Tomáš Kepka; Petr Němec

Quasitrivial semimodules. IV.

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 54 (2013), No. 1, 3–22

Persistent URL: `http://dml.cz/dmlcz/143708`

QUASITRIVIAL SEMIMODULES IV

TOMÁŠ KEPKA, PETR NĚMEC

Praha

Received September 19, 2012

In the paper, almost quasitrivial and critical semimodules are studied.

The foregoing parts [1], [2] and [3] are continued and the notation introduced in these parts is used. More attention is paid to almost quasitrivial and critical semimodules.

1. Preliminaries (A)

Let S be a non-trivial semiring and M be a (left S-)semimodule. We put $R(M) = \{ x \in M \mid rtx = stx \text{ for all } r, s, t \in S \}$.

1.1 Lemma. (i) *Either* $R(M) = \emptyset$ *or* $R(M)$ *is a subsemimodule of* M.
(ii) $P(M) \subseteq Q(M) \subseteq R(M)$.

Proof. Easy to check. □

1.2 Lemma. (i) $SR(M) \subseteq Q(M)$.
(ii) $SQ(M)=P(M)$.

Department of Algebra, MFF UK, Sokolovská 83, 186 75 Praha 8, Czech Republic (T. Kepka)
Department of Mathematics, Czech University of Life Sciences, Kamýcká 129, 165 21 Praha 6 – Suchdol, Czech Republic (P. Němec)

Supported by the Grant Agency of the Czech Republic, grant GAČR 201/09/0296.

2000 Mathematics Subject Classification. 16Y60

Key words and phrases. semiring, semimodule, almost quasitrivial, critical

E-mail address: kepka@karlin.mff.cuni.cz, nemec@tf.czu.cz

(iii) $SSR(M) = P(M)$.
(iv) *If* $R(M) \neq \emptyset$ *then* $P(M) \neq \emptyset$.

Proof. (i) If $x \in R(M)$ and $r, s, t \in S$ then $rtx = stx$, and hence $tx \in Q(M)$. Thus $SR(M) \subseteq Q(M)$.
(ii) If $x \in Q(M)$ and $r, s \in S$ then $r(sx) = (rs)x = sx$. Thus $sx \in P(M)$ and $SQ(M) \subseteq P(M)$. Of course, $SP(M) = P(M)$.
(iii) Combining (i) and (ii), we get $SSR(M) \subseteq SQ(M) \subseteq P(M)$.
(iv) This follows immediately from (iii). □

1.3 Lemma. *Assume that for all* $r, s \in S$, $r \neq s$, *there are elements* $r_i, s_i, t_i \in S$, $i = 1, 2, \dots, n$, *such that* $r = \sum r_i t_i$ *and* $s = \sum s_i t_i$. *Then:*
(i) $ra = sa$ *for every* $a \in R(M)$.
(ii) $R(M) = Q(M)$.

Proof. It is easy. □

1.4 Lemma. *If* $St = S$ *for at least one* $t \in S$ *(e.g., if* t *is right multiplicatively neutral) then* $R(M) = Q(M)$.

Proof. Use 1.3(ii). □

1.5 Proposition. *Let the semimodule* M *be minimal. Then just one of the following four cases takes place:*

(1) $R(M) = \emptyset$ *(then* $Q(M) = P(M) = \emptyset$*) or, equivalently, for every* $x \in M$ *there are elements* $r, s, t \in S$ *such that* $rtx \neq stx$*;*
(2) $R(M) = Q(M) = P(M) = \{w\}$, *where* $Sw = w$ *and* $2w = w$ *(then* $Sx = M$ *for every* $x \in M \setminus \{w\}$*);*
(3) $R(M) = Q(M) = M$ *and* $P(M) = \{w\}$ *(then* $SM = \{w\}$*);*
(4) $R(M) = Q(M) = P(M) = M$ *(then* $rx = x$ *for all* $r \in S$, $x \in M$ *and* $M(+)$ *is idempotent).*

Proof. If $R(M) = \emptyset$ then $Q(M) = \emptyset = P(M)$ by 1.2(ii) and (1) is true. If $R(M) = \{w\}$ then $Q(M) = P(M) = \{w\}$ follows from 1.1(ii) and 1.2(iv) and (2) is true. If $|R(M)| \geq 2$ then $R(M) = M$, since M is minimal and, using 1.2(iv), we get $Q(M) \neq \emptyset \neq P(M)$. If $P(M) = M$ then $Q(M) = M$ and (4) is true.

Assume, finally, that $P(M) \neq M$. Since M is minimal, we get $P(M) = \{w\}$. If $Q(M) = M$ then (3) is true (use 1.2(ii)). On the other hand, if $Q(M) \neq M$ then $Q(M) = P(M) = \{w\}$ and, by 1.2(i), $SM = SR(M) = \{w\}$. But then $Q(M) = M$, a contradiction. □

1.6 Corollary. *Let the semimodule M be strictly minimal. Then 1.5(1) is true.* □

1.7 Proposition. *Let M be minimal of type 1.5(3). Then just one of the following three cases takes place:*

(1) *$M(+)$ is a two-element semilattice;*
(2) *$M(+)$ is a two-element constant semigroup;*
(3) *$M(+)$ is a finite cyclic group of prime order.*

Proof. Let M be of type 1.5(3). Then $SM = \{w\}$, and hence every subsemigroup N of $M(+)$ such that $w \in N$ is a subsemimodule of M. Since M is minimal, it follows that $\{w\}$ and M are the only subsemigroups of $M(+)$ and the rest is clear. □

1.8 Proposition. *Let M be minimal of type 1,5(4). Then $M(+)$ is a two-element semilattice.*

Proof. $M(+)$ is idempotent and $rx = x$ for all $r \in S$ and $x \in M$. Thus every subsemigroup of $M(+)$ is a subsemimodule of M and the rest is clear. □

The semimodule M will be called

– *cs-quasitrivial* if $|SM| = 1$;
– *id-quasitrivial* if $P(M) = M$ (i.e., $rx = x$ for all $r \in S$ and $x \in M$);
– *quasitrivial* if $Q(M) = M$;
– *almost quasitrivial* if $R(M) = M$.

