(u)\big)$$

and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Let $u, v \in \mathbb{R}^\ell$, and let $\Xi_u = \Xi(\cdot, u)$ and $\Xi_v = \Xi(\cdot, v)$ denote the corresponding vector fields. Then $\phi_*[\Xi_u, \Xi_v] = [\phi_*\Xi_u, \phi_*\Xi_v]$ and so

$$T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), \Xi_v(\mathbf{1})] = [\Xi'_{\varphi(u)}(\mathbf{1}), \Xi'_{\varphi(v)}(\mathbf{1})] = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})]\,.$$

As the elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^\ell$, generate the Lie algebra, it follows that $T_{\mathbf{1}}\phi$ is a Lie algebra isomorphism. Conversely, suppose we have a Lie algebra isomorphism ψ such that $\psi \cdot \Gamma = \Gamma'$. Then there exist neighbourhoods N and N' of $\mathbf{1}$ and a (local) group isomorphism $\phi \colon N \to N'$ such that $T_{\mathbf{1}}\phi = \psi$ (see, e.g., [21]). The equation $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$ defines an affine isomorphism $\varphi \colon \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$. Consequently

$$T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}}L_{\phi(g)} \cdot \psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\phi(g), \varphi(u))\,.$$

Hence Σ and Σ' are detached feedback equivalent. $\square$

In this paper, we shall find it convenient to use the above characterization as the definition of equivalence. More precisely, we say that two (not necessarily full-rank) systems Σ and Σ' are *equivalent* if there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \Gamma'$. In particular, if $\Gamma = \Gamma'$, then we say that Σ' is a *reparametrization* of Σ. Notice that if two systems are equivalent, then they are detached feedback equivalent. (The converse, however, does not hold.) Any two equivalent systems are either both controllable or neither is controllable whenever the full-rank condition is equivalent to controllability.

3 The orthogonal group SO(4)

The orthogonal group

$$\mathsf{SO}(4) = \big\{g \in \mathsf{GL}(4, \mathbb{R}) : g^\top g = \mathbf{1},\ \det g = 1\big\}$$

is a six-dimensional semisimple compact connected Lie group. Its Lie algebra

$$\mathfrak{so}(4) = \big\{A \in \mathbb{R}^{4\times 4} : A^\top + A = \mathbf{0}\big\}$$

is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Let

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{E}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{E}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the standard (ordered) basis for $\mathfrak{so}(3)$. The map $\varsigma \colon \mathfrak{so}(3) \oplus \mathfrak{so}(3) \to \mathfrak{so}(4)$, given by

$$\left(\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \right)$$
$$\mapsto \frac{1}{2} \begin{bmatrix} 0 & x_3 - y_3 & x_2 - y_2 & x_1 - y_1 \\ -x_3 + y_3 & 0 & x_1 + y_1 & -x_2 - y_2 \\ -x_2 + y_2 & -x_1 - y_1 & 0 & x_3 + y_3 \\ -x_1 + y_1 & x_2 + y_2 & -x_3 - y_3 & 0 \end{bmatrix}$$

is a Lie algebra isomorphism. The *natural basis* of $\mathfrak{so}(4)$ is given by

$$\begin{aligned} E_i &= \varsigma \cdot (\mathbf{E}_i, \mathbf{0}) && i = 1,2,3 \\ E_j &= \varsigma \cdot (\mathbf{0}, \mathbf{E}_{j-3}) && j = 4,5,6. \end{aligned}$$

(This choice of basis proves to be the most convenient, especially for expressing the group of automorphisms.) The commutator table for $\mathfrak{so}(4)$ is given below.

	E_1	E_2	E_3	E_4	E_5	E_6
E_1	0	E_3	$-E_2$	0	0	0
E_2	$-E_3$	0	E_1	0	0	0
E_3	E_2	$-E_1$	0	0	0	0
E_4	0	0	0	0	E_6	$-E_5$
E_5	0	0	0	$-E_6$	0	E_4
E_6	0	0	0	E_5	$-E_4$	0

The group of inner automorphisms of $\mathfrak{so}(4)$ is given by

$$\mathsf{Int}(\mathfrak{so}(4)) = \left\{ \begin{bmatrix} \psi_1 & \mathbf{0} \\ \mathbf{0} & \psi_2 \end{bmatrix} : \psi_1, \psi_2 \in \mathsf{SO}(3) \right\}.$$

Proposition 2 ([1]). *The group of automorphisms* $\mathsf{Aut}(\mathfrak{so}(4))$ *is generated by* $\mathsf{Int}(\mathfrak{so}(4))$ *and the swap automorphism* $\zeta = \begin{bmatrix} \mathbf{0} & I_3 \\ I_3 & \mathbf{0} \end{bmatrix}$.

Moreover, the group of automorphisms decomposes as a semi-direct product:

$$\mathsf{Aut}(\mathfrak{so}(4)) = \mathsf{Int}(\mathfrak{so}(4)) \rtimes \{\mathbf{1}, \zeta\}.$$

4 Homogeneous systems

In this section we classify the homogeneous systems on $\mathsf{SO}(4)$. We may assume that $\Xi(\mathbf{1}, 0) = \mathbf{0}$; indeed any homogeneous system is equivalent to one for which this is the case (by use of some reparametrization). We distinguish between the number ℓ of controls involved; this yields six types of systems. For each of these types we simplify an arbitrary system by successively applying automorphisms (as well as considering reparametrizations of the system). Finally, we verify that all the candidates for class representatives are distinct and non-equivalent. Families of representatives are typically parametrized by some vector $\boldsymbol{\alpha} = (\alpha_i)$ or some scalar β.

Any automorphism of $\mathfrak{so}(4)$ preserves the dot product $A \bullet B = \sum_{i=1}^6 a_i b_i$. (Here $A = \sum_{i=1}^6 a_i E_i$ and $B = \sum_{i=1}^6 b_i E_i$.) Let $\Gamma^\perp$ denote the orthogonal complement of a subspace $\Gamma \subset \mathfrak{so}(4)$.

Lemma 1. *Suppose* $\Gamma, \widetilde{\Gamma}$ *are subspaces of* $\mathfrak{so}(4)$ *and* $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$. *Then* $\psi \cdot \Gamma = \widetilde{\Gamma}$ *if and only if* $\psi \cdot \Gamma^\perp = \widetilde{\Gamma}^\perp$.

The classification of the $(6-\ell)$-input systems therefore follows from the classification of the ℓ-input systems. Hence, we need only classify the single-input, two-input, and three-input systems. The results for the four-input and five-input systems then follow as corollaries. (The classification for the six-input systems is trivial.)

When convenient, an ℓ-input homogeneous system

$$\Sigma\colon\, u_1 \sum_{i=1}^{6} b_1^i E_i + \cdots + u_\ell \sum_{i=1}^{6} b_\ell^i E_i$$

will be written (in matrix form) as

$$\Sigma\colon\, \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} b_1^1 & \dots & b_\ell^1 \\ \vdots & & \vdots \\ b_1^6 & \dots & b_\ell^6 \end{bmatrix}.$$

Here $M_1, M_2 \in \mathbb{R}^{3\times\ell}$.

