Prasanta Kumar Nandi; Ganesh Chandra Gorain; Samarjit Kar
Stability of vibrations for some Kirchhoff equation with dissipation

*Applications of Mathematics*, Vol. 59 (2014), No. 2, 205–215

Persistent URL: http://dml.cz/dmlcz/143630

# STABILITY OF VIBRATIONS FOR SOME KIRCHHOFF EQUATION WITH DISSIPATION

Prasanta Kumar Nandi, Durgapur, Ganesh Chandra Gorain, Purulia, Samarjit Kar, Durgapur

(Received April 1, 2012)

*Abstract.* In this paper we consider the boundary value problem of some nonlinear Kirchhoff-type equation with dissipation. We also estimate the total energy of the system over any time interval $[0,T]$ with a tolerance level $\gamma$. The amplitude of such vibrations is bounded subject to some restrictions on the uncertain disturbing force $f$. After constructing suitable Lyapunov functional, uniform decay of solutions is established by means of an exponential energy decay estimate when the uncertain disturbances are insignificant.

*Keywords*: Kirchhoff equation; dissipation; vibration; stabilization; energy decay estimate

*MSC 2010*: 35B35, 35L70, 45K05, 37L15

## 1. Introduction and mathematical formulation

Research in the area of vibration stabilization of flexible structures like strings, beams, and plates has been gaining importance since early seventies in studies aimed at achieving energy decay rate of the system. Understanding the linear and nonlinear vibrations is becoming increasingly important in a wide range of engineering applications. This is particularly true in the design of flexible structures such as aircraft, satellites, bridges etc. There is an increasing trend towards lighter structures with increased slenderness often made of new composite materials and requiring some form of active vibration control. There are also applications in the areas of robotics, micro electrical mechanical systems, non-destructive testing and related disciplines such as structural health monitoring.

Stability of vibrations of nonlinear equations of motion is of great importance to researchers in the field of dynamical systems. There are wide discussions on the stabilization of nonlinear vibrations of strings in the literature [7], [8], [10], [16], [21]. Here we have considered the initial boundary value problem of the Kirchhoff-type equations with dissipation. Many authors have studied the existence and uniqueness of solutions of this type of problems by using various methods [14], [18], [1]. K. Ono and K. Nishihara [20] have proved the global existence and decay structure of solutions of this type of problem using Galerkin method. Global existence for different Kirchhoff-type equations on different conditions have been proved by a number of authors [8], [19], [6], [24]. Recently, Nandi et al. [13] have established uniform exponential stabilization for flexural vibrations of a solar panel. Intensive study over nonlinear hyperbolic vibrating equations have been made by a number of authors [11], [17], [15]. Asymptotic stability for different Kirchhoff systems can be found in the literature [2], [3], [4], [5].

The mathematical formulation involved here is mainly the integro-differential equation

$$\varrho h\frac{\partial^2 y}{\partial t^2}+\delta\frac{\partial y}{\partial t}=\Big(p_0+\frac{Eh}{2L}\int_0^L\Big(\frac{\partial y}{\partial x}\Big)^2\,\mathrm{d}x\Big)\frac{\partial^2 y}{\partial x^2}+f,\quad 0<x<L,\ t>0, \tag{1}$$

where the coefficient $\varrho$ is the mass density, $h$ is the cross-section area, $L$ is the length, $p_0$ is the initial axial tension, $\delta$ is the resistance modulus, $E$ is the Young modulus and $f$ is the external force.

The boundary conditions are

$$y(0,t)=y(L,t)=0,\quad t\geqslant 0, \tag{2}$$

and the initial values are

$$y(x,0)=y_0(x)\quad\text{and}\quad\frac{\partial y}{\partial t}(x,0)=y_1(x),\quad 0\leqslant x\leqslant L. \tag{3}$$

The aim of this work is to study the stabilization of solutions of the system (1)–(3). We adopt here a direct method to achieve the results by constructing a suitable Lyapunov functional associated with the energy functional of the system.