1.9 Proposition. (i) *If M is cs-quasitrivial or id-quasitrivial then M is quasitrivial.*
(ii) *If M is quasitrivial then M is almost quasitrivial.*

Proof. It is obvious. □

1.10 Lemma. *Assume that $M = Sv$ for at least one $v \in M$. If ϱ is a congruence of M such that the factorsemimodule M/ϱ is almost quasitrivial then $\varrho = M \times M$.*

Proof. Since $M = Sv$, we have $v = tv$ for some $t \in S$. Since M/ϱ is almost quasitrivial, we see that $(rv, sv) = (rtv, stv) \in \varrho$ for all $r, s \in S$. Using the equality $M = Sv$ one more, we get $\varrho = M \times M$. □

1.11 Corollary. *Assume that M is non-trivial, $M = Sv$ for at least one $v \in M$ and that every proper factorsemimodule of M is almost quasitrivial. Then M is congruence-simple.* □

1.12 Lemma. *Assume that the subsemimodule Sx is quasitrivial for every $x \in M$. Then M is almost quasitrivial.*

Proof. It is easy. □

1.13 Lemma. *Assume that M is not almost quasitrivial and that every proper subsemimodule of M is quasitrivial. Then $R(M) \neq M$ and $Sv = M$ for every $v \in$ $\in M \setminus R(M) \neq \emptyset$.*

Proof. Since M is not almost quasitrivial, we have $R(M) \neq M$ and $M \setminus R(M) \neq \emptyset$. If $v \in M$ is such that $Sv \neq M$ then the subsemimodule Sv is quasitrivial. It means that $|Srv| = 1$ for every $r \in S$, and therefore $v \in R(M)$. □

1.14 Lemma. *Every proper subsemimodule is (almost) quasitrivial in each of the following three cases:*

(1) *M is minimal;*
(2) *M is (almost) quasitrivial;*
(3) *M is finite, not (almost) quasitrivial and the order $|M|$ of M is minimal with respect to these properties.*

Proof. It is easy. □

1.15 Remark. Notice that there is at least one semimodule of type 1.14(3) if and only if there is at least one finite semimodule that is not (almost) quasitrivial. For instance, if S is finite and not left (almost) quasitrivial.

1.16 Lemma. *Assume that $S = \{\, \sum_{i=1}^{n} r_i s_i \mid n \geq 1, r_i, s_i \in S \,\}$ (e.g., if S is ideal-simple and $|S| \geq 2$). Assume, moreover, that every proper subsemimodule of M is almost quasitrivial. Then $Sv = M$ for every $v \in M \setminus R(M)$.*

Proof. If $v \in M$ is such that $Sv \neq M$ then the subsemimodule Sv is almost quasitrivial, and so $rtpv = stpv$ for all $r, s, t, p \in S$. Now, given $q \in S$, we have $q = \sum t_i p_i$, and therefore $rqv = sqv$. Thus $v \in R(M)$. □

1.17 Proposition. *Assume that M is not almost quasitrivial, while every proper subsemimodule of M is quasitrivial and every proper factorsemimodule of M is almost quasitrivial. Then:*
(i) *The semimodule M is congruence-simple.*
(ii) *$M = Sv$ for every $v \in M \setminus R(M)$ ($\neq \emptyset$).*

Proof. By 1.13, $M \setminus R(M) \neq \emptyset$ and $M = Sv$ for every $v \in M \setminus R(M)$. Consequently, M is congruence-simple by 1.11. □

1.18 Proposition. *Assume that $S = \{\, \sum_{i=1}^{n} r_i s_i \mid n \geq 1, r_i, s_i \in S \,\}$. Assume, moreover, that M is not almost quasitrivial, while all proper subsemimodules and all proper factorsemimodules of M are almost quasitrivial. Then:*
(i) *The semimodule M is congruence-simple.*
(ii) *$Sv = M$ for every $v \in M \setminus R(M)$.*

Proof. By 1.16, $S v = M$ for every $v \in M \setminus R(M)$. Now, M is congruence-simple by 1.11. □

1.19 Lemma. *If M is a non-trivial semimodule then $S v \neq M$ for every $v \in R(M)$.*

Proof. If M is quasitrivial then $|S x| = 1$, and hence $S x \neq M$ for every $x \in M$. Now, assume that M is not quasitrivial. Then $|S u| \geq 2$ for at least one $u \in M$ and we have $ru \neq su$ for some $r, s \in S$. If $u = tv$ for some $t \in S$ and $v \in M$ then $rtv \neq stv$, and so $v \notin R(M)$. Consequently, if $v \in R(M)$ then $u \notin S v$ and $S v \neq M$ (of course, $(M \setminus Q(M)) \cap S R(M) \subseteq (M \setminus Q(M) \cap Q(M) = \emptyset)$. □

In the sequel, a semimodule M will be called *decent* if $M = S v$ for every $v \in$ $\in M \setminus R(M)$ (see 1.19).

1.20 Proposition. *A semimodule M is decent in each of the following three cases:*

(1) *M is almost quasitrivial;*

(2) *Every proper subsemimodule of M is quasitrivial and every proper factorsemimodoule of M is almost quasitrivial;*

(3) *$S = \{ \sum_{i=1}^{n} r_i s_i \,|\, n \geq 1, r_i, s_i \in S \}$ and all proper subsemimodules and all proper factorsemimodules of M are almost quasitrivial.*

Proof. See 1.17 and 1.18. □

1.21 Proposition. *Let M be a decent semimodule such that M is not almost quasitrivial, but every proper factorsemimodule of M is almost quasitrivial. Then M is congruence-simple.*

Proof. See 1.11. □

1.22 Lemma. *Let $w \in M$ and $N = \{ a \in M \,|\, w \notin S a \}$. Then:*

(i) *$S N \subseteq N$.*

(ii) *If $N = \emptyset$ then $w \in \bigcap S a$, $a \in M$.*

(iii) *If $N = M$ then $w \notin \bigcup S a$, $a \in M$.*

(iv) *If $N = M$ and $w = x + y$ then, for every $z \in M$, either $x \notin S z$ or $y \notin S z$.*

(v) *If $N = \{v\}$ then $S v = \{v\}$.*

(vi) *If $w \notin M + (M \setminus \{w\})$ then $N + M \subseteq N$.*

Proof. It is easy. □

1.23 Lemma. *Let $w \in M$ be such that $w \notin M + (M \setminus \{w\})$. Put $N = \{ a \in M \,|\, w \notin S a \}$. Then:*

(i) *Either $N = \emptyset$ or N is an ideal of the semimodule M.*

(ii) *$(N \times N) \cup \mathrm{id}_M$ is a congruence of the semimodule M.*

(iii) *If $N = \{v\}$ then $v = o_M$, $S o_M = \{o_M\}$ and $w \in S a$ for every $a \neq o_M$.*

Proof. Use 1.22. □

1.24 Proposition. *Let M be an ideal-simple semimodule such that $0_M \in M$ and $0_M \notin K+K$, where $K = M \setminus \{0_M\}$ (e.g., M idempotent). Then just one of the following three cases takes place:*

(1) $0_M \in Sa$ *for every* $a \in M$*;*

(2) $o_M \in M$, $So_M = \{o_M\}$ *and* $0_M \in Sb$ *for every* $b \neq o_M$*;*

(3) $0_M \notin Sc$ *for every* $c \in M$.

Proof. Use 1.23. □

1.25 Proposition. *Let M be a non-quasitrivial minimal ideal-simple semimodule such that $0_M \in M$ and $0_M \notin K + K$, where $K = M \setminus \{0_M\}$ (e.g., M idempotent). Then just one of the following three cases takes place:*

(1) *M is strictly minimal and idempotent;*

(2) $Q(M) = P(M) = \{0_M\}$ *and* $Sa = M$ *for every* $a \in K$ *(either M is idempotent or 0_M is the only idempotent element);*

(3) $o_M \in M$, $Q(M) = P(M) = \{o_M\}$, *M is idempotent and* $Sa = M$ *for every* $a \in K$.