The evaluation $\psi \cdot \Xi(1, \boldsymbol{u})$ then becomes a matrix multiplication. Accordingly, two ℓ-input homogeneous systems $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ and $\Sigma'\colon \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$ are equivalent if and only if there exist an automorphism $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$ and $K \in \mathsf{GL}(\ell, \mathbb{R})$ such that

$$\psi \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} K = \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}.$$

(K corresponds to a reparametrization $\Xi(\mathbf{1}, Ku)$ of the system Σ.) More precisely, Σ and Σ' are equivalent if and only if there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(\ell, \mathbb{R})$ such that

$$\begin{aligned} &(R_1 M_1 K = M_1' \quad \text{and} \quad R_2 M_2 K = M_2') \\ \text{or} \qquad &(R_1 M_2 K = M_1' \quad \text{and} \quad R_2 M_1 K = M_2'). \end{aligned}$$

The *singular value decomposition* (SVD) turns out to be useful in classifying systems. For any matrix $M \in \mathbb{R}^{m\times n}$ of rank r, there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $D \in \mathbb{R}^{r\times r} = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$ such that $M = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^\top$ with $\sigma_1 \geq \cdots \geq \sigma_r > 0$. Specialized forms of the SVD (stated as lemmas) will be used in classifying the two-input and three-input homogeneous systems.

Theorem 1. *Any single-input homogeneous system is equivalent to*

$$\Sigma_\beta^{(1,0)}:\; u_1(E_1 + \beta E_4)$$

for some $0 \leq \beta \leq 1$*. Here* β *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 1. Clearly, no single-input homogeneous system is controllable.

Proof. Let $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ be a single-input system. (Here $M_1, M_2 \in \mathbb{R}^{3\times 1}$.) We may assume that $M_1 \neq \mathbf{0}$. (If not, consider $\Sigma\colon \zeta \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$.) There exist $R_1, R_2 \in \mathsf{SO}(3)$ such that

$$R_1 M_1 \frac{1}{\|M_1\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad R_2 M_2 \frac{1}{\|M_1\|} = \begin{bmatrix} \frac{\|M_2\|}{\|M_1\|} \\ 0 \\ 0 \end{bmatrix}.$$

Thus Σ is equivalent to $\Sigma'\colon u_1(E_1 + \frac{\|M_2\|}{\|M_1\|} E_4)$. If $\frac{\|M_2\|}{\|M_1\|} > 1$, then we have

$$\zeta \cdot \left\langle E_1 + \frac{\|M_2\|}{\|M_1\|} E_4 \right\rangle = \left\langle E_1 + \frac{\|M_1\|}{\|M_2\|} E_4 \right\rangle ,$$

and so Σ is equivalent to $\Sigma''\colon u_1(E_1 + \frac{\|M_1\|}{\|M_2\|} E_4)$. Hence Σ is equivalent to $\Sigma_\beta^{(1,0)}$ for some $0 \leq \beta \leq 1$.

Suppose $\Sigma_\beta^{(1,0)}$ and $\Sigma_{\beta'}^{(1,0)}$ are equivalent. Then there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $k \neq 0$ such that

$$\begin{aligned} &\left(R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad R_2 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} \beta' \\ 0 \\ 0 \end{bmatrix} \right) \\ \text{or} \quad &\left(R_1 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad R_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} \beta' \\ 0 \\ 0 \end{bmatrix} \right). \end{aligned}$$

Therefore $|\beta| = |\beta'|$ or $|\beta\beta'| = 1$. Thus, as $0 \leq \beta, \beta' \leq 1$, we get $\beta = \beta'$. $\square$

Corollary 1. *Any five-input homogeneous system is equivalent to*

$$\Sigma_\beta^{(5,0)} : \ u_1(E_4 - \beta E_1) + u_2 E_2 + u_3 E_3 + u_4 E_5 + u_5 E_6$$

for some $0 \leq \beta \leq 1$. Here β parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Remark 2. Every five-input homogeneous system is controllable.

Lemma 2. *For any $M \in \mathbb{R}^{3\times 2}$ there exist orthogonal matrices $R_1 \in \mathsf{SO}(3)$ and $R_2 \in \mathsf{O}(2)$ such that $R_1 M R_2 = \begin{bmatrix} D \\ 0 \quad 0 \end{bmatrix}$, where $D = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ and $a_1 \geq a_2 \geq 0$. If*

$$R_1 \begin{bmatrix} D \\ 0 \quad 0 \end{bmatrix} R_2 = \begin{bmatrix} D' \\ 0 \quad 0 \end{bmatrix}$$

for some $R_1 \in \mathsf{SO}(3)$ and $R_2 \in \mathsf{O}(2)$, then $D = D'$ (provided that D and D' are diagonal matrices such that $a_1 \geq a_2 \geq 0$ and $a_1' \geq a_2' \geq 0$).

Theorem 2. *Any two-input homogeneous system is equivalent to exactly one of the systems*

$$\Sigma_1^{(2,0)} : u_1 E_1 + u_2 E_4$$
$$\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)} : u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$, *where* $0 = \alpha_2 \leq \alpha_1$ *or* $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ *or* $0 < \alpha_2 \leq \alpha_1 < 1$. *Here* $\boldsymbol{\alpha}$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 3. $\Sigma_1^{(2,0)}$ is not controllable. $\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)}$ is not controllable exactly when $\alpha_2 = 0$ or $\alpha_1 = \alpha_2 = 1$.

Proof. Let $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ be a two-input system. (Here $M_1, M_2 \in \mathbb{R}^{3\times 2}$.) Now either $\operatorname{rank}(M_1) = \operatorname{rank}(M_2) = 1$ or $\max\{\operatorname{rank}(M_1), \operatorname{rank}(M_2)\} = 2$. Suppose $\operatorname{rank}(M_1) = \operatorname{rank}(M_2) = 1$. Then there exists $K \in \mathsf{GL}(2,\mathbb{R})$ such that

$$M_1 K = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \\ b_3 & 0 \end{bmatrix} \qquad \text{and} \qquad M_2 K = \begin{bmatrix} 0 & b_4 \\ 0 & b_5 \\ 0 & b_6 \end{bmatrix}.$$

Hence there exists $R_1, R_2 \in \mathsf{SO}(3)$ such that

$$R_1 \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \\ b_3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{b_1^2+b_2^2+b_3^2}} & 0 \\ 0 & \frac{1}{\sqrt{b_4^2+b_5^2+b_6^2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{and} \qquad R_2 \begin{bmatrix} 0 & b_4 \\ 0 & b_5 \\ 0 & b_6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{b_1^2+b_2^2+b_3^2}} & 0 \\ 0 & \frac{1}{\sqrt{b_4^2+b_5^2+b_6^2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore Σ is equivalent to $\Sigma_1^{(2,0)}$.

