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QUASITRIVIAL SEMIMODULES VII

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The paper continues the investigation of quasitrivial semimodules and related problems. In particular, strong endomorphisms of semilattices are studied.

This part is a continuation of [1], [2], [3], [4], [5] and [6] with main emphasis on strong endomorphisms of semilattices. The notation introduced in the preceding parts is used. All the results collected here are fairly basic and we will not attribute them to any particular source.

1. Congruences

Let M be a non-trivial semilattice. Let $a, b, c \in M$ and denote by $\pi_{a,b}$ the congruence of M generated by the pair (a, b).

1.1 Proposition. (i)$\pi_{a,a} = \mathrm{id}_M$.
(ii)) $\pi_{a,b} = \pi_{b,a}$.
(iii) $\pi_{a+c,b+c} \subseteq \pi_{a,b}$.
(iv) $\pi_{a,a+b} \cup \pi_{b,a+b} \subseteq \pi_{a,b}$.

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Proof. It is easy. □

1.2 Proposition. *Let $u, v \in M$, $u \neq v$. Then $(u, v) \in \pi_{a,b}$ if and only if the following two conditions are satisfied:*

(1) $u + a + b = v + a + b$;

(2) $u \in \{u + a, u + b\}$ *and* $v \in \{v + a, v + b\}$.

Proof. Define a relation π on M by $(u, v) \in \pi$ iff $u+a+b = v+a+b$ and either $u = v$ or $u \in \{u + a, u + b\}$ and $v \in \{v + a, v + b\}$. One checks easily that π is a congruence of M and $(a, b) \in \pi$. Consequently, $\pi_{a,b} \subseteq \pi$.

Conversely, let $(u, v) \in \pi$, $u \neq v$. Then $u + a + b = v + a + b$ and we have to distinguish the following four cases:

Let $u + a = u$ and $v + a = v$. Then $u + b = u + a + b = v + a + b = v + b$, $(u+b, v+b) \in \pi_{a,b}$, $(u, u+b) = (u+a, u+b) \in \pi_{a,b}$ and $(v, v+b) = (v+a, v+b) \in \pi_{a,b}$. Consequently, $(u, v) \in \pi_{a,b}$.

Let $u + b = u$ and $v + b = v$. This case is symmetric to the preceding one.

Let $u + a = u$ and $b + v = v$. Then $u + b = u + a + b = v + a + b = v + a$, $(u+b, v+a) \in \pi_{a,b}$, $(u, u+b) = (u+a, u+b) \in \pi_{a,b}$ and$(v+a, v) = (v+a, v+b) \in \pi_{a,b}$. Consequently, $(u, v) \in \pi_{a,b}$.

Let $u + b = u$ and $v + a = v$. This case is symmetric to the preceding one.

We have proved that $(u, v) \in \pi_{a,b}$, and so $\pi \subseteq \pi_{a,b}$. □

1.3 Corollary. *Assume that $a \leq b$. Then $(u, v) \in \pi_{a,b}$ if and only if $u + b = v + b$ and either $u = v$ or $u + a = u$, $v + a = v$.* □

1.4 Lemma. *Assume that $a \leq b$. Then the interval $\{\, c \mid a \leq c \leq b \,\}$ is a block of the congruence $\pi_{a,b}$.*

Proof. Easy to see. □

1.5 Proposition. *Let $a, b, c, d \in M$. Then $\pi_{a,b} = \pi_{c,d}$ if and only if $\{a, b\} = \{c, d\}$.*

Proof. If $\{a, b\} = \{c, d\} = A$ and if $|A| = 1$ then $a = b$, $c = d$ and $\pi_{a,b} = \mathrm{id}_M = \pi_{c,d}$. If $|A| = 2$ then $a \neq b$, $c \neq d$ and either $(a, b) = (c, d)$ or $(a, b) = (d, c)$. In both cases, the equality $\pi_{a,b} = \pi_{c,d}$ is clear.

Conversely, assume that $\pi_{a,b} = \pi_{c,d}$. If $a = b$ then $\pi_{a,b} = \mathrm{id}_M$, and hence $\pi_{c,d} = \mathrm{id}_M$ and $c = d$. In the remaining part of the proof, we will asume that $a \neq b$ and $c \neq d$.

By 1.2, we get the following two equalities:

(α) $a + b + c = a + b + d$,

(β) $a + c + d = b + c + d$.

Furthermore, at least one of the following four cases takes place:

(1) $a + c = c$, $a + d = d$;

(2) $b + c = c$, $b + d = d$;

(3) $a + c = c$, $b + d = d$;

(4) $b + c = c$, $a + d = d$.

Symmetrically, at least one of the following four cases takes place:

(a) $a + c = a$, $b + c = b$;

(b) $a + d = a$, $b + d = b$;

(c) $a + c = a$, $b + d = b$;

(d) $a + d = a$, $b + c = b$.

The rest of the proof is divided into ten parts.

(i) Let (1a) be true. Then $a = c$, $a \le d$ and $a \le b$. Using (α), we get $b = a + b + c = = a + b + d = b + d$ and $d \le b$. Using (β), we get $d = a + c + d = b + c + d = b + d$ and $b \le d$. Thus $a = c$ and $b = d$.

(ii) Let (1b) be true. Then $a = d$, $a \le c$ and $a \le b$. Using (α), we get $b+c = a+b+c = = a + b + d = b$. Using ($\beta$), we get $c = a + c + d = b + c + d = b + c$. Thus $a = d$ and $b = c$.

(iii) Let (1c) be true. Then $a = c$, $a \le d \le b$. Using (β), we get $d = a + c + d = = b + c + d = b + d$, $b \le d$. Thus $a = c$ and $b = d$.

(iv) Let (1d) be true. Then $a = d$, $a \le c \le b$. Using (β), we get $c = a + c + d = = b + c + d = b + c = b$. Thus $a = d$ and $b = c$.

(v) The cases (2a),...,(2d) are dual to the preceding four cases.

(vi) Let (3a) be true. Then $a = c \le b \le d$. Using (α), we get $b = a + b + c = = a + b + d = d$. Thus $a = c$ and $b = d$.

(vii) Let (3b) be true. Then $b = d \le a \le c$. Using (α), we get $c = a + b + c = = a + b + d = a$. Thus $a = c$ and $b = d$.

(viii) Let (3c) be true. Then $a = c$ and $b = d$.

(ix) Let (3d) be true. Then $b \le d \le a \le c \le b$, and hence $a = b = c = d$ (a contradiction, in fact).

(x) The cases (4a),...,(4d) are dual to the preceding four cases. □

1.6 Remark. If M is finite, $|M| = n \ge 2$, then the number of non-identical principal congruences of M is just $\frac{n(n-1)}{2}$.

1.7 Remark. (i) If $0_M, o_M \in M$ then $\pi_{0,o} = M \times M$ follows from 1.2. According to 1.5, we have $\pi_{a,b} = M \times M$ iff $\{a, b\} = \{0, o\}$.

(ii) Assume that $a, b \in M$ are such that $a \ne b$, a, b are minimal in M and, for every $x \in M$, either $a \le x$ or $b \le x$. If, moreover, $o_M \in M$ and $a+b = o_M$ then $\pi_{a,b} = M\times M$ follows from 1.2.

(iii) Let $a, b \in M$ be such that $\pi_{a,b} = M \times M$. Since $\pi_{a,b} \ne \mathrm{id}_M$, we have $a \ne b$.

First, assume that $a < b$. Using 1.2(2), we get $a = 0_M$. Using 1.3(1), we get $b = o_M$. Similarly, if $b < a$ then $b = 0_M$ and $a = o_M$.

Now, assume that the elements a, b are not comparable. Using 1.2(1), we get $a + b = o_M$. If $u \in M$, $u \ne o_M$, the either $a \le u$, $b \nleq u$ or $b \le u$, $a \nleq u$. Consequently, a and b are minimal elements (see (ii)).