Proof. Since M is minimal, we have $Sa = M$ for every $a \in M \setminus Q(M)$. Furthermore, either $Q(M) = \emptyset$ or $Q(M) = \{w\}$ is a one-element set. We have $0_M \in \mathrm{Id}(M)$, and hence either M is idempotent or $\mathrm{Id}(M) = \{0_M\}$. If M is strictly minimal then M is idempotent.

Now, assume that 1.24(1) is true. Then $P(M) \subseteq \{0_M\}$. If $Q(M) = \emptyset$ then M is strictly minimal and (1) is true. If $Q(M) = \{w\}$ then $w = 0_M$ and (2) is true.

Next, let 1.24(2) be true. Then $P(M) = Q(M) = \{o_M\}$, $\{o_M, 0_M\} \subseteq \mathrm{Id}(M)$, hence M is idempotent and (3) is true.

Finally, let 1.24(3) be satisfied. Then $Sc \neq M$ for every $c \in M$ and, since M is minimal, it follows that $|Sc| = 1$ and M is quasitrivial, a contradiction. □

1.26 Remark. Let M be as in 1.24 and, moreover, assume that M is minimal and quasitrivial (cf. 1.25). Using [1, 4.1], we see that $|M| = 2$ and M is isomorphic to one of the semimodules $Q_{1,S}$, $Q_{2,S}$ and $Q_{3,S}$. Consequently, M is idempotent and either id-quasitrivial or cs-quasitrivial.

1.27 Remark. Let M be a minimal semimodule.
(i) If $0_S \in S$ and M is not quasitrivial then $Sa = M$ for at least one $a \in M$ and we have $0_M = 0_S a \in M$.
(ii) If M is not quasitrivial then (by [1, 6.3]) there is at least one congruence ϱ of M such that the factorsemimodule $N = M/\varrho$ is minimal, congruence-simple (and hence ideal-simple) and not quasitrivial. Of course, if $0_M \in M$ then $0_n = 0_M/\varrho \in N$.
(iii) If $0_M \in M$ and $K = M \setminus \{0_M\}$ then $L = \{\, a \in M \,|\, 0_M \in M + a \,\}$ is a subgroup of $M(+)$. If M is idempotent then $L = \{0_M\}$ and $0_M \notin K + K$. If $S0_M = \{0_M\}$ then L

is a subsemimodule of M. Then either $L = M$ and $M(+)$ is a group or $L = \{0_M\}$ and $0_M \notin K + K$ again.

1.28 Lemma. *Let M be a finite strictly minimal semimodule. Then for every $w \in M$ there is at least one $r \in S$ with $rM = \{w\}$.*

Proof. We have $Sx = M$ for every $x \in M$. Consequently, $r_x x = o_M \in M$ for some $r_x \in S$ and, setting $r = \sum r_x$, $x \in M$, we get $rM = \{o_M\}$. But $S o_M = M$, $w = so_M$ and $rM = \{w\}$. □

2. Preliminaries (B)

2.1 Proposition. *Let M be a finite semimodule that is not quasitrivial and whose order $|M|$ is minimal possible. Then:*
(i) *All proper subsemimodules as well as all proper factorsemimodules of M are quasitrivial.*
(ii) *M is decent.*
(iii) *M is a one-generated semimodule.*
(iv) *If M is not almost quasitrivial then M is congruence-simple.*
(v) *M is subdirectly irreducible.*

Proof. (i) This is obvious.
(ii) Combine (i) and 1.20(2).
(iii) If $u \in M \setminus Q(M)$ then $\langle u\rangle \nsubseteq Q(M)$, and hence $\langle u\rangle = M$.
(iv) Combine (ii) and 1.11.
(v) This is obvious. □

2.2 Proposition. *Let M be a finite semimodule that is not almost quasitrivial and whose order $|M|$ is minimal possible. Then:*
(i) *All proper subsemimodules as well as all proper factorsemimodules of M are almost quasitrivial.*
(ii) *If $M = Sv$ for at least one $v \in M$ then M is congruence-simple.*
(iii) *M is subdirectly irreducible.*
(iv) *M is a one-generated semimodule.*
(v) *If $M \neq Sx$ for every $x \in M$ then $rtpx = stpx$ for all $r, s, t, p \in S$.*

Proof. It is easy. □

2.3 Proposition. *Define a relation ν on M by $(x, y) \in \nu$ if and only if $rx = ry$ for all $r \in S$. Then:*
(i) *ν is a congruence of the semimodule M.*
(ii) *If $\nu = M \times M$ and $R(M) \neq \emptyset$ then M is cs-quasitrivial.*
(ii) *If $u, v \in P(M)$ are such that $(uv) \in \nu$ then $u = v$ (i.e., $\nu|P(M) = \mathrm{id}$).*

(iv) *If* $u, v \in Q(M)$ *are such that* $r_0 u = s_0 v$ *for some* $r_0, s_0 \in S$ *then* $(u, v) \in \nu$ *and* $ru = sv$ *for all* $r, s \in S$.
(v) *If* $u, v \in R(M)$ *are such that* $r_0 p_0 u = s_0 q_0 v$ *for some* $r_0, s_0, p_0, q_0 \in S$ *then* $(p_0 u, q_0 v) \in \nu$ *and* $r p_0 u = s q_0 v$ *for all* $r, s \in S$.

Proof. (i) Easy to see.
(ii) Since $R(M) \neq \emptyset$, we have $P(M) \neq \emptyset$ by 1.2(iv). Now, if $w \in P(M)$ then $(x, v) \in \nu$ for every $x \in M$, and hence $rx = rw = w$ for every $r \in S$. Thus $SM = \{w\}$.
(iii) We have $u = rurv = v$.
(iv) We have $ru = r_0 u = s_0 v = sv$.
(v) We have $rp_0 u = r_0 p_0 u = s_0 q_0 v = sq_0 v$. □

2.4 Lemma. *If* $\nu = \mathrm{id}_M$ *then* $SR(M) = P(M)$.

Proof. If $u \in R(M)$ and $r, s, t \in S$ then $rstu = rtu$. It means that $(stu, tu) \in \nu = \mathrm{id}_M$, $stu = tu$ and $tu \in P(M)$. □

2.5 Lemma. *If* $\nu = \mathrm{id}_M$ *then* $Q(M) = P(M)$.

Proof. If $u \in Q(M)$ and $r, s \in S$ then $rsu = ru$. It means that $(su, u) \in \nu = \mathrm{id}_M$, $su = u$ and $u \in P(M)$. □

2.6 Proposition. *Assume that* M *is congruence-simple. Then* $SR(M) = Q(M) =$ $= P(M)$.

Proof. If M is cs-quasitrivial then $SR(M) = Q(M) = P(M) = M$. If $R(M) = \emptyset$ then $SR(M) = Q(M) = P(M) = \emptyset$. Assume, therefore, that $R(M) \neq \emptyset$ and M is not cs-quasitrivial. According to 2.3(ii), we have $\nu \neq M \times M$. Since M is congruence-simple, we have $\nu = \mathrm{id}_M$ and we can use 2.4 and 2.5. □

2.7 Proposition. *Assume that* $M = Sv$ *for at least one* $v \in M$ *and that every proper subsemimodule of* M *is almost quasitrivial. Then* $SR(M) = Q(M) = P(M)$.