On the other hand, suppose $\operatorname{rank}(M_1) = 2$ or $\operatorname{rank}(M_2) = 2$. We may assume $\operatorname{rank}(M_1) = 2$. (If not, consider $\Sigma\colon \zeta \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$.) There exists $R_1 \in \mathsf{SO}(3)$ such that $R_1 M_1 = \begin{bmatrix} M_1' \\ 0 \quad 0 \end{bmatrix}$. Hence, there exists $K \in \mathsf{GL}(2,\mathbb{R})$ such that $R_1 M_1 K = I_{2,0}$, where $I_{2,0} = \begin{bmatrix} I_2 \\ 0 \quad 0 \end{bmatrix}$. Thus Σ is equivalent to $\Sigma'\colon \begin{bmatrix} I_{2,0} \\ M_2' \end{bmatrix}$. By lemma 2, there exist $R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{O}(2)$ such that

$$\begin{bmatrix} K^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \det K \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R_2 M_2' K = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix}$$

for some $\alpha_1 \geq \alpha_2 \geq 0$. If $\alpha_2 = 0$ or $0 \leq \alpha_2 \leq \alpha_1 < 1$, then Σ is equivalent to $\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)}$.

Suppose $1 < \alpha_2 \leq \alpha_1$. Then

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\alpha_1} \\ \frac{1}{\alpha_2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\alpha_1} \\ \frac{1}{\alpha_2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha_2} & 0 \\ 0 & \frac{1}{\alpha_1} \\ 0 & 0 \end{bmatrix}$$

with $0 < \frac{1}{\alpha_1} \leq \frac{1}{\alpha_2} < 1$. Thus Σ is equivalent to $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}'}$ for some $0 < \alpha_2' \leq \alpha_1' < 1$. Suppose $\alpha_2 \leq 1 \leq \alpha_1$. If $\frac{1}{\alpha_2} \leq \alpha_1$, then we are done. If $\frac{1}{\alpha_2} > \alpha_1$, then Σ is likewise equivalent to $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}'}$ for some $1 \leq \frac{1}{\alpha_2'} \leq \alpha_1'$.

We now verify that none of the class representatives are equivalent. As the traces of $\Sigma^{(2,0)}_1$ and $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}}$, respectively, do not generate the same subalgebra (for any $\alpha_1, \alpha_2 \in \mathbb{R}$), they cannot be equivalent. We claim that $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}}$ and $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}'}$ are equivalent only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. Indeed, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(2,\mathbb{R})$ such that

$$R_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R_2 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} \alpha_1' & 0 \\ 0 & \alpha_2' \\ 0 & 0 \end{bmatrix}.$$

Then $K \in \mathsf{O}(2)$ and so, by lemma 2, it follows that $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$. On the other hand, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(2,\mathbb{R})$ such that

$$R_1 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} \alpha_1' & 0 \\ 0 & \alpha_2' \\ 0 & 0 \end{bmatrix}.$$

Then $\alpha_2 \neq 0$ and $\alpha_2' \neq 0$. Hence, we need only consider the cases:

(i) $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ and $0 < \alpha_2' \leq \alpha_1' < 1$,

(ii) $0 < \alpha_2 \leq \alpha_1 < 1$ and $0 < \alpha_2' \leq \alpha_1' < 1$,

(iii) $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ and $1 \leq \frac{1}{\alpha_2'} \leq \alpha_1'$.

Assume (i) holds. It follows that $R_1 = \begin{bmatrix} S_1 & 0 \\ 0 & \det S_1 \end{bmatrix}$ and $R_2 = \begin{bmatrix} S_2 & 0 \\ 0 & \det S_2 \end{bmatrix}$ for some $S_1, S_2 \in \mathsf{O}(2)$. Thus $K = \begin{bmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix} S_1^{-1}$ and so

$$S_2 \begin{bmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix} S_1^{-1} = \begin{bmatrix} \alpha_1' & 0 \\ 0 & \alpha_2' \end{bmatrix}.$$

By applying the mapping $A \mapsto AA^\top$, we get

$$S_2 \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\alpha_2^2} \end{bmatrix} S_2^\top = \begin{bmatrix} {\alpha_1'}^2 & 0 \\ 0 & {\alpha_2'}^2 \end{bmatrix}.$$

As $\frac{1}{\alpha_2} \geq \frac{1}{\alpha_1} \geq 0$ and $\alpha_1' \geq \alpha_2' \geq 0$, it follows that $\alpha_1^2 {\alpha_2'}^2 = 1$ and ${\alpha_1'}^2 \alpha_2^2 = 1$. Hence $\alpha_1' \geq 1$, a contradiction.

Similarly, if (ii) or (iii) hold, then we arrive at a contradiction. $\square$

Corollary 2. *Any four-input homogeneous system is equivalent to exactly one of the systems*

$$\begin{aligned} &\Sigma_1^{(4,0)} : u_1 E_2 + u_2 E_3 + u_3 E_5 + u_4 E_6 \\ &\Sigma_{2,\boldsymbol{\alpha}}^{(4,0)} : u_1(E_4 - \alpha_1 E_1) + u_2(E_5 - \alpha_2 E_2) + u_3 E_3 + u_4 E_6 \end{aligned}$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$, *where* $0 = \alpha_2 \leq \alpha_1$ *or* $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ *or* $0 < \alpha_2 \leq \alpha_1 < 1$. *Here* $\boldsymbol{\alpha}$ *parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 4. $\Sigma_1^{(4,0)}$ is controllable. $\Sigma_{2,\boldsymbol{\alpha}}^{(4,0)}$ is not controllable exactly when $\alpha_1 = \alpha_2 = 0$.

Lemma 3. *For any* $M \in \mathbb{R}^{3\times 3}$ *there exist* $R_1, R_2 \in \mathsf{SO}(3)$ *such that*

$$R_1 M R_2 = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)\,,$$

where $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$. *Moreover, if* $\operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$ *and* $\operatorname{diag}(\alpha_1', \alpha_2', \alpha_3')$ *are two such matrices and*

$$R_1 \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3) R_2 = \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3')$$

for some $R_1, R_2 \in \mathsf{SO}(3)$, *then* $\alpha_1 = \alpha_1'$, $\alpha_2 = \alpha_2'$, *and* $\alpha_3 = \alpha_3'$.

Theorem 3. *Any three-input homogeneous system is equivalent to exactly one of the systems*

$$\begin{aligned} &\Sigma_{1,\beta}^{(3,0)} : u_1(E_1 + \beta E_4) + u_2 E_2 + u_3 E_6 \\ &\Sigma_{2,\boldsymbol{\alpha}}^{(3,0)} : u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5) + u_3(E_3 + \alpha_3 E_6) \end{aligned}$$

for some $\alpha_1, \alpha_2, \alpha_3, \beta \in \mathbb{R}$, *where* $0 \leq \beta \leq 1$ *and* $0 = \alpha_3 \leq \alpha_2 \leq \alpha_1$ *or* $0 < |\alpha_3| \leq \alpha_2 < 1 \wedge \alpha_2 \leq \alpha_1$ *or* $\alpha_2 = 1 \leq \frac{1}{|\alpha_3|} \leq \alpha_1$. *Here* $\boldsymbol{\alpha}$ *and* β *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 5. $\Sigma_{1,\beta}^{(3,0)}$ is controllable exactly when $\beta > 0$. $\Sigma_{2,\boldsymbol{\alpha}}^{(3,0)}$ is not controllable exactly when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ or $\alpha_2 = 0$.