## 2. Energy of the system

The total energy E(t) of the system (1)–(3) at time $t\geqslant 0$ is defined by

$$E(t)=\frac{1}{2}\int_0^L\Big[\varrho h\Big(\frac{\partial y}{\partial t}\Big)^2+p_0\Big(\frac{\partial y}{\partial x}\Big)^2\Big]\,\mathrm{d}x+\frac{Eh}{8L}\Big(\int_0^L\Big(\frac{\partial y}{\partial x}\Big)^2\,\mathrm{d}x\Big)^2. \tag{4}$$

Differentiating (4) with respect to $t$ and using (1), we obtain

(5)

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_0^L \Big[\varrho h \frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2} + p_0 \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\Big]\,\mathrm{d}x + \frac{Eh}{2L}\int_0^L \Big(\frac{\partial y}{\partial x}\Big)^2 \mathrm{d}x \int_0^L \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\,\mathrm{d}x \\
&= \int_0^L \Big[\frac{\partial y}{\partial t}\Big\{ -\delta \frac{\partial y}{\partial t} + \Big(p_0 + \frac{Eh}{2L}\int_0^L \Big(\frac{\partial y}{\partial x}\Big)^2 \mathrm{d}x\Big)\frac{\partial^2 y}{\partial x^2} + f\Big\} + p_0 \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\Big]\,\mathrm{d}x \\
&\quad + \frac{Eh}{2L}\int_0^L \Big(\frac{\partial y}{\partial x}\Big)^2 \mathrm{d}x \int_0^L \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\,\mathrm{d}x \\
&= p_0 \int_0^L \Big[\frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\Big]\,\mathrm{d}x - \delta \int_0^L \Big(\frac{\partial y}{\partial t}\Big)^2 \mathrm{d}x + \int_0^L f\Big(\frac{\partial y}{\partial t}\Big)\,\mathrm{d}x \\
&= p_0 \int_0^L \frac{\partial}{\partial x}\Big(\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\Big)\,\mathrm{d}x - \delta \int_0^L \Big(\frac{\partial y}{\partial t}\Big)^2 \mathrm{d}x + \int_0^L f\Big(\frac{\partial y}{\partial t}\Big)\,\mathrm{d}x \\
&= \int_0^L f\Big(\frac{\partial y}{\partial t}\Big)\,\mathrm{d}x - \delta \int_0^L \Big(\frac{\partial y}{\partial t}\Big)^2 \mathrm{d}x \leqslant 0 \quad \text{for } t \geqslant 0,\end{aligned}$$

where the integration is performed by parts and the boundary conditions (2) are used. As the input disturbance $f$ is present in (5), the energy of the system will be neither dissipating nor conserving. Integrating (5) over [0,t], we get

$$E(t) = E(0) - 2\int_0^t\!\!\int_0^L \delta\Big(\frac{\partial y}{\partial t}\Big)^2 \mathrm{d}x\,\mathrm{d}t + \int_0^t\!\!\int_0^L f\Big(\frac{\partial y}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t \quad \text{for } t \geqslant 0, \tag{6}$$

where

$$E(0) = \frac{1}{2}\int_0^L \Big[\varrho h y_1^2(x) + p_0\Big(\frac{\partial y_0}{\partial x}\Big)^2\Big]\,\mathrm{d}x + \frac{Eh}{8L}\Big(\int_0^L \Big(\frac{\partial y_0}{\partial x}\Big)^2 \mathrm{d}x\Big)^2. \tag{7}$$

We see from (6) that when the input disturbance $f \equiv 0$, the energy of the system is dissipating with time, satisfying $E(t) \leqslant E(0)$ for every $t \geqslant 0$. Now the estimate (7) implies that if $y_0 \in H_0^1[0,L]$ and $y_1 \in L^2[0,L]$, where

$$H_0^1[0,L] := \{F;\ F \in H^1[0,L] \text{ and } F(0) = F(L) = 0\} \tag{8}$$

is the subspace of the classical Sobolev space

$$H^1[0,L] = \{F;\ F \in L^2[0,L], F' \in L^2[0,L]\} \tag{9}$$

of real valued functions of order one, then $E(t) \leqslant E(0) < +\infty$ for every $t \geqslant 0$.