1.8 Lemma. *Let $a, b \in M$. Then $\pi_{a,b} = \pi_{a,a+b}$ (or $\pi_{a,b} \subseteq \pi_{a,a+b}$) iff $a \le b$ (or $a+b = b$).*

Proof. We have $\pi_{a,a+b} \subseteq \pi_{a,b}$. Thus $\pi_{a,b} = \pi_{a,a+b}$ iff $(a,b) \in \pi_{a,a+b}$. Our result now follows from 1.3 (or 1.5). □

1.9 Lemma. *Let $a, b, c \in M$, $a \leq c < b$. Then $\pi_{a.c} \subsetneqq \pi_{a,b}$.*

Proof. Use 1.3 (or 1.5). □

1.10 Proposition. *The following conditions are equivalent for a congruence ϱ of M:*

(i) *ϱ is a minimal congruence of M.*

(ii) *$\varrho = \pi_{a,b}$, where $a, b \in M$, $a < b$ and $b \leq c$ for every $c \in M$ such that $a < c$ (equivalently, $a + c = b + c$ for every $c \in M$ such that $a + c \neq a$).*

Proof. (i) implies (ii). Let $(a,b) \in \varrho$, $a \neq b$. Assume that $b \nleq a$, the other case being symmetric. Since ϱ is minimal and $\mathrm{id}_M \neq \pi_{a,b} \subseteq \varrho$, we have $\varrho = \pi_{a,b}$. Moreover, since $b \nleq a$, we get $a < a + b$, $\mathrm{id}_M \neq \pi_{a,a+b} \subseteq \pi_{a,b}$ and $\pi_{a,a+b} = \pi_{a,b}$. By 1.8, $a < b$. Let $c \in M$ be such that $a < c$. If $b \nleq c$ then $c < b + c$ and $\pi_{c,b+c} \neq \mathrm{id}_M$. Anyway, $(c, b+c) = (a+c, b+c) \in \pi_{a,b}$, and hence $\pi_{c,b+c} \subseteq \varrho = \pi_{a,b}$. Since ϱ is minimal, we have $\pi_{c,b+c} = \pi_{a,b}$ and $(a,b) \in \pi_{c,b+c}$. Using 1.2(2), we get $a \in \{a+c, a+b+c\} = \{c, b+c\}$, a contradiction. Thus $b \leq c$.
(ii) implies (i). Since $a \neq b$, we have $\varrho = \pi_{a,b} \neq \mathrm{id}_M$. Now, let $(c,d) \in \varrho$, $c \neq d$. It follows from 1.2(2) that $a \leq c$ and $a \leq d$, and hence $c + b = d + b$ by 1.2(1). If $a \neq c$ then $b \leq c$, $c = b + c = b + d$. Similarly, if $a \neq d$ then $b \leq d$, $d = b + d = b + c$. Since $c \neq d$, we have either $a = c$ or $a = d$. If $a = c$ then $b = a + b = c + b = d + b$, $a < d \leq b$ and $b \leq d$, so that $b = d$. Similarly, if $a = d$ then $b = c$. We have proved that $\{a,b\} = \{c,d\}$. The rest is clear. □

1.11 Remark. Let ϱ be a minimal congruence of M. By 1.10 (and its proof), there are $a, b \in M$ such that $\varrho = \pi_{a,b}$, $a < b$ and $a + c = b + c$ for every $c \in M$ such that $a + c \neq a$. Then b covers a and $\varrho = \mathrm{id}_M \cup \{(a,b), (b,a)\}$. (In fact, if 1.10(ii) is true then $\mathrm{id}_M \cup \{(a,b), (b,a)\}$ is a congruence and, of course, it is minimal.)

1.12 Remark. (i) If $o_M \in M$ and $a \in M \setminus \{o_M\}$ is maximal in $M \setminus \{o_M\}$ then π_a, o is a minimal congruence of M.
(ii) If $0_M \in M$ and the set $M \setminus \{0_M\}$ has the smallest element, say a, then $\pi_{0,a}$ is a minimal congruence of M.

1.13 Remark. Let $a, b \in M$, $a \leq b$. Then $\pi_{a,b} = \ker\lambda_b \cap \xi_a$ (see [6, 2.8, 2.13].

2. The semiring of strong endomorphisms (a)

An endomorphism $f \in \underline{E} = \mathrm{End}(M(+))$ is called *strong* if $f(\varrho) \subseteq \varrho$ for every congruence ϱ of M. The set of all strong endomorphisms of M will be denoted by $\underline{E}^{(\sigma)}$.

2.1 Proposition. (i) *The set* $\underline{E}^{(\sigma)}$ *is a subsemiring of* $\underline{E}$.
(ii) $\underline{E}^{(\beta)} \subseteq \underline{E}^{(\sigma)}$.

Proof. It is easy. □

2.2 Proposition. *The semiring* $\underline{E}^{(\sigma)}$ *is not ideal-simple.*

Proof. $\underline{E}^{(1)}$ is a proper non-trivial ideal of the semiring $\underline{E}^{(\sigma)}$. □

2.3 Proposition. *The following conditions are equivalent for* $f \in \underline{E}$*:*

(i) $f \in \underline{E}^{(\sigma)}$.
(ii) $f(\pi_{a,b}) \subseteq \pi_{a,b}$ *for all* $a, b \in M$, $a < b$.
(iii) *If* $a, b, c, d \in M$ *are such that* $a < b$, $a \le c$, $a \le d$, $b+c = b+d$ *and* $f(c) \ne f(d)$ *then* $a \le f(c)$, $a \le f(d)$ *and* $b + f(c) = b + f(d)$.

Proof. Clearly, (i) implies (ii) and (ii) is equivalent to (iii) (see 1.3). It remains to show that (ii) implies (i). For, let ϱ be a congruence of M and let $(a, b) \in \varrho$, $a \ne b$. If $a < b$ then $(f(a), f(b) \in f(\pi_{a,b}) \subseteq \pi_{a,b} \subseteq \varrho$. The case $b < a$ is similar. Finally, if $a \nleq b$ and $b \nleq a$ then $a < c$, $b < c$, where $c = a + b$, $(f(a), f(c)) \in \varrho$ and $(f(b), f(c)) \in \varrho$. Thus $(f(a), f(b)) \in \varrho$. □

2.4 Proposition. *The following conditions are equivalent for* $f \in \underline{E}$*:*

(i) $f \in \underline{E}^{(\sigma)}$ *and* $a \le f(a)$ *for every* $a \in M$.
(ii) $f(\ker\lambda_a) \subseteq \ker\lambda_a$ *and* $a \le f(a)$ *for every* $a \in M$.
(iii) $f(a + b) = a + f(b)$ $(= b + f(a))$ *for all* $a, b \in M$.