Proof. Let $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ be a three-input system. (Here $M_1, M_2 \in \mathbb{R}^{3\times 3}$.) Clearly either $\max\{\operatorname{rank}(M_1), \operatorname{rank}(M_2)\} = 3$ or $\max\{\operatorname{rank}(M_1), \operatorname{rank}(M_2)\} = 2$. Suppose, $\operatorname{rank}(M_1) = 3$ or $\operatorname{rank}(M_2) = 3$. We may assume $\operatorname{rank}(M_1) = 3$. (If not, consider

$\Sigma\colon \zeta \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$.) Then there exists $K \in \mathsf{GL}(3,\mathbb{R})$ such that $M_1K = I_3$. Thus Σ is equivalent to $\Sigma'\colon \begin{bmatrix} I_3 \\ M_2' \end{bmatrix}$. By lemma 3, there exist $R_2, K \in \mathsf{SO}(3)$ such that

$$R_2 M_2' K = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$$

for some $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$.

If $\alpha_3 = 0$ or $|\alpha_3| \leq \alpha_2 < 1$ or $1 = \alpha_2 \leq \frac{1}{|\alpha_3|} \leq \alpha_1$, then we are done. Suppose $1 < |\alpha_3| \leq \alpha_2 \leq \alpha_1$ or $0 < |\alpha_3| < 1 < \alpha_2 \leq \alpha_1$. If $\alpha_3 > 0$, then

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{1}{\alpha_1} \\ 0 & \frac{1}{\alpha_2} & 0 \\ \frac{1}{\alpha_3} & 0 & 0 \end{bmatrix} = I_3$$

$$\text{and} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{1}{\alpha_1} \\ 0 & \frac{1}{\alpha_2} & 0 \\ \frac{1}{\alpha_3} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha_3} & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 \\ 0 & 0 & \frac{1}{\alpha_1} \end{bmatrix}.$$

If $\alpha_3 < 0$, then

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{\alpha_1} \\ 0 & \frac{1}{\alpha_2} & 0 \\ -\frac{1}{\alpha_3} & 0 & 0 \end{bmatrix} = I_3$$

$$\text{and} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{\alpha_1} \\ 0 & \frac{1}{\alpha_2} & 0 \\ -\frac{1}{\alpha_3} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\alpha_3} & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 \\ 0 & 0 & -\frac{1}{\alpha_1} \end{bmatrix}.$$

In both cases $0 < \frac{1}{\alpha_1} \leq \frac{1}{\alpha_2} \leq \frac{1}{|\alpha_3|}$. Thus Σ is equivalent to some system $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}'}$ with $0 < |\alpha_3'| \leq \alpha_2' < 1$ and $\alpha_2' \leq \alpha_1'$. Likewise, if $\frac{1}{|\alpha_3|} \geq \alpha_1 \geq \alpha_2 = 1$, then Σ is equivalent to some system $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}'}$ with $1 = \alpha_2' \leq \frac{1}{|\alpha_3'|} \leq \alpha_1'$.

On the other hand, suppose $\operatorname{rank}(M_1) = 2$ and $\operatorname{rank}(M_2) \in \{1, 2\}$. Again, a simple argument shows that Σ is equivalent to some system $\Sigma'\colon \begin{bmatrix} I_{2,0} \\ M_1' \end{bmatrix}$, where $I_{2,0} = \begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$. If $\operatorname{rank}(M_1') = 1$, it is easy to show that Σ is equivalent to $\Sigma^{(3,0)}_{1,0}$. Assume that $\operatorname{rank}(M_1') = 2$. Then there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3,\mathbb{R})$ such that

$$R_1 I_{2,0} K = I_{2,0} \qquad \text{and} \qquad R_2 M_1' K = \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By the SVD there exist $S_1, S_2 \in \mathsf{O}(2)$ such that $S_2 \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} S_1 = \operatorname{diag}(\beta, 0)$ for some $\beta \geq 0$. Let

$$K' = \begin{bmatrix} S_1 & \mathbf{0} \\ \mathbf{0} & \det S_1 \end{bmatrix} \in \mathsf{SO}(3) \qquad \text{and} \qquad R_2' = \begin{bmatrix} S_2 & \mathbf{0} \\ \mathbf{0} & \det S_2 \end{bmatrix} \in \mathsf{SO}(3)\,.$$

Now

$$(K')^{-1} I_{2,0} K' = I_{2,0} \qquad \text{and} \qquad R_2' \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} K' = \begin{bmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $\beta \leq 1$, then we are done (i.e., Σ is equivalent to $\Sigma^{(3,0)}_{1,\beta}$). Suppose that $\beta > 1$. Then

$$\zeta \cdot \langle E_1 + \beta E_4, E_2, E_6 \rangle = \left\langle \tfrac{1}{\beta} E_4 + E_1, E_5, E_3 \right\rangle .$$

It is a simple matter to show that there exists an automorphism ψ such that

$$\psi \cdot \left\langle \tfrac{1}{\beta} E_4 + E_1, E_5, E_3 \right\rangle = \left\langle E_1 + \tfrac{1}{\beta} E_4, E_2, E_6 \right\rangle .$$

Thus Σ is equivalent to $\Sigma^{(3,0)}_{1,\beta'}$ for some $0 \leq \beta' \leq 1$.

We now verify that none of these class representatives are equivalent. As the traces of $\Sigma^{(3,0)}_{1,\beta}$ and $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}}$, respectively, do not generate the same subalgebra (for any $\beta, \alpha_1, \alpha_2 \in \mathbb{R}$), they cannot be equivalent. Suppose two systems $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}}$ and $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}'}$, with $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$ and $\alpha_1' \geq \alpha_2' \geq |\alpha_3'| \geq 0$, are equivalent. We claim that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. Indeed, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3, \mathbb{R})$ such that $R_1 I_3 K = I_3$ and $R_2 \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) K = \mathrm{diag}(\alpha_1', \alpha_2', \alpha_3')$. Then, by lemma 3, it follows that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. On the other hand, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3, \mathbb{R})$ such that $R_1 \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) K = I_3$ and $R_2 I_3 K = \mathrm{diag}(\alpha_1', \alpha_2', \alpha_3')$. Then $\alpha_1^2 {\alpha_3'}^2 = 1$, $\alpha_2^2 {\alpha_2'}^2 = 1$ and $\alpha_3^2 {\alpha_1'}^2 = 1$. Clearly, $\alpha_3, \alpha_3' \neq 0$. Three possibilities remain, either

(i) $0 < |\alpha_3| \leq \alpha_2 < 1$ and $0 < |\alpha_3'| \leq \alpha_2' < 1$, or
(ii) $0 < |\alpha_3| \leq \alpha_2 < 1$ and $0 < |\alpha_3'| \leq \alpha_2' < 1 \ \wedge \ \alpha_2' \leq \alpha_1'$, or
(iii) $0 < |\alpha_3| \leq \alpha_2 < 1 \ \wedge \ \alpha_2 \leq \alpha_1$ and $0 < |\alpha_3'| \leq \alpha_2' < 1 \ \wedge \ \alpha_2' \leq \alpha_1'$.