Proof. We can assume that M is non-trivial. Then M is congruence-simple by 1.11 and we can use 2.6. □

2.8 Proposition. *Assume that* M *is not almost quasitrivial, every proper subsemimodule of* M *is quasitrivial and every proper factorsemimodule of* M *is almost quasitrivial. Then* $SR(M) = Q(M) = P(M)$.

Proof. Just combine 2.7 and 1.17. □

2.9 Proposition. *Assume that* $S = \{\sum_{i=1}^{n} r_i s_i\}$. *Let* M *be not almost quasitrivial. If all proper subsemimodules and all proper factorsemimodules of* M *are almost quasitrivial then* $SR(M) = Q(M) = P(M)$.

Proof. Just combine 2.7 and 1.18. □

3. Preliminaries (C)

A semimodule M is called *faithful* if for all $r, s \in S$, $r \neq s$, there is at least one $x \in M$ with $rx \neq sx$.

3.1 Lemma. *Let* M *be a faithful semimodule such that* $o_M \in M$. *If* $r \in S$ *is such that* $rM = \{o_M\}$ *then* $r = o_S$ *is additively absorbing in* S.

Proof. We have $(r + s)x = rx + sx = o_M + sx = o_M = rx$ for all $s \in S$ and $x \in M$. Since M is faithful, we get $r + s = r$, and hence $r = o_S$. □

3.2 Lemma. *Let* M *be a faithful semimodule such that* $0_M \in M$. *If* $r \in S$ *is such that* $rM = \{0_M\}$ *then* $r = 0_S$ *is additively neutral in* S.

Proof. We have $(r + s)x = rx + sx = 0_M + sx = sx$ for all $s \in S$ and $x \in M$. Since M is faithful, we get $r + s = s$, and hence $r = 0_S$. □

3.3 Lemma. *Let* M *be a faithful semimodule such that* $0_M \in M$ *and* $o_M \in M$. *Then:*
(i) *If* $S o_M = \{o_M\}$ *and if* $r \in S$ *is such that* $r(M \setminus \{o_M\}) \subseteq \{0_M\}$ *then* $r = 0_S$.
(ii) *If* $S 0_M = \{0_M\}$ *and if* $r \in S$ *is such that* $r(M \setminus \{0_M\}) \subseteq \{o_M\}$ *then* $r = o_S$.

Proof. It is easy. □

3.4 Lemma. *Assume that there is a faithful semimodule* M *such that* M *is almost quasitrivial. Then* $rt = st$ *for all* $r, s, t \in S$ *(i.e., the (left* S*-)semimodule* ${}_S S$ *is quasitrivial.*

Proof. We have $rtx = stx$ for every $x \in M$. □

3.5 Lemma. *Define a relation* μ_S *on* S *by* $(r, s) \in \mu_S$ *if and only if* $rt = st$ *for every* $t \in S$. *Then:*
(i) μ_S *is a congruence of the semiring* S.
(ii) $\mu_S = \mathrm{id}_S$ *if and only if the (left* S*-)semimodule* ${}_S S$ *is faithful.*
(iii) $\mu_S = S \times S$ *if and only if* $rt = st$ *for all* $r, s, t \in S$.

Proof. It is easy. □

3.6 Proposition. *Let S be a congruence-simple semiring. Then just one of the following five cases takes place:*

(1) *The left S-semimodule ${}_SS$ is faithful;*
(2) *S is a zero multiplication ring of finite prime order;*
(3) *$S(+)$ is a two-element semilattice and $ab = b$ for all $a, b \in S$;*
(4) *$S(+)$ is a two-element semilattice and $SS = \{w\}$ (there are two non-isomorphic cases);*
(5) *$S(+)$ is a two element constant semigroup and $S + S = \{\} = SS$.*

Proof. In view of 3.5(ii), assume that $\mu_S \neq \mathrm{id}_S$. Then $\mu_S = S \times S$ and $rt = st$ for all $r, s, t \in S$. That is, there is a transformation α of S such that $ab = \alpha(b)$ for all $a, b \in S$. One checks readily that α is an endomorphism of the additive semigroup $S(+)$ and $\alpha^2 = \alpha = 2\alpha$. Consequently, α is an endomorphism of the semiring S and $\ker(\alpha)$ is a congruence of S.

Assume first that $\ker(\alpha) = \mathrm{id}_S$. Then α is injective and $\alpha^2 = \alpha$ implies $\alpha = \mathrm{id}_S$ and $ab = b$ for all $a, b \in S$. We get $a = (a + a)a = aa + aa = a + a$, so that $S(+)$ is a semilattice. Besides, every congruence of $S(+)$ is a congruence of the semiring S. Thus $S(+)$ is a congruence-simple semilattice and $|S| = 2$ immediately follows. This means that (3) is true.

Next, assume that $\ker(\alpha) \neq \mathrm{id}_S$. Then $\ker(\alpha) = S \times S$, α is constant and $SS = \{w\}$. Clearly, $2w = w$ and every congruence of $S(+)$ is a congruence of the semiring S. Thus $S(+)$ is a congruence-simple (commutative) semigroup and the rest is clear. □

3.7 Corollary. *Let S be a congruence-simple semiring such that $|SS| \geq 2$ and either $|S| \geq 3$ or $ab \neq b$ ($ab \neq a$, resp.) for some $a, b \in S$. Then the left (right, resp.) semimodule ${}_SS$ (S_S, resp.) is faithful.* □

3.8 Proposition. *Let S be a congruence-simple semiring. Then every semimodule is either faithful or quasitrivial.*

Proof. The map $r \mapsto (x \mapsto rx)$ is a semiring homomorphism of the semiring S into the full endomorphism semiring $\mathrm{End}(M(+))$ of the additive semigroup $M(+)$. This homomorphism is injective if and only if M is faithful and it is constant if and only if M is quasitrivial. □

4. Critical semimodules (A)

A semimodule M will be called *1-critical* if it is faithful but none of proper subsemimodules and proper factorsemimodules of M is faithful.

4.1 Proposition. *Let M be a finite faithful semimodule whose order $|M|$ is minimal. Then M is 1-critical.*

Proof. It is obvious. □

4.2 Corollary. *If there is at least one finite faithful semimodule then there is at least one finite 1-critical semimodule.* □

4.3 Proposition. *Let the semiring S be congruence-simple, finite and not left quasitrivial. Then there is at least one finite 1-critical semimodule.*

Proof. It follows from 3.2 that ${}_SS$ is faithful and we can use 4.2. □

A semimodule M will be called *2-critical* if M is not quasitrivial, but all proper subsemimodules and all proper factorsemimodules of M are quasitrivial.