Proof. (i) implies (ii). Thus implication is trivial.
(ii) implies (iii). We have $(b, a + b) \in \lambda_a$, and hence $(f(b), f(a + b)) \in \lambda_a$ and $a + f(b) = a + f(a + b) = a + f(a) + f(b) = f(a) + f(b) = f(a + b)$.
(iii) implies (i). First, $f(a) = f(a + a) = a + f(a)$ yields $a \le f(a)$. Further, if $(c, d) \in \pi_{a,b}$, where $c \ne d$ and $a \le b$, then $a \le c$, $a \le d$ and $c + b = d + b$ (use 1.3). From this, $a \le f(a) \le f(c)$, $a \le f(a) \le f(d)$, $f(c)+b = f(c+b) = f(d+b) = f(d)+b$ by (iii), and hence $(f(c), f(d)) \in \pi_{a,b}$. By 2.3, we get $f \in \underline{E}^{(\sigma)}$. □

2.5 Proposition. (i) *The set* $\underline{E}^{(\sigma 1)} = \{ f \in \underline{E} \,|\, a \le f(a)$ for every $a \in M \}$ *is a subsemiring of* $\underline{E}^{(\sigma)}$.
(ii) $\underline{E}^{(\sigma 1)} + \underline{E}^{(\sigma)} \subseteq \underline{E}^{(\sigma 1)}$.
(iii) $\underline{E}^{(\gamma)} \cup \{\mathrm{id}_M\} \subseteq \underline{E}^{(\sigma 1)}$ *(notice that* $\underline{E}^{(\gamma)} \cup \{\mathrm{id}_M\}$ *is a subsemiring of* $\underline{E}$*).*

Proof. It is easy. □

2.6 Proposition. *Let $f \in \underline{E}^{(\sigma)}$ be such that $P_f = \{a \in M \,|\, a \nleq f(a)\} \neq \emptyset$. Then:*
(i) *$f(P_f) = \{w_f\}$ is a one-element set and $P_f = \{a \in M \,|\, a \nleq w_f\}$ (so that P_f is a principal prime ideal of M).*
(ii) *$w_f \neq o_M$ and $w_f \in Q_f = \{a \in M \,|\, a \leq f(a)\}$.*
(iii) *$Q_f = \{a \in M \,|\, a \leq w_f\}$ is a subsemilattice of M and $w_f = o_{Q_f}$.*
(iv) *$f(M) \subseteq Q_f$.*
(v) *$f(w_f) = w_f$.*
(vi) *$f(a+b) = a + f(b)$ for all $a, b \in Q_f$ (and $f|Q_f$ is a strong endomorphism of Q_f).*
(vii) *$a + f(b) = a + w_f$ for all $a \in P_f$ and $b \in M$.*
(viii) $f(a+b) = a + f(b)$ for all $a \in Q_f$, $b \in M$.

Proof. First, take $a \in P_f$ and put $N_a = M + a$. Then N_a is an ideal of M, $\varrho_a = (N_a \times N_a) \cup \mathrm{id}_M$ is a congruence of M and $(a, a+x) \in \varrho_a$ for every $x \in M$. Since $f \in \underline{E}^{(\sigma)}$, we get $(f(a), f(a) + f(x)) \in \varrho_a$. If $f(a) \neq f(a) + f(x)$ then $f(a) \in N_a$, $f(a) = a + y$ for some $y \in M$ and $a \leq f(a)$, a contradiction with $a \in P_f$. Thus $f(a) = f(a) + f(x)$ and $f(x) \leq f(a)$. We have proved that $f(M) \leq f(a)$ for every $a \in P_f$. In particular, we have $f(P_f) = \{w_f\}$. Since $a \nleq f(a) = w_f$, we have $a \nleq w_f$ for every $a \in P_f$. Conversely, if $a \in M$ is such that $a \nleq w_f$ then $a \nleq f(a)$, since $f(a) \leq w_f$. Consequently, $P_f = \{a \in M \,|\, a \nleq w_f\}$ is a principal prime ideal, $Q_f = \{a \in M \,|\, a \leq w_f\}$ and $w_f = o_{Q_f} \in Q_f$. Since $f(M) \leq f(a) = w_f$ for all $a \in P_f$, we see that $f(M) \subseteq Q_f$. Since $P_f \neq \emptyset$, we see that $w_f \neq o_M$. Since $w_f \in Q_f$, we have $w_f \leq f(w_f) \in Q_f$, and therefore $w_f = f(w_f$. We have proved that (i),...,(v) are true.

Let $a, b \in M$. Then $(b, a+b) \in \ker\lambda_a$, and hence $(f(b), f(a) + f(b)) \in \ker\lambda_a$ and $a + f(b) = a + f(a) + f(b)$. If $a \in Q_f$ then $a + f(a) = f(a)$ and we get $a + f(b) = f(a) + f(b)$ (i.e., (vi) and (viii) are true). If $a \in P_f$ then $a + f(b) = a + f(a) + f(b) = a + f(a+b) = a + w_f$, since $a + b \in P_f$ (i.e., (vii) is true). □

2.7 Proposition. *$f^2 = f$ for every $f \in \underline{E}^{(\sigma)}$.*

Proof. If $a \in M$ is such that $a \leq f(a)$ then $f^2(a) = f(f(a)+a) = f(a)+f(a) = f(a)$ by 2.6(vi). If $a \nleq f(a)$ then $f(a) = w_f$ and $f^2(a) = f(w_f) = w_f = f(a)$ (use 2.6 again). Thus $f^2 = f$. □

2.8 Proposition. *The following conditions are equivalent for $f \in \underline{E}$:*

(i) $f \in \underline{E}^{(\sigma)}$.

(ii) $f(\ker\lambda_a) \subseteq \ker\lambda_a$ *and* $f(\xi_a) \subseteq \xi_a$ *for every $a \in M$.*

(iii) *The following are true:*

(iii1) *If $a \in M$ is such that $|f(M+a)| \geq 2$ then $f(M+a) \subseteq M + a$;*

(iii2) *If $a, b, c \in M$ are such that $a + b = a + c$ then $a + f(b) = a + f(c)$.*

Proof. (i) implies (ii). This is trivial.
(ii) implies (iii). If $|f(M+a)| \geq 2$ then, for every $b \in M$, there is $c \in M$ with $f(b+a) \neq f(c+a)$. Of course, $(b+a, c+a) \in \xi_a$, and so $(f(b+a)f(c+a)) \in \xi_a$.

But $f(b+a) \neq f(c+a)$ implies $f(b+a) \in M+a$. This is (iii1), and (iii2) follows immediately from the inclusion $f(\ker\lambda_a \subseteq \lambda_a$.
(iii) impliews (ii). We can proceed conversely.
(ii) implies (i). Using 1.13, we get $f(\pi_{a,b}) \subseteq \pi_{a,b}$ for all $a, b \in M$, $a \leq b$. Now $f \in \underline{E}^{(\sigma)}$ by 2.3. □

2.9 Proposition. *The following conditions are equivalent for $f \in \underline{E}$:*

(i) $f \in \underline{E}^{(\sigma)}$.

(ii) *The following are true:*

(ii1) $f(a+b) = a + f(b)$ *for all* $a, b \in M$, $a \leq f(a)$*;*

(ii2) $a + f(b) = a + f(a)$ *for all* $a, b \in M$, $a \nleq f(a)$*;*

(ii3) $f(a) = f(a+b)$ *for all* $a, b \in M$, $a \nleq f(a)$.

Proof. (i) implies (ii). See 2.6(viii),(vii),(i).
(ii) implies (i). We are going to check the conditions 2.8(iii1),(iii2). Let $a \in M$. If $a \leq f(a)$ then $f(M+a) \subseteq M+a$ by (ii1). If $a \nleq f(a)$ then $f(M+a) = \{f(a)\}$ by (ii3). Now, 2.8(iii1) is clear.

Let $a, b, c \in M$ be such that $a+b = a+c$. Then $f(a+b) = f(a+c)$, and so $a + f(b) = a + f(c)$ by (ii1), provided that $a \leq f(a)$. On the other hand, if $a \nleq f(a)$ then $a + f(b) = a + f(a) = a + f(c)$ by (ii2). Now, 2.8(iii2) is clear. □

2.10 Construction. Let $w \in M$, $w \neq o_M$, $P = \{\, x \in M \,|\, x \nleq w \,\}$ and $Q = M \setminus P = \{\, y \in$ $\in M \,|\, y \leq w \,\}$. Then P is a principal prime ideal of M, $w \in Q$, Q is a subsemilattice of M and $w = o_Q$. Let g be an endomorphism of Q satisfying the following two conditions:

(1) $g(a+b) = a + g(b)$ for all $a, b \in Q$,

(2) $a + g(b) = a + w$ for all $a \in P$ and $b \in Q$.