Again (as in theorem 2), each case leads to a contradiction. □

Remark 6. There is only one six-dimensional affine subspace of $\mathfrak{so}(4)$, namely $\mathfrak{so}(4)$. Therefore any six-input system is equivalent to the system

$$\Sigma^{(6,0)} \colon u_1 E_1 + u_2 E_2 + u_3 E_3 + u_4 E_4 + u_5 E_5 + u_6 E_6 .$$

Clearly, this system is controllable.

5 Inhomogeneous systems

We now proceed to the classification of the inhomogeneous systems on $\mathsf{SO}(4)$. This classification is, in part, based on our classification of homogeneous systems. As before, we distinguish between the number ℓ of controls involved; this yields five types of systems. (Clearly there are no six-input inhomogeneous systems.) Suppose

$$\Sigma \colon A + u_1 B_1 + \cdots + u_\ell B_\ell$$

is an inhomogeneous system. Then the corresponding homogeneous system

$$\widetilde{\Sigma} \colon u_1 B_1 + \cdots + u_\ell B_\ell$$

is equivalent to exactly one homogeneous class representative Σ^0. Consequently, Σ is equivalent to a system Σ' with parametrization map $\Xi'(\mathbf{1}, u) = A' + \Xi^0(\mathbf{1}, u)$. Such an (arbitrary) system is then further simplified by applying automorphisms preserving the trace Γ^0 of Σ^0. Accordingly, for each homogeneous class representative Σ^0, representatives for the associated class of inhomogeneous systems are identified. We will, in addition, use vectors $\boldsymbol{\varepsilon} = (\varepsilon_i)$ to parametrize class representatives.

Again, it is convenient to write the condition of equivalence in matrix form. An ℓ-input inhomogeneous system specified by

$$\Sigma\colon \sum_{i=1}^{6} a^i E_i + u_1 \sum_{i=1}^{6} b_1^i E_i + \cdots + u_\ell \sum_{i=1}^{6} b_\ell^i E_i$$

will be written (in matrix form) as

$$\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} a^1 & b_1^1 & \dots & b_\ell^1 \\ \vdots & \vdots & & \vdots \\ a^6 & b_1^6 & \dots & b_\ell^6 \end{bmatrix}.$$

Here $M_1, M_2 \in \mathbb{R}^{3\times(\ell+1)}$. Two ℓ-input inhomogeneous systems $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ and $\Sigma'\colon \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$ are equivalent if and only if there exist an automorphism $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$ and $K \in \mathsf{Aff}(\ell, \mathbb{R})$ such that

$$\psi \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} K = \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}.$$

Here

$$\mathsf{Aff}(\ell, \mathbb{R}) = \left\{ \begin{bmatrix} 1 & \mathbf{0} \\ \boldsymbol{v} & N \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^{\ell\times 1},\ N \in \mathsf{GL}(\ell, \mathbb{R}) \right\}.$$

For an inhomogeneous system

$$\Sigma\colon A + u_1 B_1 + \cdots + u_\ell B_\ell\,,$$

with $A = \sum_{i=1}^{6} \varepsilon_i E_i$, it follows that $\sum_{i=1}^{6} \varepsilon_i^2 \neq 0$. We omit this condition in the statements of the theorems throughout this section. A proof sketch is provided for theorem 4 to elucidate the approach used in the classification of inhomogeneous systems. More details are provided in the proof of theorem 5. The proofs of theorems 6, 7, and 8 are similar and shall therefore be omitted.

Theorem 4. *Every single-input inhomogeneous system is equivalent to exactly one of the systems*

$$\Sigma_{\beta\boldsymbol{\varepsilon}}^{(1,1)}\colon A + u_1(E_1 + \beta E_4)$$

for some $0 \leq \beta \leq 1$, *where* *(i) if* $\beta = 0$ *then*

$$A = \varepsilon_2 E_2 + \varepsilon_4 E_4$$

with $\varepsilon_2, \varepsilon_4 \geq 0$, *and*

(ii) if $0 < \beta \leq 1$ *then*

$$A = \varepsilon_2 E_2 + \varepsilon_4 E_4 + \varepsilon_5 E_5$$

with $\varepsilon_2, \varepsilon_4, \varepsilon_5 \geq 0$ *and* $\big((\beta = 1 \wedge \varepsilon_4 = 0) \Rightarrow \varepsilon_2 \geq \varepsilon_5\big)$.

Here β *and* $\boldsymbol{\varepsilon}$ *parametrize a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 7. If $\beta = 0$, then $\Sigma^{(1,1)}_{\beta\boldsymbol{\varepsilon}}$ is not controllable. If $\beta > 0$, then $\Sigma^{(1,1)}_{\beta\boldsymbol{\varepsilon}}$ is not controllable exactly when $\varepsilon_2 = 0$ or $\varepsilon_5 = 0$ or $(\varepsilon_2 = \varepsilon_5 \wedge \varepsilon_4 = 0 \wedge \beta = 1)$.

Proof. Let $\Sigma\colon A + u_1 B_1$ be a single-input system. Then, by theorem 1, Σ is equivalent to a system

$$\widehat{\Sigma}\colon \sum_{i=2}^{6} \varepsilon_i E_i + u_1(E_1 + \beta E_4)$$

for some $0 \leq \beta \leq 1$. Suppose $\beta > 0$. Now

$$R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad R_1, R_2 \in \mathsf{SO}(3)$$

exactly when $k = \det S_1 = \det S_2$, $R_1 = \begin{bmatrix} \det S_1 & 0 \\ 0 & S_1 \end{bmatrix}$, $R_2 = \begin{bmatrix} \det S_2 & 0 \\ 0 & S_2 \end{bmatrix}$, and $S_1, S_2 \in \mathsf{O}(2)$. Accordingly, there exist $S_1, S_2 \in \mathsf{O}(2)$ such that

$$\begin{bmatrix} \det S_1 & 0 \\ 0 & S_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \varepsilon_2 & 0 \\ \varepsilon_3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \det S_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \varepsilon_2' & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{and} \qquad \begin{bmatrix} \det S_2 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} \varepsilon_4 & \beta \\ \varepsilon_5 & 0 \\ \varepsilon_6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \det S_1 \end{bmatrix} = \begin{bmatrix} \varepsilon_4' & \beta \\ \varepsilon_5' & 0 \\ 0 & 0 \end{bmatrix}$$

for some $\varepsilon_2', \varepsilon_4', \varepsilon_5' \geq 0$. Therefore Σ is equivalent to the system

$$\Sigma'\colon \varepsilon_2' E_2 + \varepsilon_4' E_4 + \varepsilon_5' E_5 + u_1(E_1 + \beta E_4)\,.$$

Moreover, if $\beta = 1$ and $\varepsilon_4' = 0$, then Σ can be shown to be equivalent to a system

$$\Sigma''\colon \varepsilon_2'' E_2 + \varepsilon_5'' E_5 + u_1(E_1 + E_4)$$

for some $\varepsilon_2'' \geq \varepsilon_5'' \geq 0$.