Yamazaki [23] considers the initial-boundary value problem for the Kirchhoff equations in exterior domains of dimension three, showing that these problems admit time-global solutions. Yamazaki [22] also proves the unique global solvability of initial-boundary value problem for the Kirchhoff equations in exterior domains or in the whole Euclidean space for dimension larger than three, showing time decay estimate of the linear wave equation.

## 3. Stability results

The main result of this paper can be stated in the following two theorems.

**Theorem 1.** *Let $y(x,t)$ be a solution of the system* (1)–(3) *with the initial values $(y_0, y_1) \in H_0^1[0,L] \times L^2[0,L]$. Then for every $T > 0$,*

$$\int_0^T E(t)\,\mathrm{d}t \leqslant ME(0) + \gamma \int_0^T \|f\|^2_{L^2(0,L)}\,\mathrm{d}t, \tag{10}$$

*where $M$ and $\gamma$ are defined later in* (37).

**Theorem 2.** *Let $y(x,t)$ be a solution of the system* (1)–(3) *with the initial values $(y_0, y_1) \in H_0^1[0,L] \times L^2[0,L]$. Then the total energy of the system decays uniformly exponentially with time, that means, the energy $E(t)$ satisfies the relation*

$$E(t) \leqslant M\mathrm{e}^{-\mu t}E(0) \quad \forall t \geqslant 0 \tag{11}$$

*for some finite reals $M > 1$ and $\mu > 0$, both being independent of time $t$.*

In the sequel, we need the following two inequalities.

For any real number $\alpha > 0$, we have a trivial inequality

$$|u \cdot v| \leqslant \frac{1}{2\alpha}(|u|^2 + \alpha^2|v|^2). \tag{12}$$

By Poincaré-type Scheeffer's inequality [12], we can write

$$\int_0^L y^2\,\mathrm{d}x \leqslant \frac{L^2}{\pi^2}\int_0^L \Big(\frac{\partial y}{\partial x}\Big)^2\,\mathrm{d}x. \tag{13}$$

for every $y(x,t)$ satisfying boundary conditions (2).

Next we consider the following lemmas:

**Lemma 1.** *For every solution $y(x,t)$ of the system* (1)–(3), *the time derivative of the functional $G$ (cf. G. C. Gorain* [7], *V. Komornik and E. Zuazua* [9]*) defined by*

$$G(t)=\int_0^L \varrho h y\frac{\partial y}{\partial t}\,\mathrm{d}x+\frac{1}{2}\int_0^L \delta y^2\,\mathrm{d}x \quad \text{for } t\geqslant 0 \tag{14}$$

*satisfies*

$$\frac{\mathrm{d}G}{\mathrm{d}t}=\int_0^L \varrho h\Big(\frac{\partial y}{\partial t}\Big)^2\mathrm{d}x-\int_0^L\left(p_0+\frac{Eh}{2L}\int_0^L\Big(\frac{\partial y}{\partial x}\Big)^2\mathrm{d}x\right)\Big(\frac{\partial y}{\partial x}\Big)^2\mathrm{d}x+\int_0^L yf\,\mathrm{d}x. \tag{15}$$

P r o o f. Differentiating (14) with respect to $t$ and using the equation (1), we obtain

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}t}&=\int_0^L \varrho h\Big(\frac{\partial y}{\partial t}\Big)^2\mathrm{d}x+\int_0^L \varrho h y\frac{\partial^2 y}{\partial t^2}\,\mathrm{d}x+\int_0^L \delta y\frac{\partial y}{\partial t}\,\mathrm{d}x\\&=\int_0^L \varrho h\Big(\frac{\partial y}{\partial t}\Big)^2\mathrm{d}x\\&\quad+\int_0^L y\left\{-\delta\frac{\partial y}{\partial t}+\left(p_0+\frac{Eh}{2L}\int_0^L\Big(\frac{\partial y}{\partial x}\Big)^2\mathrm{d}x\right)\frac{\partial^2 y}{\partial x^2}+f\right\}\mathrm{d}x\\&\quad+\int_0^L \delta y\frac{\partial y}{\partial t}\,\mathrm{d}x.\end{aligned}\tag{16}$$