According to 2.4, g is a strong endomorphism of Q and $a \leq g(a)$ for every $a \in Q$. Now, define a transformation f of M by $f(P) = \{w\}$ and $f|Q = g$. One checks easily that $f \in \underline{E}$, $P = \{\, a \in M \,|\, a \nleq f(a) \,\}$ and $Q = \{\, a \in M \,|\, a \leq f(a) \,\}$. Also the conditions 2.9(ii1),(ii2) and (ii3) are clear. Thus $f \in \underline{E}^{(\sigma)}$.

2.11 Remark. If $f \in \underline{E}^{(\sigma)}$ is such that $a \nleq f(a)$ for at least one $a \in M$ then f is obtained just in the way described in 2.10 (see 2.6).

2.12 Construction. First, put $N = M$ if $o_M \in M$ and $N = M \cup \{o_N\}$ if $o_M \notin M$. Further, let A denote the set of ordered pairs $(a, b) \in N \times N$ such that $a \in M$, $a \leq b$ and $a + x = b + x$ for every $x \in N$, $x \nleq b$ (and $b \nleq x$).
(i) Clearly, $(a, a) \in A$ and $(a, o_N) \in A$ for every $a \in M$.
(ii) If $(a, b), (c, d) \in A$ then $(a + c, b + d) \in A$.

Indeed, we have $a + c \in M$, $a + c \leq b + d$ and if $x \nleq b + d$ then $x \nleq b$, $x \nleq d$, $a + x = b + x$, $c + x = d + x$ and, finally, $a + c + x = a + d + x = b + d + x$.
(iii) If $(a, b), (c, d) \in A$ and $c \leq b$, $d \nleq b$ then $(a + c, b) \in A$.

Indeed, $a + c \in M$, $a + c \le b$ and if $x \nleq b$ then $a + x = b + x$ and $a + c + x = $ $= b + c + x = b + x$.
(iv) Let $(a, b) \in A$. Define a transformation $\kappa_{a,b}$ of M by $\kappa_{a,b}(x) = a + x$ if $x \in M$ and $x \le b$, and $\kappa_{a,b}(x) = b$ if $x \in M$ and $x \nleq b$. Using 2.10, we get $\kappa_{a,b} \in \underline{E}^{(\sigma)}$. Put $\underline{E}^{(\kappa)} = \{\kappa_{a,b} \,|\, (a, b) \in A\}$. Clearly, $\kappa_{a,a} = \sigma_a$ for every $a \in M$ and $\kappa_{a,o} = \lambda_a$. Consequently, $\underline{E}^{(1)} \subseteq \underline{E}^{(\kappa)}$ and $\underline{E}^{(\gamma)} \subseteq \underline{E}^{(\kappa)}$.
(v) Let $(a, b), (c, d) \in A$. Then $\kappa_{a,b} + \kappa_{c,d} = \kappa_{a+c,b+d}$.
Indeed, $(a + c, b + d) \in A$ by (ii). If $x \nleq b + d$ then $x \nleq b$, $x \nleq d$ and $(\kappa_{a,b} + $ $+ \kappa_{c,d})(x) = \kappa_{a,b}(x) + \kappa_{c,d}(x) = b + d = \kappa_{a+c,b+d}(x)$. If $x \le b + d$ and $x \nleq b$, $x \nleq d$ then $(\kappa_{a,b}+\kappa_{c,d})(x) = \kappa_{a,b}(x)+\kappa_{c,d}(x) = b+d = x+b+d = x+a+c = \kappa_{a+c,b+d}(x)$. If $x \le b+d$, $x \nleq b$ and $x \le d$ then $(\kappa_{a,b}+\kappa_{c,d})(x) = \kappa_{a,b}(x)+\kappa_{c,d}(x) = b+x+c = x+a+c = \kappa_{a+c,b+d}(x)$. The case $x \le b + d$, $x \le b$ and $x \nleq d$ is similar. Finally, if $x \le b$ and $x \le d$ then $(\kappa_{a,b} + \kappa_{c,d})(x) = \kappa_{a,b}(x) + \kappa_{c,d}(x) = x + a + c = \kappa_{a+c,b+d}(x)$.
(vi) Let $(a, b), (c, d) \in A$, $d \le b$. Then $\kappa_{a,b}\kappa_{c,d} = \kappa_{a+c.a+d}$.
Indeed, $(a + c, a + d) \in A$ by (i) and (ii). If $x \nleq a + d$ then $x \nleq a$, $x \nleq d$ and $\kappa_{a,b}\kappa_{c,d}(x) = \kappa_{a,b}(d) = a + d = \kappa_{a+c,a+d}(x)$. If $x \le d$ then $\kappa_{a,b}\kappa_{c,d}(x) = \kappa_{a,b}(x + c) = $ $= x + a + c = \kappa_{a+c,a+d}(x)$. If $x \le a + d$ and $x \nleq d$ then $\kappa_{a,b}\kappa_{c,d}(x) = \kappa_{a,b}(d) = a + d = $ $= a + d + x = a + c + x = \kappa_{a+c,a+d}(x)$.
(vii) Let $(a, b), (c, d) \in A$, $d \nleq b$, $c \le b$. Then $\kappa_{a,b}\kappa_{c,d} = \kappa_{a+c,b}$.
Indeed, $(a + c, b) \in A$ by (iii). If $x \nleq b$ then $\kappa_{c,d} \nleq b$ and $\kappa_{a,b}\kappa_{c,d}(x) = b = \kappa_{a+c,b}(x)$. If $x \le b$, $x \le d$ then $\kappa_{a,b}\kappa_{c,d}(x) = \kappa_{a,b}(x + c) = x + a + c = \kappa_{a+c,b}(x)$. Finally, if $x \le b$, $x \nleq d$ then $x + c = x + d \nleq b$ and $x + c \le b$, a contradiction.
(viii) Let $(a, b), (c, d) \in A$, $c \nleq b$. Then $\kappa_{a,b}\kappa_{c,d} = \kappa_{b,b} = \sigma_b$.
Indeed, $\kappa_{c,d}(x) \nleq b$ for every $x \in M$.
(ix) Using (v),...,(viii), we see that $\underline{E}^{(\kappa)}$ is a subsemiring of $\underline{E}^{(\sigma)}$. Notice that $\underline{E}^{(1)} \cup$ $\cup\, \underline{E}^{(\gamma)} \subseteq \underline{E}^{(\kappa)}$ and $\underline{E}^{(1)} \cup \underline{E}^{(\gamma)}$ is a subsemiring of $\underline{E}^{(\kappa)}$.
(x) $\mathrm{id}_M \in \underline{E}^{(\kappa)}$ iff $0_M \in M$. Then $\mathrm{id}_M = \kappa_{0.o}$.
(xi) Define an addition and a multiplication on A by $(a, b) + (c, d) = (a + c, b + d)$, and $(a, b)(c, d) = (a + c, a + d)$ if $d \le b$, $(a, b)(c, d) = (a + c, b)$ if $d \nleq b$, $c \le b$ and $(a, b)(c, d) = (b, b)$ if $c \nleq b$. The mapping $(a, b) \mapsto \kappa_{a,b}$ is an isomorphism of the algebraic structure A onto the semiring $\underline{E}^{(\kappa)}$. Thus A becomes a semiring isomorphic to $\underline{E}^{(\kappa)}$.
(xii) Define an addition and multiplication on $M \times \{0, 1\}$ by $(a, 0) + (b, 0) = (a + b, 0)$, $(a, 1) + (b, 1) = (a + b, 1)$, $(a, 0) + (b, 1) = (a + b, 1) = (a, 1) + (b, 0)$, $(a, 0)(b, 0) = $ $= (a, 0) = (a, 0)(b, 1)$, $((a, 1)(b, 1) = (a + b, 1)$, $(a, 1)(b, 0) = (a + b, 0)$. In this way, we obtain a semiring that is an isomorphic copy of the semiring $\underline{E}^{(1)} \cup \underline{E}^{(\gamma)}$. This semiring is bi-ideal-simple but neither ideal-simple nor congruence-simple.