Likewise, if $\beta = 0$, it follows that Σ is equivalent to a system

$$\Sigma'\colon \varepsilon_2' E_2 + \varepsilon_4' E_4 + u_1 E_1$$

for some $\varepsilon_2', \varepsilon_4' \geq 0$. (Again, as in the homogeneous case, one verifies that all the systems obtained are distinct and non-equivalent.) $\square$

Theorem 5. *Every two-input inhomogeneous system is equivalent to exactly one of the systems*

(1) $$\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}} \colon \varepsilon_2 E_2 + \varepsilon_5 E_5 + u_1 E_1 + u_2 E_4 \quad \text{with} \quad \varepsilon_2 \geq \varepsilon_5 \geq 0$$

(2) $$\Sigma^{(2,1)}_{2,\boldsymbol{\alpha\varepsilon}} \colon A + u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)$$

with $\alpha_1 \geq \alpha_2 = 0$ *or* $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ *or* $0 < \alpha_2 \leq \alpha_1 < 1$, *where*

(i) *if* $\alpha_1 = \alpha_2 = 0$ *then*
$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4$$
with $\varepsilon_3, \varepsilon_4 \geq 0$, *and*

(ii) *if* $\alpha_1 = \alpha_2 > 0$ *then*
$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_6 E_6$$
with $\varepsilon_3 = 0 \Rightarrow \varepsilon_6 \geq 0$, $\varepsilon_6 \in \mathbb{R}$, $\varepsilon_3, \varepsilon_4 \geq 0$, *and*

(iii) *if* $\alpha_1 > \alpha_2 = 0$ *then*
$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5 + \varepsilon_6 E_6$$
with $(\varepsilon_4 = 0 \vee \varepsilon_5 = 0) \Rightarrow \varepsilon_6 \geq 0$, $\varepsilon_6 \in \mathbb{R}$, $\varepsilon_3, \varepsilon_4, \varepsilon_5 \geq 0$, *and*

(iv) *if* $\alpha_1 > \alpha_2 > 0$ *then*
$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5 + \varepsilon_6 E_6$$
with $(\varepsilon_3, \varepsilon_4 > 0) \vee (\varepsilon_3 > 0 \wedge \varepsilon_5 \geq 0) \vee (\varepsilon_4, \varepsilon_5 \geq 0) \vee (\varepsilon_5, \varepsilon_6 \geq 0)$, $\varepsilon_5, \varepsilon_6 \in \mathbb{R}$, $\varepsilon_3, \varepsilon_4 \geq 0$.

Here $\boldsymbol{\alpha}$ *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 8. $\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}}$ is controllable exactly when $\varepsilon_5 \neq 0$. $\Sigma^{(2,1)}_{2,\boldsymbol{\alpha\varepsilon}}$ is not controllable exactly when $\alpha_2 = 0 \wedge (\alpha_1 = 0 \vee \varepsilon_5 = \varepsilon_6 = 0)$ or $\alpha_1 = \alpha_2 = 1 \wedge \varepsilon_4 = 0 \wedge \varepsilon_3 = \varepsilon_6$.

Proof. Let $\Sigma \colon A + u_1 B_1 + u_2 B_2$ be a two-input system. Then, by theorem 2, Σ is equivalent either to
$$\widehat{\Sigma}_1 \colon \sum_{i=1}^{6} \varepsilon_i E_i + u_1 E_1 + u_2 E_4$$
or
$$\widehat{\Sigma}_2 \colon \sum_{i=3}^{6} \varepsilon_i E_i + u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)\,.$$
It is easy to show that $\widehat{\Sigma}_1$ is equivalent to $\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}}$. Suppose Σ is equivalent to $\widehat{\Sigma}_2$ and $\alpha_1 > \alpha_2 > 0$. Now
$$R_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad R_2 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix}$$

$R_1, R_2 \in \mathsf{SO}(3)$, and $N \in \mathsf{GL}(2,\mathbb{R})$ exactly when $N = S$, $R_1 = R_2 = \begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix}$, and $S = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$, $\sigma_1, \sigma_2 \in \{-1, 1\}$. Accordingly, (a tedious but straightforward computation shows that) there exists $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3' & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} \begin{bmatrix} \varepsilon_4 & \alpha_1 & 0 \\ \varepsilon_5 & 0 & \alpha_2 \\ \varepsilon_6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \varepsilon_4' & \alpha_1 & 0 \\ \varepsilon_5' & 0 & \alpha_2 \\ \varepsilon_6' & 0 & 0 \end{bmatrix}$$

where $\varepsilon_3', \varepsilon_4' \geq 0$ and $(\varepsilon_3' = 0 \vee \varepsilon_4' = 0) \Rightarrow \varepsilon_5' \geq 0$ and $\varepsilon_3' = \varepsilon_4' = 0 \Rightarrow (\varepsilon_5', \varepsilon_6' \geq 0)$ and $\varepsilon_3' = \varepsilon_5' = 0 \Rightarrow \varepsilon_6' \geq 0$. These conditions are equivalent to those given in the theorem.

On the other hand, suppose Σ is equivalent to $\widehat{\Sigma}_2$ and $\alpha_1 = \alpha_2 > 0$. Then

$$R_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad R_2 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \\ 0 & 0 \end{bmatrix}$$

$R_1, R_2 \in \mathsf{SO}(3)$, and $N \in \mathsf{GL}(2,\mathbb{R})$ exactly when $N = S^\top$, $R_1 = R_2 = \begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix}$, and $S \in \mathsf{O}(2)$. Therefore there exists $S \in \mathsf{O}(2)$ such that

$$\begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & S^\top \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3' & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix} \begin{bmatrix} \varepsilon_4 & \alpha_1 & 0 \\ \varepsilon_5 & 0 & \alpha_1 \\ \varepsilon_6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & S^\top \end{bmatrix} = \begin{bmatrix} \varepsilon_4' & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \\ \varepsilon_6' & 0 & 0 \end{bmatrix}$$

where $\varepsilon_3', \varepsilon_4' \geq 0$ and $\varepsilon_3' = 0 \Rightarrow \varepsilon_6' \geq 0$.