Integrating by parts and using the boundary conditions (2) and the energy identity (4), we get

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}t}&=\int_0^L \varrho h\Big(\frac{\partial y}{\partial t}\Big)^2\mathrm{d}x+\left[y\left(p_0+\frac{Eh}{2L}\int_0^L\Big(\frac{\partial y}{\partial x}\Big)^2\mathrm{d}x\right)\frac{\partial y}{\partial x}\right]_0^L\\&\quad-\int_0^L\left(p_0+\frac{Eh}{2L}\int_0^L\Big(\frac{\partial y}{\partial x}\Big)^2\mathrm{d}x\right)\Big(\frac{\partial y}{\partial x}\Big)^2\mathrm{d}x+\int_0^L yf\,\mathrm{d}x\\&=\int_0^L \varrho h\Big(\frac{\partial y}{\partial t}\Big)^2\mathrm{d}x-\int_0^L\left(p_0+\frac{Eh}{2L}\int_0^L\Big(\frac{\partial y}{\partial x}\Big)^2\mathrm{d}x\right)\Big(\frac{\partial y}{\partial x}\Big)^2\mathrm{d}x\\&\quad+\int_0^L yf\,\mathrm{d}x.\end{aligned}\tag{17}$$

Hence, the lemma is proved. $\square$

**Lemma 2.** *For every solution $y(x,t)$ of the system* (1)–(3), *an estimate of the functional $G$ is given by*

$$-\lambda_0 E(t) \leqslant G(t) \leqslant \lambda_1 E(t) \quad \textit{for } t \geqslant 0, \tag{18}$$

*where*

$$\lambda_0 = \frac{L}{\pi}\sqrt{\frac{\varrho h}{p_0}}, \qquad \lambda_1 = \frac{\delta}{p_0}\,\frac{L^2}{\pi^2}. \tag{19}$$

Proof. We can estimate the first term of (14) as

$$\begin{aligned} \left|\int_0^L \varrho h y \frac{\partial y}{\partial t}\,\mathrm{d}x\right| &= \int_0^L \left|\sqrt{\varrho h}\frac{\partial y}{\partial t}\right| \left|\sqrt{\varrho h} y\right| \mathrm{d}x \\ &\leqslant \frac{1}{2\alpha}\int_0^L \left[\varrho h\Big(\frac{\partial y}{\partial t}\Big)^2 + \alpha^2 \varrho h y^2\right] \mathrm{d}x, \quad \text{using (12)} \\ &\leqslant \frac{1}{2\alpha}\int_0^L \left[\varrho h\Big(\frac{\partial y}{\partial t}\Big)^2 + \alpha^2 \varrho h \frac{L^2}{\pi^2}\Big(\frac{\partial y}{\partial x}\Big)^2\right] \mathrm{d}x, \quad \text{using (13)} \\ &= \frac{1}{2}\,\frac{L}{\pi}\sqrt{\frac{\varrho h}{p_0}}\int_0^L \left[\varrho h\Big(\frac{\partial y}{\partial t}\Big)^2 + p_0\Big(\frac{\partial y}{\partial x}\Big)^2\right] \mathrm{d}x = \lambda_0 E(t), \end{aligned} \tag{20}$$

by choosing

$$\alpha = \frac{\pi}{L}\sqrt{\frac{p_0}{\varrho h}} = \frac{1}{\lambda_0}. \tag{21}$$