2.13 Proposition. *The following conditions are equivalent:*

(i) $0_M \in M$.
(ii) $\underline{E}^{(\sigma)} = \underline{E}^{(\kappa)}$.
(iii) $\underline{E}^{(\sigma 1)} = \underline{E}^{(\gamma)}$.
(iv) $\mathrm{id}_M \in \underline{E}^{(\kappa)}$.

(v) $\mathrm{id}_M \in \underline{E}^{(\gamma)}$.

(vi) *The semiring* $\underline{E}^{(\kappa)}$ *has a left or right multiplicatively neutral element.*

(vii) *The semiring* $\underline{E}^{(\gamma)}$ *has a left or right multiplicatively neutral element.*

(viii) *The semiring* $\underline{E}^{(\kappa)}$ *(*$\underline{E}^{(\sigma)}$*, resp.) has the additively neutral element.*

(ix) *The semiring* $\underline{E}^{(\gamma)}$ *has the additively neutral element.*

If these conditions are satisfied then $\kappa_{0,0} = \sigma_0$ *is additively neutral in* $\underline{E}^{(\kappa)}$, $\kappa_{0,o} = \lambda_0 = \mathrm{id}_M$ *is multiplicatively neutral both in* $\underline{E}^{(\kappa)}$ *and* $\underline{E}^{(\gamma)}$ *and* $\kappa_{0,o}$ *is additively neutral in* $\underline{E}^{(\gamma)}$.

Proof. (i) implies (ii),...,(ix). First, let $f \in \underline{E}^{(\sigma)}$. If $a \leq f(a)$ for every $a \in M$ (i.e., if $f \in \underline{E}^{(\sigma 1)}$) then the equality $f = \lambda_{f(0)}$ follows from 2.9(ii1). Thus $\underline{E}^{(\sigma 1)} = \underline{E}^{(\gamma)}$. Next, if $P_f = \{ a \in M \,|\, a \not\leq f(a) \} \neq \emptyset$ then $0 \in M \setminus P_f = Q_f$ and $f = \kappa_{f(0),w_f}$ (use 2.6(i),(vi),(vii)). It follows that $\underline{E}^{(\sigma)} = \underline{E}^{(\kappa)}$ (of course, $\lambda_{f(0)} = \kappa_{f(0),o}$).

Clearly, $\mathrm{id}_M = \kappa_{0,o} = \lambda_0$ is multiplicatively neutral both in $\underline{E}^{(\kappa)}$ and in $\underline{E}^{(\gamma)}$. Besides, λ_0 is additively neutral in $\underline{E}^{(\gamma)}$ and $\kappa_{0,0} = \sigma_0$ is additively neutral in $\underline{E}^{(\kappa)}$.

(ii) implies (iv) and (vi). We have $\mathrm{id}_M \in \underline{E}^{(\sigma)}$.

(iii) implies (v) and (vii). We have $\mathrm{id}_M \in \underline{E}^{(\sigma 1)}$.

(iv) implies (ii) and (v) implies (vii) trivially.

(vi) implies (i). First, let $(a, b) \in A$ be such that $\kappa_{a,b}$ is left multiplicatively neutral in $\underline{E}^{(\kappa)}$. If $c \in M$ is such that $c \not\leq b$ then $\kappa_{c,c} = \kappa_{a,b}\kappa_{c,c} = \kappa_{b,b}$ (see 2.12(viii)), and hence $c = b$, a contradiction. It follows that $b = o_N$. Now, by 2.12(vi), $\kappa_{d,d} = \kappa_{a,b}\kappa_{d,d} = \kappa_{a+d,a+d}$ for every $d \in M$. It follows that $a = 0_M \in M$ and $\kappa_{a,b} = \kappa_{0,o} = \lambda_0 = \mathrm{id}_M$.

Next, let $\kappa_{a,b}$ be right multiplicatively neutral in $\underline{E}^{(\kappa)}$. Then $\kappa_{c,o} = \kappa_{c,o}\kappa_{a,b} = \kappa_{a+c,b+c}$ for every $c \in M$ (see 2.12(vi)), and hence $a + c = c$, $b + c = o_N$. It follows that $a = 0_M \in M$ and $b = o_N$. Again, $\kappa_{a,b} = \kappa_{0,o} = \lambda_0 = \mathrm{id}_M$.

(vii) implies (i). We have $\lambda_a\lambda_b = \lambda_{a+b}$ for all $a, b \in M$ and the rest is clear.

(viii) implies (i). Let $\kappa_{a,b}$ be additively neutral in $\underline{E}^{(\kappa)}$. Then $\kappa_{c,c} = \kappa_{a,b} + \kappa_{c,c} = \kappa_{a+c,a+c}$ for every $c \in M$. Thus $a + c = c = b + c$ and $a = 0_M = b$.

(ix) implies (i). We have $\lambda_a + \lambda_b = \lambda_{a+b}$ for all $a, b \in M$ and the rest is clear. □

2.14 Proposition. *The semiring* $\underline{E}^{(\sigma)}$ *is bi-ideal-simple if and only if* $0_M \in M$.

Proof. By [6, 3.1], $\underline{E}^{(\sigma)}$ is bi-ideal-simple iff $S \subseteq \underline{E}^{(\alpha)}$. Since $\mathrm{id}_M \in \underline{E}^{(\sigma)}$, the result follows from [6, 3.4] □

2.15 Remark. We have $\sigma_a = \kappa_{a,a} \leq \kappa_{a,b}$ for all $(a, b) \in A$. It means that $\underline{E}^{(\kappa)} \subseteq \underline{E}^{(\alpha)}$. By [6, 3.1], the semiring $\underline{E}^{(\kappa)}$ is bi-ideal-simple (cf. 2.14).

2.16 Corollary. *The semiring* $\underline{E}^{(\sigma)}$ *is bi-ideal-simple if and only if* $\underline{E}^{(\sigma)} = \underline{E}^{(\kappa)}$ *(and if and only if* $0_M \in M$*).* □

2.17 Example. (i) Let M be a chain (i.e., $a + b \in \{a, b\}$ for all $a, b \in M$). Then $A = \{ (a, b) \,|\, a \in M, b \in N, a \leq b \}$ (see 2.12).

(ii) Let M be an antichain (i.e., $o_M \in M$ and $a + b = o_M$ for all $a, b \in M$, $a \neq b$). Then $A = \{ (a, a), (a, o_M) \,|\, a \in M \}$. Consequently, $\underline{E}^{(\kappa)} = \underline{E}^{(1)} \cup \underline{E}^{(\gamma)}$ (see 2.12(vii)).