4.4 Proposition. *Let M be a finite non-quasitrivial semimodule whose order $|M|$ is minimal. Then M is 2-critical.*

Proof. It is obvious. □

4.5 Corollary. *If there is at least one finite non-quasitrivial semimodule then there is at least one finite 2-critical semimodule.* □

4.6 Proposition. *Let S be a finite semiring. Then:*
(i) *If for all $r, s \in S$, $r \neq s$, there is at least one $t \in S$ with $rt \neq st$ then there is at least one finite 1-critical semimodule.*
(ii) *If $rt \neq st$ for some $r, s, t \in S$ then there is at least one finite 2-critical semimodule.*

Proof. (i) The left semimodule ${}_SS$ is faithful and we use 4.2.
(ii) The left semimodule ${}_SS$ is not quasitrivial and we use 4.5. □

4.7 Lemma. *Let M be a semimodule and let $N = \{ \sum_{i=1}^{n} r_i x_i \,|\, n \geq 1, r_i \in S, x_i \in M \}$. Then:*
(i) *N is a subsemimodule of M.*
(ii) *If M is minimal then either $N = M$ or $|N| = 1$.*
(iii) *If N is faithful then the left semimodule ${}_SS$ is faithful.*
(iv) *If N is not quasitrivial then the left semimodule ${}_SS$ is not quasitrivial.*
(v) *N is quasitrivial if and only if M is almost quasitrivial.*

Proof. It is easy. □

4.8 Construction. Let S be a semiring and $\alpha \notin S$. Put $T = S \cup \{\alpha\}$ and $\alpha = 0_T$, where α is additively neutral and multiplicatively absorbing in T. Then T becomes a semiring. T is additively idempotent if and only if S is so. Similarly, T is commutative if and only if S is commutative, T is finite if and only if S is finite, etc.

4.9 Construction. Let S be a semiring. Put $R = S \times \{0, 1\}$ and define an addition and multiplication on R by the following rules: $(a, 0) + (b, i) = (b, i) + (a, 0) = (a + b, i)$,

$(a,1)+(b,i) = (b,i)+(a,1) = (a+b,1)$, $(a,0)(b,0) = (ab,0)$, $(a,0)(b,1) = (ab+a,0)$, $(a,1)(b,0) = (ab+b,0)$ and $(a,1)(b,1) = (ab+a+b,1)$.

Clearly, the addition is both associative and commutative. As concerns the multiplication, we have $(a,0)((b,0)(c,0)) = (abc,0) = ((a,0)(b,0))(c,0)$, $(a,0)((b,0)(c,1)) = = (a,0)(bc+b,0) = (abc+ab,0) = (ab,0)(c,1) = ((a,0)(b,0))(c,1)$, $(a,0)((b,1)(c,0)) = (a,0)(bc+c,0) = (abc+ac,0) = (ab+a,0)(c,0) = ((a,0)(b,1))(c,0)$, $(a,1)((b,0)(c,0)) = (a,1)(bc,0) = (abc+bc,0) = (ab+b,0)(c,0) = ((a,1)(b,0))(c,0)$, $(a,0)((b,1)(c,1)) = (a,0)(bc+b+c,1) = (abc+ab+ac+a,0) = (ab+a,0)(c,1) = = ((a,0)(b,1))(c,1)$, $(a,1)((b,0)(c,1)) = (a,1)(bc+b,0) = (abc+ab+bc+b,0) = = (ab+b,0)(c,1) = ((a,1)(b,0))(c,1)$, $(a,1)((b,1)(c,0)) = (a,1)(bc+c,0) = (abc+ + ac+bc+c,0) = (ab+a+b,1)(c,0) = ((a,1)(b,1))(c,0)$ and $(a,1)((b,1)(c,1)) = = (a,1)(bc+b+c,1) = (abc+ab+ac+bc+a+b+c,1) = (ab+a+b,1)(c,1) = = ((a,1)(b,1))(c,1)$. We have checked that the multiplication is associative. Furthermore, $(a,0)((b,0)+(c,0)) = (a,0)(b+c,0) = (ab+ac,0) = (ab,0)+(ac,0) = = (a,0)(b,0)+(a,0)(c,0)$, $(a,0)((b,0)+(c,1)) = (a,0)(b+c,1) = (ab+ac+a,0) = = (ab,0)+(ac+a,0) = (a,0)(b,0)+(a,0)(c,1)$, $(a,1)((b,0)+(c,0)) = (a,1)(b+ + c,0) = (ab+ac+b+c,0) = (ab+b,0)+(ac+c,0)+(a,1)(b,0)+(a,1)(c,0)$ and $(a,1)((b,1)+(c,0)) = (a,0)(b+c,1) = (ab+ac+a+b+c,1) = (ab+a+ + b,1)+(ac+c,0) = (a,1)(b,1)+(a,1)(c,0)$. On the other hand, $(a,0)((b,1)+(c,1)) = = (a,0)(b+c,1)+(ab+ac+a,0)$ and $(a,0)(b,1)+(a,0)(c,1) = (ab+a,0)+(ac+a,0) = = (ab+ac+2a,0)$, $(a,1)((b,1)+(c,1)) = (a,1)(b+c,1) = (ab+ac+a+b+c,1)$ and $(a,1)(b,1)+(a,1)(c,1) = (ab+a+b,1)+(ac+a+c,1) = (ab+ac+2a+b+c,1)$. Consequently, the algebraic structure $R = R(+,\cdot)$ is a semiring if and only if $ab+ac+2a = = ab+ac+a$ and $ab+ac+a+b+c = ab+ac+2a+b+c$ for all $a,b,c \in S$. Of course, these equations are satisfied if the semiring S is additively idempotent.

If $0_S \in S$ then $(0_S,0) = 0_R$ is additively neutral in R. If $o_S \in S$ then $(o_S,1) = o_R$ is additively absorbing in R. If $0_S \in S$ and 0_S is multiplicatively absorbing in S then $(0_S,1) = 1_R$ is multiplicatively neutral in R. If $w \in S$ is multiplicatively absorbing in S then $(w,0)$ is multiplicatively absorbing in R. If S is additively idempotent then R is so.

4.10 Proposition. *Let S be an additively idempotent semiring. Then S is a subsemiring of a semiring R such that:*

(1) *R is additively idempotent;*
(2) *$0_R \in R$, 0_R is multiplicatively absorbing;*
(3) *$1_R \in R$;*
(4) *If $o_S \in S$ then $o_R \in R$;*
(5) *If S is finite then $|R| \le 2|S|+2$.*

Proof. Combine 4.8 and 4.9. □

4.11 Proposition. *Let a semiring S be a subsemiring of a semiring R such that $1_R \in R$. Put $Q = S \cup (S+1_R) \cup \{1_R\}$. Then Q is a subsemirign of R, $1_R = 1_Q \in Q$ and S is an ideal of the semiring Q.*

Proof. It is easy. □

4.12 Proposition. *Let S be a finite additively idempotent semiring. Then there is a finite 1-critical semimodule M such that $|M| \leq 2|S| + 1$.*

Proof. By 4.10 and 4.11, S is a subsemiring of a finite additively idempotent semiring Q such that $|Q| \leq 2|S| + 1$ and $1_Q \in Q$. Of course, ${}_SQ$ is a faithful left S-semimodule. □

4.13 Lemma. $|S| = 1$ *if and only if there is a faithful quasitrivial semimodule.*

Proof. It is obvious. □

4.14 Lemma. *Asuume that $|S| \geq 2$. Then every faithful 2-critical semimodule is 1-critical.*

Proof. Use 4.13. □

5. Critical semimodules (B)

Throughout this section, let S be a congruence-simple semiring.