Again, we can estimate the second term of (14) as

$$\begin{aligned} 0 &\leqslant \frac{1}{2}\int_0^L \delta y^2\,\mathrm{d}x \leqslant \frac{1}{2}\delta\int_0^L y^2\,\mathrm{d}x, \\ &\leqslant \frac{1}{2}\delta\frac{L^2}{\pi^2}\int_0^L \Big(\frac{\partial y}{\partial x}\Big)^2 \mathrm{d}x, \quad \text{using (13)} \\ &\leqslant \frac{1}{2}\,\frac{\delta}{p_0}\,\frac{L^2}{\pi^2}\int_0^L p_0\Big(\frac{\partial y}{\partial x}\Big)^2 \mathrm{d}x, \\ &\leqslant \frac{\delta}{p_0}\,\frac{L^2}{\pi^2}E(t), \quad \text{using (4)}. \end{aligned} \tag{22}$$

In view of (20) and (22), the lemma follows immediately. □

P r o o f of Theorem 1. Proceeding as in G. C. Gorain [8] and V. Komornik and E. Zuazua [9], we define energy like a Lyapunov functional $V$ by

$$V(t) = E(t) + \varepsilon G(t) \quad \text{for } t \geqslant 0, \tag{23}$$

where $\varepsilon > 0$ is a small constant.

In view of Lemma 2, the functional $V$ defined by (23) can be estimated as

$$(1 - \varepsilon\lambda_0)E(t) \leqslant V(t) \leqslant (1 + \varepsilon(\lambda_0 + \lambda_1))E(t). \tag{24}$$

Since $\varepsilon > 0$, we may assume that

$$0 < \varepsilon < \frac{1}{\lambda_0} \tag{25}$$

so that $V(t) > 0$, for every $t \geqslant 0$.

Differentiating (23) with respect to $t$, and using (5) and (15), we obtain

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} = &- (\delta - \varepsilon\varrho h) \int_0^L \Big(\frac{\partial y}{\partial t}\Big)^2 \,\mathrm{d}x + \int_0^L f\Big(\frac{\partial y}{\partial t}\Big) \,\mathrm{d}x \\ &- \varepsilon \int_0^L \Big(p_0 + \frac{Eh}{2L}\int_0^L \Big(\frac{\partial y}{\partial x}\Big)^2 \,\mathrm{d}x\Big)\Big(\frac{\partial y}{\partial x}\Big)^2 \,\mathrm{d}x + \varepsilon \int_0^L yf \,\mathrm{d}x. \end{aligned} \tag{26}$$

By the inequality (12) we can estimate

$$\int_0^L f\Big(\frac{\partial y}{\partial t}\Big) \,\mathrm{d}x \leqslant \frac{1}{2}\int_0^L \Big[\frac{\varrho h\varepsilon}{2}\Big(\frac{\partial y}{\partial t}\Big)^2 + \frac{2f^2}{\varrho h\varepsilon}\Big] \,\mathrm{d}x, \tag{27}$$

$$\int_0^L yf \,\mathrm{d}x \leqslant \frac{1}{2}\int_0^L \Big[\frac{f^2}{p_0k_0} + p_0\Big(\frac{\partial y}{\partial x}\Big)^2\Big] \,\mathrm{d}x, \tag{28}$$