3. The semiring of strong endomorphisms (b)

3.1 Proposition. *Let $a, b \in M$, $a < b$, and let P be a prime ideal of M. Then $\varrho_{a,b,P} \in \underline{E}^{(\sigma 1)}$ if and only if $b = o_M$, a is maximal in $M \setminus \{o_M\}$ and $P = \{ x \in M \,|\, x \neq a \}$ (i.e., P is principal, $b \in P$ and $a \in M \setminus P = \{ y \in M \,|\, y \leq a \}$).*

Proof. First, let $\varrho_{a,b,P} \in \underline{E}^{(\sigma 1)}$. By 2.4(iii), $\varrho_{a,b,P}(x + y) = x + \varrho_{a,b,P}(y)$ for all $x, y \in M$. If $x, y \in M \setminus P$ then we get $a = x + a$, and hence $M \setminus P = \{ u \,|\, u \leq a \}$ and $P = \{ v \,|\, v \nleq a \}$. Consequently, $b \in P$, $M \setminus P \leq a < b$, $v \leq \varrho_{a,b,P}(v) = b$, $P \leq b$ and, finally, $M \leq b$. Thus $b = o_M$. If $a < z \leq b$ then $o_m = b = \varrho_{a,b,P}(z + a) = z + \varrho_{a,b,P}(a) = = z + a = z$. It means that z is maximal in $M \setminus \{o_M\}$. Conversely, assume that $b = o_M$, a is maximal in $M \setminus \{o_M\}$ and $P = \{ x \in M \,|\, x \nleq a \}$. Due to 2.4(iii), we need to check that $\varrho_{a,b,P}(x + y) = x + \varrho_{a,b,P}(y)$ for all $x, y \in M$. If $x, y \notin P$ then $x + y \notin P$ and $\varrho_{a,b,P}(x + y) = a = x + a = x + \varrho_{a,b,P}(y)$, since $M \setminus P = \{ u | u \leq a \}$. If $y \in P$ then $x + y \in P$ and $\varrho_{a,b,P}(x + y) = b = o_M = x + o_M = x + b = x + \varrho_{a,b,P}(y)$. If $x \in P$, $y \notin P$, then $x + y \in P$ and $\varrho_{a,b,P}(x + y) = b = o_M = x + a = x + \varrho_{a,b,P}(y)$, since $x \nleq a$ and a is maximal. □

3.2 Proposition. *Let $a, b \in M$, $a < b$, and let P be a prime ideal of M. Then $\varrho_{a,b,P} \in \underline{E}^{(\sigma)} \setminus \underline{E}^{(\sigma 1}$ if and only if $b \neq o_M$, $P = \{ x \,|\, x \nleq a \}$ and $a + x = b + x$ for every $x \in P$ (i.e., P is principal, $b \in P$ and $a \in M \setminus P = \{ y \,|\, y \leq a \}$).*

Proof. First, let $\varrho_{a,b,P} \in \underline{E}^{(\sigma)} \setminus \underline{E}^{(\sigma 1)}$. We have $P = \{ x \,|\, \varrho_{a,b,P}(x) = b \}$ and $b = = \varrho_{a,b,P}(x) = \varrho^2_{a,b,P}(x) = \varrho_{a,b,P}(b)$ by 2.7. Thus $b \in P$. Similarly, $M \setminus P = \{ y \,|\, \varrho_{a,b,P}(y) = = a \}$ and $a = \varrho_{a,b,P}(y) = \varrho^2_{a,b,P}(y) = \varrho_{a,b,P}(a)$. Thus $a \in M \setminus P$. Since $\varrho_{a,b,P} \notin \underline{E}^{(\sigma 1)}$, the set $P_1 = \{ u \,|\, u \nleq \varrho_{a,b,P}(u) \}$ is non-empty. By 2.6(i), there is $w \in M$ such that $\varrho_{a,b,P}(P_1) = \{w\}$. That is, $P_1 = \{ u \,|\, u \nleq w \}$. But $w \in \{a, b\}$. If $w = a$ then $b \in P_1$ and $b = \varrho_{a,b,P}(b) = w = a$, a contradiction. It follows that $w = b$ and $P_1 \subseteq P$. By 2.6(ii), we have $b \neq o_M$. If $x \in P_1$ then $x + a = x + \varrho_{a,b,P}(a) = x + b$ by 2.6(vii). If $x \nleq b$ then $x \in P_1$. Now, let $a < x \leq b$. If $\varrho_{a,b,P}(x) = a$ then $x \in P_1$. If $\varrho_{a,b,P}(x) = b$ then $x \notin P_1$ and $b = \varrho_{a,b,P}(x) = \varrho_{a,b,P}(x + a) = x + \varrho_{a,b,P}(a) = x + a = x$ by 2.6(viii) and we get $x + a = x + b$ trivially. Finally, if $x \nleq a$ then either $x \nleq b$ and $x + a = x + b$ or $x \leq b$, $a < a + x \leq b$ and $a + x = a + a + x = b + a + x = b = b + x$ again. We have proved that $x + a = x + b$ whenever $x \nleq a$. Moreover, $\varrho_{a,b,P}(x) + a = \varrho_{a,b,P}(x + a) = \varrho_{a,b,P}(x + b) = = \varrho_{a,b,P}(x) + b$, so that $\varrho_{a,b,P}(x) = b$ and $\{ x \,|\, x \nleq a \} \subseteq P$. On the other hand, if $y \leq a$ then $\varrho_{a,b,P}(y) \leq \varrho_{a,b,P}(a) = a$, $\varrho_{a,b,P}(y) = a$ and $y \in M \setminus P$. Thus $P = \{ x \,|\, x \nleq a \}$.

Now, conversely, assume that $b \neq o_M$ and $a + x = b + x$ for every $x \in P = \{ u \,|\, u \nleq a \}$. It follows from 2.1 that $\varrho_{a,b,P} \notin \underline{E}^{(\sigma 1}$. To show that $\varrho_{a,b,P} \in \underline{E}^{(\sigma)}$, we check the conditions 2.9(ii1,2,3).

Let $x \in M$ be such that $x \leq \varrho_{a,b,P}(x) = a$. If $\varrho_{a,b,P}(y) = a$ then $x + y \in M \setminus P$ and $\varrho_{a,b,P}(x + y) = a = x + a = x + \varrho_{a,b,P}(y)$. If $\varrho_{a,b,P}(y) = b$ then $x + y \in P$ and $\varrho_{a,b,P}(x + y) = b = x + b = x + \varrho_{a,b,P}(y)$.

Let $x \in M$ be such that $x \leq \varrho_{a,b,P}(x) = b$. Then $\varrho_{a,b,P}(x + y) = b = x + b$. If $x \leq a$ then $b = \varrho_{a,b,P}(x) \leq \varrho_{a,b,P}(a) = a$, a contradiction. Thus $x \nleq a$ and $x + a = x + b$. The equality $b = x + \varrho_{a,b,P}(y)$ is clear and we have checked the condition 2.9(ii1).

Let $x \in M$ be such that $x \nleq \varrho_{a,b,P}(x) = a$. Then $x + a = x + b$, and hence $x + a = \varrho_{a,b,P}(y)$ for every $y \in M$. Further, $x \in P$ and $\varrho_{a,b,P}(x) = b = \varrho_{a,b,P}(x + y)$.

Let $x \in M$ be such that $x \nleq \varrho_{a,b,P}(x) = b$. Then $x+a = x+b$, $x+\varrho_{a,b,P}(x) = x+b = x + \varrho_{a,b,P}(y)$ and $\varrho_{a,b,P}(x) = b = \varrho_{a,b,}(x + y)$ for every $y \in M$. We have checked the conditions 2.9(ii2,3). □

3.3 Proposition. *Let $a, b \in M$, $a < b$, and let P be a prime ideal of M. Then $\varrho_{a,b,P} \in \underline{E}^{(\sigma)}$ if and only if $P = \{ x \,|\, x \nleq a \}$ and $b \leq P + a$ (then P is principal, $b \in P$, $a \in M \setminus P$ and b covers a; $\varrho_{a,b,P} \in \underline{E}^{(\sigma 1)}$ iff $b = o_M$).*

Proof. Combine 2.1 and 2.2. □

3.4 Remark. (i) Let N be a subsemilattice of M such that N is not upwards cofinal in M. Then the set P of $x \in M$ such that $x \nleq u$ for every $u \in N$ is non-empty. One checks easily that P is a prime ideal of M.