5.1 Lemma. *Let M be a semimodule. Then just one of the following two cases holds:*
(1) *M is faithful;*
(2) *M is quasitrivial.*

Proof. Due to 3.8, at least one of the two cases is true. On the other hand, if M were both faithful and quasitrivial then $|S| = 1$, a contradiction. □

5.2 Lemma. *Assume that S is not left quasitrivial. Let M be a semimodule. Then just one of the following two cases holds:*
(1) *M is faithful and not almost quasitrivial;*
(2) *M is quasitrivial.*

Proof. Combine 5.1 nad 3.4. □

5.3 Proposition. *A semimodule is 1-critical if and only if it is 2-critical.*

Proof. This follows immediately from 5.1. □

A semimodule satisfying the equivalent conditions of 5.3 will be called *critical*.

5.4 Proposition. *Assume that S is not left quasitrivial. The following conditions are equivalent for a seminmodule M:*
(i) *M is critical.*
(ii) *M is not almost quasitrivial, but all proper subsemimodules and all proper factorsemimodules of M are quasitrivial.*

(iii) *M is not almost quasitrivial, but all proper subsemimodules and all proper factorsemimodules of M are almost quasitrivial.*

Proof. Combine 5.2 and 5.3. □

5.5 Proposition. *Assume that S is not left quasitrivial. Let M be a critical semimodule. Then:*
(i) *M is faithful and not almost quasitrivial.*
(ii) *M is congruence-simple.*
(iii) *$R(M) = Q(M) = P(M) \neq M$ and $M = Sv$ for every $v \in M \setminus P(M)$.*
(iv) *Every proper subsemimodule of M is id-quasitrivial and contained in $P(M)$.*
(v) *Either $P(M) = \emptyset$ and M is strictly minimal or $P(M)$ is the greatest proper subsemimodule of M.*
(vi) *M is minimal if and only if $|P(M)| = 1$.*

Proof. M is faithful and not almost quasitrivial by 5.2. By 1.20, M is decent, i.e. $M = Sv$ for every $v \in M \backslash R(M)$, and M is congruence-simple by 1.21. By 2.6, we have $Q(M) = P(M)$. Now, if N is a proper subsemimodule of M then N is quasitrivial, and hence $N \subseteq Q(M) = P(M)$ and N is id-quasitrivial. Since M is not almost quasitrivial, we have $R(M) \neq M$ and $R(M) \subseteq P(M)$. Thus $R(M) = Q(M) = P(M)$ and the rest is clear. □

5.6 Lemma. *Let M be a minimal semimodule that is congruence-simple and not quasitrivial* (see [1, 4.1]). *Then M is critical.*

Proof. It is easy. □

5.7 Lemma. *Let M be a minimal semimodule that is not quasitrivial* (see [1, 4.1]). *Then there is at least one congruence ϱ on M such that the factorsemimodule M/ϱ is minimal, congruence-simple and critical.*

Proof. Combine [1, 6.3] and 5.6. □

5.8 Lemma. *Let M be an almost minimal semimodule that is congruence-simple and $|M| \geq 3$. Then M is critical.*

Proof. Use [3, 1.1]. □

5.9 Lemma. *Let M be an almost minimal semimodule such that $|M| \geq 3$. Then there is at least one congruence η of M such that the factorsemimodule M/η is almost minimal, congruence-simple and critical.*

Proof. Combine [3, 1.4] and 5.8. □

6. A few observations

Let S be a semiring and M be a (left S-)semimodule. For all $u, v \in M$ define a relation $\alpha_{u,v}$ on M by $(a, b) \in \alpha_{u,v}$ if and only if $\{u, v\} \nsubseteq \{ra, rb\}$ for every $r \in S$.

6.1 Lemma. (i) *$\alpha_{u,v}$ is symmetric.*
(ii) *If $u \neq v$ then $\alpha_{u,v}$ is reflexive.*
(iii) *If $u = v$ then $\alpha_{u,u}$ is reflexive if and only if $u \notin \bigcup S a$, $a \in M$.*

Proof. (i) This follows immediately from the definition of the relation $\alpha_{u,v}$.
(ii) We have $|\{ra\}| = 1$ and $|\{u, v\}| = 2$.
(ii) This is obvious. □

6.2 Lemma. *If $(a, b) \in \alpha_{u,v}$ then $(ra, rb) \in \alpha_{u,v}$ for every $r \in S$.*

Proof. It is easy. □

6.3 Lemma. *Assume that $u \neq v$ and that the following two conditions are satisfied:*
(a) *$u \notin M + N$, where $N = M \setminus \{u\}$;*
(b) *$v \notin K + u$, where $K = M \setminus \{v\}$.*
Then $(a + c, b + c) \in \alpha_{u,v}$ for all $(a, b) \in \alpha_{u,v}$ and $c \in M$.

Proof. Let, on the contrary, $u = r(a + c)$ and $v = r(b + c)$ for some $r \in S$. Then $ra + rc = u$ and, using (a), we get $ra = u = rc$. Further, $rb + u = rb + rc = v$, and hence $rb = v$ by (b). Thus $ra = u$ and $rb = v$, $(u, v) \in \alpha_{u,v}$, a contradiction. □

Let $\beta_{u,v}$ denote the transitive closure of $\alpha_{u,v}$. Clearly, $\beta_{u,v}$ is symmetric.

6.4 Lemma. *If $u \neq v$ then $\beta_{u,v}$ is an equivalence.*

Proof. Use 6.1(i),(ii). □

6.5 Lemma. *If $(a, b) \in \beta_{u,v}$ then $(ra, rb) \in \beta_{u,v}$ for every $r \in S$.*

Proof. Use 6.2. □

6.6 Lemma. *Assume that $u \neq v$ and the the conditions 6.3(a),(b) are satisfied. Then $\beta_{u,v}$ is a congruence of the semimodule M. In particular, if M is congruence-simple then either $\alpha_{u,v} = \beta_{u,v} = \mathrm{id}_M$ or $\beta_{u,v} = M \times M$.*

Proof. Use 6.4, 6.3 and 6.5. □

6.7 Lemma. *Assume $u \neq v$. The following conditions are equivalent:*
(i) $\alpha_{u,v} = \mathrm{id}_M$.
(ii) $\beta_{u,v} = \mathrm{id}_M$.