choosing $k_0 = \pi^2/L^2$, so that the differential relation (26) reduces to

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} \leqslant &- (\delta - \varepsilon\varrho h) \int_0^L \Big(\frac{\partial y}{\partial t}\Big)^2 \,\mathrm{d}x - \varepsilon \int_0^L \Big(p_0 + \frac{Eh}{2L}\int_0^L \Big(\frac{\partial y}{\partial x}\Big)^2 \,\mathrm{d}s\Big)\Big(\frac{\partial y}{\partial x}\Big)^2 \,\mathrm{d}x \\ &+ \frac{1}{2}\int_0^L \Big[\frac{\varrho h\varepsilon}{2}\Big(\frac{\partial y}{\partial t}\Big)^2 + \frac{2f^2}{\varrho h\varepsilon}\Big] \,\mathrm{d}x + \frac{1}{2}\int_0^L \Big[\frac{\varepsilon f^2}{p_0k_0} + p_0\varepsilon\Big(\frac{\partial y}{\partial x}\Big)^2\Big] \,\mathrm{d}x, \\ = &\Big(-\delta + \frac{5}{4}\varepsilon\varrho h\Big)\int_0^L \Big(\frac{\partial y}{\partial t}\Big)^2 \,\mathrm{d}x - \frac{1}{2}\int_0^L p_0\varepsilon\Big(\frac{\partial y}{\partial x}\Big)^2 \,\mathrm{d}x \\ &- \frac{\varepsilon Eh}{2L}\Big[\int_0^L \Big(\frac{\partial y}{\partial x}\Big)^2 \,\mathrm{d}x\Big]^2 + \Big[\frac{1}{\varepsilon\varrho h} + \frac{\varepsilon}{2p_0k_0}\Big]\int_0^L f^2 \,\mathrm{d}x \\ = &\Big(-\delta + \frac{7}{4}\varepsilon\varrho h\Big)\int_0^L \Big(\frac{\partial y}{\partial t}\Big)^2 \,\mathrm{d}x - \varepsilon\Big[\frac{1}{2}\int_0^L \Big[\varrho h\Big(\frac{\partial y}{\partial t}\Big)^2 + p_0\Big(\frac{\partial y}{\partial x}\Big)^2\Big] \,\mathrm{d}x \\ &+ \frac{Eh}{8L}\Big(\int_0^L \Big(\frac{\partial y}{\partial x}\Big)^2 \,\mathrm{d}x\Big)^2\Big] + \Big[\frac{1}{\varepsilon\varrho h} + \frac{\varepsilon}{2p_0k_0}\Big]\int_0^L f^2 \,\mathrm{d}x. \end{aligned} \tag{29}$$

Choosing

$$\varepsilon < \frac{4\delta}{7\varrho h}, \tag{30}$$

the relation (29) reduces to

$$\frac{\mathrm{d}V}{\mathrm{d}t} + \varepsilon E(t) \leqslant C\|f\|^2_{L^2(0,L)}, \tag{31}$$

where

$$C = \frac{1}{\varepsilon\varrho h} + \frac{\varepsilon}{2p_0k_0}. \tag{32}$$

In view of (24), the relation (31) reduces to

$$\frac{\mathrm{d}V}{\mathrm{d}t} + \mu V(t) \leqslant C\|f\|^2_{L^2(0,L)}, \tag{33}$$

where

$$\mu = \frac{\varepsilon}{1+\varepsilon(\lambda_0+\lambda_1)}. \tag{34}$$

Multiplying (33) by $\mathrm{e}^{\mu t}$ and integrating from 0 to $t$, we obtain

$$V(t) \leqslant \mathrm{e}^{-\mu t}\left[V(0) + C\int_0^t \|f\|^2_{L^2(0,L)}\mathrm{e}^{\mu\tau}\,\mathrm{d}\tau\right] \quad \text{for } t \geqslant 0. \tag{35}$$

In view of (24), the relation (35) reduces to

$$E(t) \leqslant \mathrm{e}^{-\mu t}\left[ME(0) + \mu\gamma\int_0^t \|f\|^2_{L^2(0,L)}\mathrm{e}^{\mu\tau}\,\mathrm{d}\tau\right], \quad \forall t \geqslant 0, \tag{36}$$

where

$$M = \frac{1+\varepsilon(\lambda_0+\lambda_1)}{1-\varepsilon\lambda_0} \quad \text{and} \quad \gamma = C\mu(1-\varepsilon\lambda_0). \tag{37}$$

Integrating the above relation over $[0,T]$, we get

$$\int_0^T E(t)\,\mathrm{d}t \leqslant ME(0)\int_0^T \mathrm{e}^{-\mu t}\,\mathrm{d}t + \mu\gamma\int_0^T \mathrm{e}^{-\mu t}F(t)\,\mathrm{d}t \quad \text{for } t \geqslant 0, \tag{38}$$

where

$$F(t) = \int_0^t \|f\|^2_{L^2(0,L)}\mathrm{e}^{\mu\tau}\,\mathrm{d}\tau. \tag{39}$$