Now, assume that $P = \{ x \,|\, x \nleq a \}$ for some $a \in M$ ($a \neq o_M$). If $u \in N$ then $u \leq u$, $u \notin P$, $u \leq a$. It means that $N \leq a$ and $N \subseteq M \setminus P$. Moreover, $a \notin P$ and $a \leq w$ for at least one $w \in N$. It is clear that $a = o_K$, where $K = \{ v \,|\, v \leq u \text{ for some } u \in N \} = M \setminus P$ is a subsemilattice of M. Of course, $a = w$, and so $a \in N$ and $a = o_N$ as well. Conversely, if $o_N \in N$ then $P = \{ x \,|\, x \nleq o_N \}$.
(ii) If $a_1 < a_1 < a_3 < \dots$ is an infinite strictly increasing chain then $N = \{a_1, a_2, a_3, \dots \}$ is a subsemilattice of M and $o_N \notin N$.
(iii) Let N be a subsemilattice of M such that $o_N \notin N$. Choose $a_1 \in N$ arbitrarily. Then $a_1 \neq o_N$, and hence $a_1 < a_2$ for some $a_2 \in N$, etc. We get an infinite chain $a_1 < a_2 < a_3 < \dots$ of elements from N. If N is not upwards cofinal in M then the chain is not upwards cofinal either.
(iv) Let P be a prime ideal of M. Then $N = M \setminus P$ is a subsemilattice of M. If P is principal then $P = \{ x \,|\, x \nleq a \}$ for some $a \in M$, $N = \{ y \,|\, y \leq a \}$, $a \neq o_M$, and hence $a = o_N$. Conversely, if $o_N \in N$ then $P = \{ x \,|\, x \notin N \} = \{ x \,|\, x \nleq o_N \}$, so that P is principal.
(v) Let $a_1 < a_2 < a_3 < \dots$ be an infinite strictly increasing chain of elements from M. Put $P = \{ x \,|\, x \nleq a_i \text{ for every } i \}$ and assume that $P \neq \emptyset$ (e.g., if $o_M \in M$), i.e., that the chain is not upwards cofinal. Then P is a prime ideal and P is not principal.
(vi) The following conditions are equivalent:

(vi1) Every prime ideal of M is principal.
(vi2) If N is a subsemilattice of M such that $o_N \notin N$ then N is upwards cofinal in M.
(vi3) If $a_1 < a_2 < a_3 < \dots$ is an infinite strictly increasing chain of elements from M then the chain is upwards cofinal in M.

Moreover, if $o_M \in M$ then these conditions are equivalent to:

(vi4) There is no infinite strictly increasing chain of elements from M.

3.5 Proposition. *The following conditions are equivalent:*

(i) *If ϱ is a congruence of M such that $|M/\varrho| = 2$ then $\varrho = \ker(f)$ for a strong endomorphism f of M.*
(ii) *If P is a prime ideal of M then there are $a, b \in M$ such that $a < b$ and the endomorphism $\varrho_{a,b,P}$ is strong.*
(iii) *If P is a prime ideal of M then P is principal, $P = \{\, x \,|\, x \nleq a \,\}$, $a \neq o_M$, and there is $b \in M$ such that $a < b$ and $b \leq P + a$.*
(iv) *The following two conditions are true:*
  (iv1) *Every infinite strictly increasing chain of lements from M is upwards cofinal.*
  (iv2) *For every $a \in M \setminus \{o_M\}$, the ideal $\{\, x \,|\, a < x \,\}$ has the smallest element.*
(v) *The following two conditions are true:*
  (v1) *No infinite strictly increasing chain of elements from M has an upper bound.*
  (v2) *For every $a \in M \setminus \{o_M\}$, the ideal $\{\, x \,|\, a < x \,\}$ has the smallest element.*

Proof. (i) implies (ii). The relation $\sigma = (P \times P) \cup (N \times N)$, where $N = M \setminus P$, is a congruence of M and $|M/\sigma| = 2$. Now, $\sigma = \ker(f)$ for a strong endomorphism f and it is easy to see that $f = \varrho_{a,b,P}$ for some $a, b \in M$, $a < b$.
(ii) is equivalent to (iii). This follows from 3.3
(ii) implies (i). Since $|M/\varrho| = 2$, we have $\varrho = (P \times P) \cup (N \times N)$, where P is a prime ideal and $N = M \setminus P$. The rest is clear.
(iii) implies (iv). Every prime ideal is principal and (iv1) follows from 3.4(vi3). Furthermore, given $a \in M \setminus \{o_M\}$, the set $P = \{\, x \,|\, x \nleq a \,\}$ is a prime ideal and, by (iii), there is $b \in M$ with $a < b \leq P + a$. If $a < y$ then $b \leq z + a = y$. Thus b is the smallest element of the ideal $\{\, y \,|\, a < y \,\}$.
(iv) implies (v). Clearly, (iv1) implies (v1).
(v) implies (iv). We have to show that (iv1) is true. Using (v2), for any $a \in M \setminus \{o_M\}$ we find the uniquely determined element $f(a)$ such that $a < f(a)$ and $f(a)$ is the smallest element of the set $\{\, x \,|\, a < x \,\}$. Now, suppose that $a_1 < a_2 < a_3 < \dots$ is a chain that is not upwards cofinal. Put $b_1 = a_1$ and $b_{i+1} = f^i(a_1)$ for $i \geq 1$. Then $b_1 < b_2 < b_3 < \dots$ and $f(b_i) = b_{i+1}$. Moreover, $b_1 \leq a_1$, $b_2 = f(a_1) \leq a_2$ and $b_{i+1} = f(b_i) \leq f(a_i) \leq a_{i+1}$. Consequently, the chain $b_1 < b_2 < b_3 < \dots$ is not cofinal either and there is $c \in M$ such that $c \nleq b_i$ for every i. Let $j \geq 1$ be such that $b_k \nleq c$ for $k \geq j$ (use (v1)). We get $c, b_k < b_k + c$, so that $b_{k+1} \leq b_k + c$ and $b_k + c = b_{k+1} + c$.

Thus $b_j + c = b_{j+1} + c = b_{j+2} + c = \cdots = d$ and $b_i \leq d$ for every i, a contradiction with (v1).
(iv) implies (iii). By (iv1) and 3.4(vi), P is principal, so that $P = \{\, x \,|\, x \nleq a \,\}$ for some $a \in M$, $a \neq o_M$. By (iv2), $b \leq P + a$, where b is the smallest element of the set $\{\, x \,|\, a < x \,\}$. □

3.6 Remark. Let M satisfy the equivalent conditions of 3.5.
(i) If $o_M \in M$ then every chain of elements from M is finite.
(ii) If $0_M \in M$ then $(M, \leq)$ is similar to an ordinal.
(iii) If $0_M, o_M \in M$ then M is a finite chain.

3.7 Remark. (i) Let $a \in M \setminus \{o_M\}$, $P = \{\, x \,|\, x \nleq a \,\}$. Assume that $\varrho_{a,b,P}$ is strong for every b such that $a < b$. It follows from 3.3 that b is uniquely determined and $\{b\} = \{\, y \,|\, a < y \,\}$. In particular, $b = o_M$ and a is maximal in $M \setminus \{o_M\}$.
(ii) Assume that the endomorphism $\varrho_{a,b,P}$ is strong whenever $a, b \in M$, $a < b$ and $P = \{\, x \,|\, x \nleq a \,\}$. Then $o_M \in M$ and every element from $M \setminus \{o_M\}$ is an atom.
(iii) $\varrho_{a,b,P} \in \underline{E}^{(\sigma)}$ for all $a, b \in M$ with $a < b$ and all prime ideals P iff $|M| = 2$.

4. The semiring of strong endomorphisms (c)

4.1 Lemma. *Let $a \leq b$ and $f \in \underline{E}$ be such that* $\ker(f) = \pi_{a,b}$. *Then:*
(i) $f(a) = f(b)$.
(ii) $f(c) < f(a)$ *for* $c < a$.
(iii) $f(b) < f(d)$ *for* $b < d$.
(iv) $f(a) = f(b) = f(e)$ *for* $a \leq e \leq b$.