The (families of) equivalence representatives 2(i) and 2(iii) are obtained similarly. (Again, as in the homogeneous case, one verifies that all the systems obtained are distinct and non-equivalent.) □

Theorem 6. *Every three-input inhomogeneous system is equivalent to exactly one of the systems*

(1) $\quad \Sigma^{(3,1)}_{1,\beta\boldsymbol{\varepsilon}} : A + u_1(E_1 + \beta E_4) + u_2 E_2 + u_3 E_6 \quad$ *with* $\quad 0 \leq \beta \leq 1$, *where*

(i) if $\beta = 0$ *then*

$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 \quad \textit{with} \quad \varepsilon_3, \varepsilon_4 \geq 0\,,$$

(ii) if $0 < \beta \leq 1$ *then*

$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5$$

with $\big((\varepsilon_4 = 0 \wedge \beta = 1) \Rightarrow \varepsilon_3 \geq \varepsilon_5\big)$, $\varepsilon_3, \varepsilon_4, \varepsilon_5 \geq 0$.

(2) $$\Sigma^{(3,1)}_{2,\boldsymbol{\alpha\varepsilon}} \colon A + u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5) + u_3(E_3 + \alpha_3 E_6)$$
with $0 = \alpha_3 \leq \alpha_2 \leq \alpha_1$ *or* $0 < |\alpha_3| \leq \alpha_2 < 1 \wedge \alpha_2 \leq \alpha_1$ *or* $\alpha_2 = 1 \leq \frac{1}{|\alpha_3|} \leq \alpha_1$, *where*

(i) if $\alpha_1 = \alpha_2 = |\alpha_3|$ *then*
$$A = \varepsilon_4 E_4 \quad \text{with} \quad \varepsilon_4 \geq 0\,,$$

(ii) if $\alpha_1 > \alpha_2 = |\alpha_3|$ *then*
$$A = \varepsilon_4 E_4 + \varepsilon_5 E_5 \quad \text{with} \quad \varepsilon_4, \varepsilon_5 \geq 0\,,$$

(iii) if $\alpha_1 = \alpha_2 > |\alpha_3|$ *then*
$$A = \varepsilon_4 E_4 + \varepsilon_6 E_6 \quad \text{with} \quad \varepsilon_4, \varepsilon_6 \geq 0\,,$$

(iv) if $\alpha_1 > \alpha_2 > |\alpha_3|$ *then*
$$A = \varepsilon_4 E_4 + \varepsilon_5 E_5 + \varepsilon_6 E_6 \quad \text{with} \quad \varepsilon_6 \in \mathbb{R}\,,\ \varepsilon_4, \varepsilon_5 \geq 0\,.$$

Here $\boldsymbol{\alpha}$, β *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 9. $\Sigma^{(3,1)}_{1,\beta\boldsymbol{\varepsilon}}$ is controllable exactly when $\beta \neq 0$ or $\varepsilon_4 \neq 0$. $\Sigma^{(3,1)}_{2,\boldsymbol{\alpha\varepsilon}}$ is not controllable exactly when $\alpha_2 = 0$ and $(\alpha_1 = 0 \vee \varepsilon_5 = 0)$.

Theorem 7. *Every four-input inhomogeneous system is equivalent to exactly one of the systems*

(1) $$\Sigma^{(4,1)}_{1,\boldsymbol{\varepsilon}} \colon \varepsilon_1 E_1 + \varepsilon_4 E_4 + u_1 E_2 + u_2 E_3 + u_3 E_5 + u_4 E_6 \quad \text{with} \quad \varepsilon_1 \geq \varepsilon_4 \geq 0$$

(2) $$\Sigma^{(4,1)}_{2,\boldsymbol{\alpha\varepsilon}} \colon A + u_1(E_4 - \alpha_1 E_1) + u_2(E_5 - \alpha_2 E_2) + u_3 E_3 + u_4 E_6$$
with $\alpha_1 \geq \alpha_2 = 0$ *or* $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ *or* $0 < \alpha_2 \leq \alpha_1 < 1$, *where*

(i) if $\alpha_1 = \alpha_2$ *then*
$$A = \varepsilon_1 E_1 \quad \text{with} \quad \varepsilon_1 \geq 0\,,$$

(ii) if $\alpha_1 > \alpha_2$ *then*
$$A = \varepsilon_1 E_1 + \varepsilon_2 E_2 \quad \text{with} \quad \varepsilon_1, \varepsilon_2 \geq 0\,.$$

Here $\boldsymbol{\alpha}$ *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 10. Every four-input inhomogeneous system is controllable.

Theorem 8. *Every five-input inhomogeneous system is equivalent to exactly one of the systems*
$$\Sigma^{(5,1)}_{\beta\boldsymbol{\varepsilon}} \colon \varepsilon_1 E_1 + u_1(E_4 - \beta E_1) + u_2 E_2 + u_3 E_3 + u_4 E_5 + u_5 E_6$$
with $0 \leq \beta \leq 1$, $\varepsilon_1 \geq 0$. *Here* β *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 11. Every five-input inhomogeneous system is controllable.

6 Conclusion

We have classified all left-invariant control affine systems on the orthogonal group $\mathsf{SO}(4)$ (cf. [1]). Specifically, we have shown that any system is equivalent to exactly one of a list of equivalence representatives. In addition, we have identified exactly which of the representative systems are controllable.

As a simple by-product of the classification of homogeneous systems, we recover a classification of subalgebras of $\mathfrak{so}(4)$. (Two subalgebras $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{so}(4)$ are equivalent if there exists $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$ such that $\psi \cdot \mathfrak{a}_1 = \mathfrak{a}_2$). Any (non-trivial) subalgebra of $\mathfrak{so}(4)$ is equivalent to exactly one of the following subalgebras

$$\begin{aligned}
\mathfrak{a}^{(1)}_\alpha &= \langle E_1 + \alpha E_4 \rangle &&= \varsigma \cdot \langle (\mathbf{E}_1, \alpha \mathbf{E}_1) \rangle \\
\mathfrak{a}^{(2)} &= \langle E_1, E_4 \rangle &&= \varsigma \cdot \langle \mathbf{E}_1 \rangle \oplus \langle \mathbf{E}_1 \rangle \\
\mathfrak{a}^{(3)}_1 &= \langle E_1, E_2, E_3 \rangle &&= \varsigma \cdot \mathfrak{so}(3) \oplus \{\mathbf{0}\} \\
\mathfrak{a}^{(3)}_2 &= \langle E_1 + E_4, E_2 + E_5, E_4 + E_6 \rangle &&= \varsigma \cdot \{(A, A) : A \in \mathfrak{so}(3)\} \\
\mathfrak{a}^{(4)} &= \langle E_1, E_2, E_3, E_4 \rangle &&= \varsigma \cdot \mathfrak{so}(3) \oplus \langle \mathbf{E}_1 \rangle .
\end{aligned}$$

Here $0 \leq \alpha \leq 1$ parametrizes a family of nonequivalent class representatives. (Only $\mathfrak{a}^{(3)}_1$ is an ideal.)