Integrating by parts, we get

$$\int_0^T E(t)\,\mathrm{d}t = M(1-\mathrm{e}^{\mu T})E(0) + \gamma\Big[F(0) - \mathrm{e}^{-\mu T}F(T) + \int_0^T \mathrm{e}^{-\mu t}F'(t)\Big] \leqslant ME(0) + \gamma\int_0^T \|f\|^2_{L^2(0,L)}\,\mathrm{d}t, \tag{40}$$

since

$$F(0) = 0 \quad \text{and} \quad F'(t) = \|f\|^2_{L^2(0,L)}\mathrm{e}^{\mu t}\,\mathrm{d}t. \tag{41}$$

Hence the theorem. □

R e m a r k 1. If $f$ is bounded in the sense $\sup\limits_{t>0}\big[\int_0^t \|f\|^2_{L^2(0,L)}\,\mathrm{d}\tau\big] < +\infty$ and for every $(y_0, y_1) \in H_0^1[0,L] \times L^2[0,L]$, then it follows from (36) that $\sup\limits_{t\geqslant 0} E(t) < +\infty$. Thus the energy of the system is uniformly bounded over time and the solution of the system (1)–(3) is bounded also for the boundedness of the input disturbances $f$ in the sense above. This signifies that the system is bounded-input bounded-output stable.

R e m a r k 2. The result of Theorem 1 is an estimate of the total energy $E(t)$ as the system evolves over a time interval $[0,T]$, for $T > 0$, due to work done by the disturbing force $f$ and dissipation. The term $\gamma$ appearing in (40) may be seen as the tolerance factor of the disturbing force $f$ on the total energy over any time interval $[0,T]$.

P r o o f of Theorem 2. We shall prove Theorem 2 with the help of Theorem 1. In the case when $f \equiv 0$, it follows from (36) that

$$E(t) \leqslant M\mathrm{e}^{-\mu t}E(0) \quad \text{for } t \geqslant 0. \tag{42}$$

Hence the theorem. □

R e m a r k 3. The result (42) shows an uniform exponential stability of the system (1)–(3) due to dissipation in an ideal case where there is no input disturbance in the system. It is obvious that the exponential decay estimate will be maximum for the largest admissible value $\varepsilon$ satisfying (30).

## 4. Conclusions

This study deals with the mathematical stability results of the boundary value problem of some nonlinear Kirchhoff-type equation with dissipation. The total energy of the system is calculated over any time interval $[0, T]$ with a tolerance level $\gamma$ of the input disturbances. It also covers that the amplitude of such vibrations is bounded subject to a bounded disturbing force $f$. Finally, uniform decay of solutions by means of an exponential energy decay estimate is achieved when the uncertain disturbances are not important enough to merit attention of the system.

Acknowledgement. The authors wish to thank the reviewers for their valuable comments and suggestions in revising the paper.

*References*


[1] *M. Aassila, A. Benaissa*: Global existence and asymptotic behavior of solutions of mildly degenerate Kirchhoff equations with nonlinear dissipative term. Funkc. Ekvacioj, Ser. Int. *44* (2001), 309–333. (In French.)

[2] *G. Autuori, P. Pucci*: Asymptotic stability for Kirchhoff systems in variable exponent Sobolev spaces. Complex Var. Elliptic Equ. *56* (2011), 715–753.

[3] *G. Autuori, P. Pucci*: Local asymptotic stability for polyharmonic Kirchhoff systems. Appl. Anal. *90* (2011), 493–514.

[4] *G. Autuori, P. Pucci, M. C. Salvatori*: Asymptotic stability for anisotropic Kirchhoff systems. J. Math. Anal. Appl. *352* (2009), 149–165.