(iii) *For all $a, b \in M$, $a \neq b$, there is at least one $r \in S$ such that either $ra = u$, $rb = v$ or $ra = v$, $rb = u$.*

Proof. It is easy. □

6.8 Lemma. *Asuume that M is idempotent and that $u \neq u + v = v$. If $a, b \in M$ are such that $a + b = b$ and $(a, b) \notin \alpha_{u,v}$ then $ra = u$ and $rb = v$ for at least one $r \in S$.*

Proof. We have $\{u, v\} = \{ra, rb\}$ for some $r \in S$. If $ra = v$ then $rb = u$ and $ra = v = u + v = rb + ra = r(b + a) = rb = u$. Thus $u = v$, a contradiction. □

6.9 Lemma. *Assume that $u = 0_M$ and $0_M \notin N + N$, where $N = M \setminus \{0_M\}$. Then the conditions 6.3(a),(b) are satisfied.*

Proof. First, if $u = 0_M = a + b$ for some $a, b \in M$ then either $a = 0_M$ or $b = 0_M$. But then $b = 0_M$ or $a = 0_M$. In both cases, we get $a = 0_M = b$ and 6.3(a) is true. The condition 6.3(b) is clear. □

6.10 Lemma. *Assume that M is idempotent and $u = 0_M$. Then the conditions 6.3(a),(b) are satisfied for every $v \in M$.*

Proof. It is easy to see that $0_M \notin N + N$, where $N = M \setminus \{0_M\}$ and 6.9 applies. □

6.11 Remark. Assume that M is idempotent and put $x \leq y$ iff $y = x + y$; then $\leq$ is a compatible relation of order on M. Now, it is clear that the condition 6.3(a) is satisfied if and only if the element u is minimal in the ordered set $M(\leq)$.

If $u \nleq v$ then 6.3(b) is true. If $u < v$ and v is irreducible then 6.3(b) is true as well. Thus 6.3(b) is satisfied if and only if either $u \nleq v$ or $u \leq v$ and $v \neq y + u$ for every $z \in M$ such that $z < v$.

6.12 Lemma. *Assume that M is idempotent and u is minimal in $M(\leq)$. If either $u \nleq v$, or $u < v$ and v is irreducible, then the conditions 6.3(a),(b) are satisfied.*

Proof. See 6.11. □

6.13 Lemma. *Assume that $Su = \{u\}$. If $(u, b) \in \alpha_{u,v}$ then $v \notin Sb$.*

Proof. If $v = rb$ for some $r \in S$ then $\{u, v\} = \{ru, rb\}$ and $(u, b) \notin \alpha_{u,v}$, a contradiction. □

6.14 Lemma. *Assume that $Sv = \{v\}$. If $(a, v) \in \alpha_{u,v}$ then $u \notin Sa$.*

Proof. Similar to that of 6.13. □

6.15 Lemma. *Assume that $Su = \{u\}$ and $v \in Sz$ for every $z \in M \setminus \{u\}$. If $(u, a) \in \beta_{u,v}$ then $a = u$.*

Proof. Assume $a \neq u$. Since $(u, a) \in \beta_{u,v}$, there is $b \in M$ with $u \neq b$ and $(u, b \in \in \alpha_{u,v}$. By 6.13, we have $v \notin Sb$, a contradiction. □

6.16 Lemma. *Assume that $Sv = \{v\}$ and $u \in Sz$ for every $z \in M \setminus \{v\}$. If $(a, v) \in \beta_{u,v}$ then $a = v$.*

Proof. Similar to that of 6.15. □

6.17 Proposition. *Assume that M is idempotent and congruence-simple. Assume further that $u \neq v$, u is minimal in $M(\leq)$, either $u \nleq v$ or $u < v$ and $v \neq x + u$ for every $x < v$ and that at least one of the following two conditions is satisfied:*

(1) *$Su = \{u\}$ and $v \in Sz$ for every $z \in M \setminus \{u\}$;*

(2) *$Sv = \{v\}$ and $u \in Sz$ for every $z \in M \setminus \{v\}$.*

Then:

(i) *For all $a, b \in M$ such that $a \neq b$ there is at least one $r \in S$ such that either $ra = u$, $rb = v$ or $ra = v$, $rb = u$.*

(ii) *If $u < v$ then for all $a, b \in M$ such that $a < b$ there is at least one $r \in S$ such that $ra = u$ and $rb = v$.*

Proof. By 6.11 (see also 6.12), the conditions 6.3(a),(b) are satisfied. Now, by 6.6, the relation $\beta_{u,v}$ is a congruence of the semimodule M. Using 6.15 or 6.16, we see that $(u, v) \notin \beta_{u,v}$, and so $\beta_{u,v} \neq M \times M$. Since M is congruence-simple, we get $\beta_{u,v} = \mathrm{id}_M$. Thus $\alpha_{u,v} = \mathrm{id}_M$ as well and (i) follows from 6.7. As for (ii), if $ru = b$ and $rv = a$ then $ru = b = a + b = rv + ru = r(v + u) = rv = a$, so that $a = b$, a contradiction. □

6.18 Proposition. *Assume that M is idempotent, congruence-simple and that $0_M \in \in M$. Assume further that $v \neq 0_M$ and at least one of the following two conditions is satisfied:*

(1) *$S0_M = \{0_M\}$ and $v \in Sz$ for every $z \neq 0_M$;*

(2) *$Sv = \{v\}$ and $0_M \in Sz$ for every $z \neq v$.*

Then:

(i) *For all $a, b \in M$, $a \neq b$, there is at least one $r \in S$ such that either $ra = 0_M$, $rb = v$ or $ra = v$, $rb = 0_M$.*

(ii) *If $a < b$ then there is at least one $r \in S$ such that $ra = 0_M$ and $rb = v$.*

Proof. Use 6.17, where $u = 0_M$. □

6.19 Proposition. *Assume that M is idempotent, congruence-simple and that $o_M \in \in M$. Assume further that u is minimal in $M(\leq)$, $o_M \neq x + u$ for every $x \neq o_M$ and that at least one of the following two conditions is satisfied:*

(1) *$Su = \{u\}$ and $o_M \in Sz$ for every $z \neq u$;*

(2) $S0_M = \{o_M\}$ *and* $u \in Sz$ *for every* $z \neq o_M$.
Then:
(i) *For all* $a, b \in M$ *such that* $a \neq b$*, there is at least one* $r \in S$ *such that either* $ra = u$*,* $rb = o_M$ *or* $ra = o_M$*,* $rb = u$*.*
(ii) *If* $a < b$ *then there is at least one* $r \in S$ *such that* $ra = u$ *and* $rb = o_M$*.*

Proof. Use 6.17, where $v = o_M$. □

7. Observations continued

Let S be a semiring and M be an idempotent and congruence-simple (left S-)semimodule.

7.1 Let $u \in M$ be minimal in $M(\leq)$ and let $v \in M$ be such that $u < v$ and $v \neq w + u$ for every $w \in M$, $w < v$. (Notice that these conditions are satisfied for $u = 0_M$.)