The classification of (controllable) systems should prove useful in the study of certain classes of invariant optimal control problems on $\mathsf{SO}(4)$. Generally, an (affine quadratic) invariant optimal control problem is given by the specification of

(1) a left-invariant control system $\Sigma = (\mathsf{G}, \Xi)$

(2) an affine quadratic cost function $\chi : \mathbb{R}^\ell \to \mathbb{R}$, $u \mapsto \mathcal{Q}(u - \mu)$ (here $\mathcal{Q}$ is assumed positive definite and $\mu \in \mathbb{R}^\ell$)

(3) boundary data (g_0, g_1, T), consisting of an initial state $g_0 \in \mathsf{G}$, a target state $g_1 \in \mathsf{G}$, and a (usually fixed) terminal time $T > 0$.

Explicitly, we want to *minimize* the functional

$$\mathcal{J} = \int_0^T \chi(u(t))\, \mathrm{d}t$$

over the trajectories of Σ subject to the boundary conditions. The equivalence of such problems has been considered in [8], [10]; this is called *cost equivalence*. It establishes a one-to-one correspondence between the associated optimal trajectories (resp. associated extremal curves) of equivalent problems. For two cost equivalent problems, the underlying left-invariant control systems must be equivalent. Hence (once a classification of systems has been found), only the transformations leaving each system invariant need be considered when investigating cost equivalence.

Some specific (invariant) optimal control problems on $\mathsf{SO}(4)$ have been studied by diverse authors in several contexts. For instance, D'Alessandro studied a particular (time) optimal control problem associated with a homogeneous three--input control affine system in the context of quantum control ([15]), whereas Puta et al. considered a particular optimal control problem for a homogeneous four--input control affine system in the broad context of motion control ([4]). Recently, Holderbaum et al. made attempts to compare different trajectories of some particular control systems on $\mathsf{SE}(3), \mathsf{SO}(1,3)$ and $\mathsf{SO}(4)$, in the context of rigid body dynamics ([5], [23]). Various variational problems associated with $\mathsf{SO}(4)$ (and its Lie algebra), like the Kowalewki's top or the integrable Suslin problem, have also been treated (see, e.g., [18], [22]). With a classification of controllable systems at hand a more unified approach to control problems on $\mathsf{SO}(4)$ may be feasible. This is a topic for future research.

Invariant optimal control problems naturally give rise to Hamilton-Poisson systems, via the Pontryagin Maximum Principle. Moreover, if two invariant optimal control problems are cost equivalent, then the associated Hamilton-Poisson systems are linearly equivalent ([8], [10]). In the context of Hamiltonian systems, Raţiu et al. studied the stability of equilibria for the $\mathfrak{so}(4)$ free rigid body ([13]). Furthermore, integrability (and explicit integration) of certain Euler equations on $\mathfrak{so}(4)$ and their physical applications were considered in [14], whereas (general) integrable quadratic Hamiltonians on $\mathfrak{so}(4)$ were also studied in [27].

Appendix: Classification of systems on SO(4) in matrix form

In the following tables, the homogeneous systems correspond to $A = 0$.

Single-input			
	$\Xi^0(\mathbf{1}, u)$	A	
$\Sigma^{(1,1)}_{\beta\boldsymbol{\varepsilon}}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \beta \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \varepsilon_2 \\ 0 \\ \varepsilon_4 \\ 0 \\ 0 \end{bmatrix}$ $\beta=0$	$\begin{bmatrix} 0 \\ \varepsilon_2 \\ 0 \\ \varepsilon_4 \\ \varepsilon_5 \\ 0 \end{bmatrix}$ $0<\beta\leq 1$

Two-input					
	$\Xi^0(\mathbf{1}, u)$	A			
$\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \varepsilon_2 \\ 0 \\ 0 \\ \varepsilon_5 \\ 0 \end{bmatrix}$			
$\Sigma^{(2,1)}_{2,\boldsymbol{\alpha\varepsilon}}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ \varepsilon_3 \\ \varepsilon_4 \\ 0 \\ 0 \end{bmatrix}$ $\alpha_1=\alpha_2=0$	$\begin{bmatrix} 0 \\ 0 \\ \varepsilon_3 \\ \varepsilon_4 \\ 0 \\ \varepsilon_6 \end{bmatrix}$ $\alpha_1=\alpha_2>0$	$\begin{bmatrix} 0 \\ 0 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$ $\alpha_1>\alpha_2=0$	$\begin{bmatrix} 0 \\ 0 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$ $\alpha_1>\alpha_2>0$

Three-input		
	$\Xi^0(\mathbf{1},u)$	A
$\Sigma^{(3,1)}_{1,\boldsymbol{\alpha\varepsilon}}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\\beta&0&0\\0&0&0\\0&0&1\end{bmatrix}$	$\underset{\beta=0}{\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\0\\0\end{bmatrix}}$ $\underset{0<\beta\leq 1}{\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\0\end{bmatrix}}$
$\Sigma^{(3,1)}_{2,\boldsymbol{\alpha\varepsilon}}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{bmatrix}$	$\underset{\alpha_1=\alpha_2=\lvert\alpha_3\rvert}{\begin{bmatrix}0\\0\\0\\\varepsilon_4\\0\\0\end{bmatrix}}$ $\underset{\alpha_1>\alpha_2=\lvert\alpha_3\rvert}{\begin{bmatrix}0\\0\\0\\\varepsilon_4\\\varepsilon_5\\0\end{bmatrix}}$ $\underset{\alpha_1=\alpha_2>\lvert\alpha_3\rvert}{\begin{bmatrix}0\\0\\0\\\varepsilon_4\\0\\\varepsilon_6\end{bmatrix}}$ $\underset{\alpha_1>\alpha_2>\lvert\alpha_3\rvert}{\begin{bmatrix}0\\0\\0\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}}$

Four-input		
	$\Xi^0(\mathbf{1},u)$	A
$\Sigma^{(4,1)}_{1,\boldsymbol{\varepsilon}}$	$\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$	$\begin{bmatrix}\varepsilon_1\\0\\0\\\varepsilon_4\\0\\0\end{bmatrix}$
$\Sigma^{(4,1)}_{2,\boldsymbol{\alpha\varepsilon}}$	$\begin{bmatrix}-\alpha_1&0&0&0\\0&-\alpha_2&0&0\\0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$	$\underset{\alpha_1=\alpha_2}{\begin{bmatrix}\varepsilon_1\\0\\0\\0\\0\\0\end{bmatrix}}$ $\underset{\alpha_1>\alpha_2}{\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\0\\0\\0\\0\end{bmatrix}}$

Five-input		
	$\Xi^0(\mathbf{1},u)$	A
$\Sigma^{(5,1)}_{\beta\boldsymbol{\varepsilon}}$	$\begin{bmatrix}-\beta&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$	$\begin{bmatrix}\varepsilon_1\\0\\0\\0\\0\\0\end{bmatrix}$

Six-input	
	$\Xi^0(\mathbf{1}, u)$
$\Sigma^{(6,0)}_{\beta\boldsymbol{\varepsilon}}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$\Sigma\colon A + \Xi^0(\mathbf{1}, u) \quad \Xi^0(\mathbf{1}, u) = u_1 B_1 + \cdots + u_\ell B_\ell \longleftrightarrow \begin{bmatrix} B_1 & \cdots & B_\ell \end{bmatrix}$

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