[5] *G. Autuori, P. Pucci, M. C. Salvatori*: Asymptotic stability for nonlinear Kirchhoff systems. Nonlinear Anal., Real World Appl. *10* (2009), 889–909.

[6] *P. D'Ancona, S. Spagnolo*: Nonlinear perturbations of the Kirchhoff equation. Commun. Pure Appl. Math. *47* (1994), 1005–1029.

[7] *G. C. Gorain*: Boundary stabilization of nonlinear vibrations of a flexible structure in a bounded domain in $R^n$. J. Math. Anal. Appl. *319* (2006), 635–650.

[8] *G. C. Gorain*: Exponential energy decay estimates for the solutions of $n$-dimensional Kirchhoff type wave equation. Appl. Math. Comput. *177* (2006), 235–242.

[9] *V. Komornik, E. Zuazua*: A direct method for the boundary stabilization of the wave equation. J. Math. Pures Appl., IX. Sér. *69* (1990), 33–54.

[10] *I. Lasiecka, J. Ong*: 2069–2107. Commun. Partial Differ. Equations *24* (1999).

[11] *G. P. Menzala*: On classical solutions of a quasilinear hyperbolic equation. Nonlinear Anal., Theory Methods Appl. *3* (1979), 613–627.

[12] *D. S. Mitrinović, J. E. Pečarić, A. M. Fink*: Inequalities Involving Functions and Their Integrals and Derivatives. Mathematics and Its Applications: East European Series 53, Kluwer Academic Publishers, Dordrecht, 1991.

[13] *P. K. Nandi, G. C. Gorain, S. Kar*: Uniform exponential stabilization for flexural vibrations of a solar panel. Appl. Math. (Irvine) *2* (2011), 661–665.

[14] *R. Narasimha*: Non-linear vibration of an elastic string. J. Sound Vib. *8* (1968), 134–146.

[15] *A. H. Nayfeh, D. T. Mook*: Nonlinear Oscillations. Pure and Applied Mathematics. A Wiley-Interscience Publication, John Wiley & Sons, New York, 1979.

[16] *W. G. Newman*: Global solution of a nonlinear string equation. J. Math. Anal. Appl. *192* (1995), 689–704.

[17] *K. Nishihara*: On a global solution of some quasilinear hyperbolic equation. Tokyo J. Math. *7* (1984), 437–459.
[18] *K. Nishihara, Y. Yamada*: On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms. Funkc. Ekvacioj, Ser. Int. *33* (1990), 151–159.
[19] *K. Ono*: Global existence, decay, and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings. J. Differ. Equations *137* (1997), 273–301.
[20] *K. Ono, K. Nishihara*: On a nonlinear degenerate integro-differential equation of hyperbolic type with a strong dissipation. Adv. Math. Sci. Appl. *5* (1995), 457–476.
[21] *S. M. Shahruz*: Bounded-input bounded-output stability of a damped nonlinear string. IEEE Trans. Autom. Control *41* (1996), 1179–1182.
[22] *T. Yamazaki*: Global solvability for the Kirchhoff equations in exterior domains of dimension larger than three. Math. Methods Appl. Sci. *27* (2004), 1893–1916.
[23] *T. Yamazaki*: Global solvability for the Kirchhoff equations in exterior domains of dimension three. J. Differ. Equations *210* (2005), 290–316.
[24] *Y. Ye*: On the exponential decay of solutions for some Kirchhoff-type modelling equations with strong dissipation. Applied Mathematics *1* (2010), 529–533.

*Authors' addresses*: *Prasanta Kumar Nandi*, Department of Mathematics, National Institute of Technology, Durgapur-713209, West Bengal, India, e-mail: pknandi.math@gmail.com; *Ganesh Chandra Gorain*, Department of Mathematics, J. K. College, Purulia-723101, West Bengal, India, e-mail: goraing@gmail.com; *Samarjit Kar*, Department of Mathematics, National Institute of Technology, Durgapur-713209, West Bengal, India, e-mail: kar_s_k@yahoo.com.