7.1.1 Proposition. *Assume that* $Su = \{u\}$ *(i.e.,* $u \in P(M)$*) and* $v \in Sz$ *for every* $z \in M \setminus \{u\}$*). Then, for all* $a, b \in M$ *such that* $b \nleq a$*, there is at least one* $r \in S$ *such that* $ra = u$ *and* $rb = v$*.*

Proof. Since $b \nleq a$, we have $a < b$ and, by 6.17(ii), there is $r \in S$ with $ra = u$ and $r(a + b) = v$. Now, $v = r(a + b) = ra + rb = u + rb$ and $rb \leq v$. According to our assumptions, we get $rb = v$. □

7.1.2 Proposition. *Assume that* $Su = \{u\}$ *and* $v \in Sz$ *for every* $z \neq u$*. Let* $a \in M$ *be such that* $a \neq o_M$ *and the set* $P_a = \{\, b \,|\, b \nleq a \,\}$ *is finite. Then there is at least one* $r \in S$ *such that* $rb \geq v$ *for every* $b \in P_a$ *and* $rc = u$ *for every* $c \in M \setminus P_a$*.*

Proof. Since $a \neq 0_M$, the set P_a is non-empty. Of course, $a \in M \setminus P_a$ and this set is non-empty as well. By 7.1.1, for every $b \in P_a$ there is $r_b \in S$ with $r_b a = u$ and $r_b b = v$. Put $r = \sum r_b$, $b \in P_a$. Then $ra = \sum r_b a = \sum u$ and, since u is minimal in $M(\leq)$, it follows that $rc = u$. Finally, $rb = r_b b + \sum \cdots \geq r_b b = v$. □

7.1.3 Corollary. *Assume that* $Su = \{u\}$*,* $o_M \in M$ *and* $o_m \in Sz$ *for every* $z \neq u$*. Let* $a \in M$ *be such that* $a \neq o_M$ *and the set* P_a *is finite. Then there is at least one* $r \in S$ *such that* $rb = o_M$ *for every* $b \in P_a$ *and* $rc = u$ *for every* $c \in M \setminus P_a$*.* □

7.1.4 Proposition. *Assume that* $Su = \{u\}$*,* $o_M \in M$*,* $o_M \in Sz$ *for every* $z \neq u$*. Let* $a \in M$ *be such that* $a \neq o_M$ *and ther set* P_a *is finite. Then for every* $s \in S$ *there is at least one* $r \in S$ *such that* $rb = so_M$ *for every* $b \in P_a$ *and* $rc = u$ *for every* $c \in M \setminus P_a$*.*

Proof. This follows easily from 7.1.3. □

7.2 Corollary. *Assume that* $0_M \in M$, $o_M \in M$, $S0_M = \{0_M\}$ *and* $o_M \in Sx$ *for every* $x \neq o_M$. *Let* $a \in M$ *be such that* $a \neq o_M$ *and the set* $P_a = \{b \mid a + b \neq a\}$ *is finite. Then there is* $r \in S$ *such that* $rb = o_M$ *for every* $b \in P_a$ *and* $rc = 0_M$ *for every* $c \in M \setminus P_a$. □

7.3 Corollary. *Assume that* $0_M \in M$, $o_M \in M$, $S0_M = \{0_M\}$, $So_M = S$ *and* $o_M \in Sx$ *for every* $x \neq 0_M$. *Let* $a \in M$ *be such that* $a \neq o_M$ *and the set* $P_a = \{b \mid a + b \neq a\}$ *is finite. Then for every* $w \in M$ *there is* $r \in S$ *such that* $rb = w$ *for every* $b \in P_a$ *and* $rc = 0_M$ *for every* $c \in M \setminus P_a$. □

7.4 Let $u \in M$ be minimal in $M(\leq)$ and let $v \in M$ be such that $u < v$ and $v \neq w + u$ for every $w \in M$, $w < v$. (Notice that these conditions are satisfied for $u = 0_M$.)

7.4.1 Proposition. *Assume that* $Sv = \{v\}$ *(i.e.,* $v \in P(M)$ *and* $u \in Sz$ *for every* $z \in M \setminus \{v\}$*). Then, for all* $a, b, \in M$*such that* $b \nleq a$*, there is at least one* $r \in S$ *such that* $ra = u$ *and* $rb = v$.

Proof. It is the same as that of 7.1.1. □

7.4.2 Proposition. *Assume that* $Sv = \{v\}$ *and* $u \in Sz$ *for every* $z \neq v$. *Let* $a \in M$ *be such that* $a \neq o_M$ *and the set* P_a *is finite. Then there is at least one* $r \in S$ *such that* $rb \geq v$ *for every* $b \in P_a$ *and* $rc = u$ *for every* $c \in M \setminus P_a$.

Proof. Using 7.4.1, we can proceed in the same way as in the proof of 7.1.2. □

7.4.3 Corollary. *Assume that* $o_M \in M$, $So_M = \{o_M\}$ *and* $u \in Sz$ *for every* $z \neq o_M$. *Let* $a \in M$ *be such that* $a \neq o_M$ *and the set* P_a *is finite. Then there is at least one* $r \in S$ *such that* $rb = o_M$ *for every* $b \in P_a$ *and* $rc = u$ *for every* $c \in M \setminus P_a$. □

7.4.4 Proposition. *Assume that* $o_M \in M$, $So_M = \{o_M\}$ *and* $u \in Sz$ *for every* $z \neq o_M$. *Let* $a \in M$ *be such that* $a \neq o_M$ *and the set* P_a *is finite. Then for every* $s \in S$ *there is at least one* $r \in S$ *such that* $rb = o_M$ *for every* $b \in P_a$ *and* $rc = su$ *for every* $c \in M \setminus P_a$.

Proof. This follows easily from 7.4.3. □

7.5 Corollary. *Assume that* $0_M \in M$, $o_M \in M$, $So_M = \{o_M\}$ *and* $0_M \in Sx$ *for every* $x \neq o_M$. *Let* $a \in M$ *be such that* $a \neq o_M$ *and the set* $P_a = \{b \mid a + b \neq a\}$ *is finite. Then there is* $r \in S$ *such that* $rb = o_M$ *for every* $b \in P_a$ *and* $rc = 0_M$ *for every* $c \in M \setminus P_a$. □

7.6 Corollary. *Assume that* $0_M \in M$, $o_M \in M$, $So_M = \{o_M\}$, $S0_M = M$ *and* $0_M \in Sx$ *for every* $x \neq o_M$. *Let* $a \in M$ *be such that* $a \neq o_M$ *and the set* $P_a = \{b \mid a + b \neq a\}$ *is finite. Then for every* $w \in M$ *there is* $r \in S$ *such that* $rb = w$ *for every* $b \in P_a$ *and* $rc = 0_M$ *for every* $c \in M \setminus P_a$. □

References

[1] K. Al-Zoubi, T. Kepka and P. Němec: *Quasitrivial semimodules I,* Acta Univ. Carolinae Math. Phys. **49/1** (2008), 3–16.

[2] K. Al-Zoubi, T. Kepka and P. Němec: *Quasitrivial semimodules II,* Acta Univ. Carolinae Math. Phys. **49/1** (2008), 17–24.

[3] K. Al-Zoubi, T. Kepka and P. Němec: *Quasitrivial semimodules III,* Acta Univ. Carolinae Math. Phys. **50/1** (2009), 3–13.