Proof. By 1.3, $(u, v) \in \pi_{a,b}$ iff $u + b = v + b$ and either $u = v$ or $a \leq u$, $a \leq v$. Since $(a, b) \in \pi_{a,b} = \ker(f)$, we have $f(a) = f(b)$. Since $(c, a) \notin \pi_{a,b}$, we have $f(c) \neq f(a)$, however $f(c) \leq f(a)$, and hence $f(c) < f(a)$. Similarly, $f(b) < f(d)$. Finally, $(a, e) \in \pi_{a,b}$ and $f(a) = f(b)$. □

4.2 Lemma. *Let $a \leq b$, $x \in M$ be arbitrary and let $f \in \underline{E}$ be such that* $\ker(f) = \pi_{a,b}$ *and $f^2 = f$. Then:*
(i) $x + b = f(x) + b$.
(ii) $x \leq b$ *iff* $f(x) \leq b$.
(iii) *If* $b \leq x$ *then* $f(x) \leq x$.
(iv) *If* $b \leq f(x)$ *then* $x \leq f(x)$.
(v) *If* $b \leq x$ *and* $b \leq f(x)$ *then* $x = f(x)$.
(vi) $f(b) \leq b$.
(vii) *If* $x \neq f(x)$ *then* $a \leq x$ *and* $a \leq f(x)$.
(viii) *If* $a \nleq x$ *or* $a \nleq f(x)$ *then* $x = f(x)$.
(ix) $a \leq f(a)$.

(x) $a \le f(a) = f(b) \le b$.
(xi) *If* b *covers* a *then either* $a = f(a) = f(b)$ *or* $f(a) = f(b) = b$.

Proof. Since $f^2(x) = f(x)$, we have $(x, f(x)) \in \ker(f) = \pi_{a,b}$ and the rest is easy (use 4.1(i)). $\square$

4.3 Lemma. *Let* $c < a \le b$ *and let* $f \in \underline{E}$ *be such that* $\ker(f) = \pi_{a,b}$, $f(\pi_{c,a}) \subseteq \pi_{c,a}$ *and* $f^2 = f$. *Then* $a = f(a) = f(b)$.

Proof. We have $(f(c), f(a)) \in f(\pi_{c,a}) \subseteq \pi_{c,a}$, and hence $a + f(c) = a + f(a) = f(a)$ (use 4.2(ix)). By 4.2(viii), $c = f(c)$, and therefore $a = a + c = a + f(c) = f(a)$. By 4.2(x), $a = f(a) = f(b)$. $\square$

4.4 Lemma. *Let* $a \le b \le d$ *and let* $f \in \underline{E}$ *be such that* $\ker(f) = \pi_{a,b}$, $f(\pi_{b,d}) \subseteq \pi_{b,d}$ *and* $f^2 = f$. *If* $f(b) \ne f(d)$ *then* $f(a) = f(b) = b$.

Proof. We have $(f(b), f(d)) \in f(\pi_{b,d}) \subseteq \pi_{b,d}$. If $f(b) \ne f(d)$ then $b \le f(b)$ and the equality $f(a) = f(b) = b$ follows from 4.2(x). $\square$

4.5 Lemma. *Let* $c < a \le b < d$ *and let* $f \in \underline{E}$ *be such that* $\ker(f) = \pi_{a,b}$, $(f(c), f(a)) \in$ $\in \pi_{c,a}$, $(f(b), f(d)) \in \pi_{b,d}$ *and* $f^2 = f$. *Then* $a = b$.

Proof. By 4.3, we have $a = f(a) = f(b)$ and 4.1(iii) implies $f(b) \ne f(d)$. Thus $f(a) = f(b) = b$ by 4.4, and so $a = b$. $\square$

4.6 Corollary. *Let* $c < a \le b < d$. *Then* $a = b$, *provided that there is* $f \in \underline{E}^{(\sigma)}$ *such that* $\ker(f) = \pi_{a,b}$. $\square$

4.7 Lemma. *Let* $c < a \le b$, $d \le b$, $a \nleq d$, *and let* $f \in \underline{E}$ *be such that* $\ker(f) = \pi_{a,b}$, $(f(c), f(a)) \in \pi_{c,a}$ *and* $f^2 = f$. *Then* $d < a$.

Proof. By 4.3, $a = f(a) = f(b)$. Since $d \le b$, we have $f(d) \le f(b) = a$. If $a \nleq f(d)$ then $d = f(d) \le a$ follows from 4.2(viii) (since $a \nleq d$, we get $d < a$). On the other hand, if $a \le f(d)$ then $f(a) = a = f(d)$ and $(a, d) \in \ker(f) = \pi_{a,b}$. Consequently, either $a = d$ or $a \le d$. In both cases, $a \le d$, a contradiction. $\square$

4.8 Lemma. *Let* $c < a < b$ *and* $d < b$. *Then* $d \le a$, *provided that there are* $f, g \in \underline{E}^{(\sigma)}$ *such that* $\ker(f) = \pi_{a,b}$ *and* $\ker(g) = \pi_{a,d}$ *(if* $a \le d$*).*

Proof. If $a \nleq d$ then $d < a$ by 4.7. If $a \le d$ then $a = d$ by 4.6. $\square$

4.9 Proposition. *Assume that for all* $a, b \in M$, $a < b$, *there is* $f \in \underline{E}^{(\sigma)}$ *such that* $\ker(f) = \pi_{a,b}$. *Then just one of the following two cases holds:*

(1) $o_M \in M$ *and every element from* $M \setminus \{o_M\}$ *is minimal;*

(2) $o_M \in M$, *the set* $M \setminus \{o_M\}$ *has just ane maximal element, say* w, *and for every* $a \in M \setminus \{o_M, w\} \neq \emptyset$, $a \leq w$ *and* a *is minimal.*

Proof. Combine 4.6 and 4.8. □

5. The semiring of strong endomorphisms (d)

5.1 Example. Let M be an antichain (i.e., $o_M \in M$ and every element from $M \setminus \{o_M\}$ is minimal).
(i) $0_M \in M$ iff $|M| = 2$ (equivalently, M is a chain).
(ii) Let f be a transformation of M such that $f(a) \in \{a, o_M\}$ for every $a \in M$. We claim that $f \in \underline{E}^{(\sigma 1)}$.

Indeed, $f(a + a) = f(a) = f(a) + f(a)$. If $a \neq b$ then $f(a + b) = f(o) = o$ and $f(a) + f(b) \in \{o, a + b\} = \{o\}$. Thus $f \in \underline{E}$ and $a \leq f(a)$ for every $a \in M$. The condition (ii1) is clearly satisfied and $f \in \underline{E}^{(\sigma 1)}$ by 2.9.
(iii) Let $f \in \underline{E}^{(\sigma 1)}$. By 2.9(ii), we have $f(a + b) = a + f(b)$ for all $a, b \in M$. In particular, $f(o) = a + f(o)$, $a \leq f(o)$ and $f(o) = o$. Furthermore, if $b \neq f(b)$ then $o = f(o) = f(f(b) + b) = f(b) + f(b) = f(b)$. Thus $f(a) \in \{a, o_M\}$ for every $a \in M$.
(iv) Combining (ii) and (iii), we conclude that $\underline{E}^{(\sigma 1)}$ is just the set of all transformations f of M such that $f(a) \in \{a, o_M\}$ for every $a \in M$ (then $f(o) = o$).
(v) Let $f \in \underline{E}^{(\sigma)} \setminus \underline{E}^{(\sigma 1)}$. Then $P_f = \{a \in M \mid a \nleq f(a)\} = \{a \in M \mid f(a) \neq a, o_M\} \neq \emptyset$. By 2.6, $f(P_f) = \{w_f\}$ and $P_f = \{a \mid a \nleq w_f\} = \{a \mid a \neq w_f\} = M \setminus \{w_f\}$. By 2.6(v), $f(w_f) = w_f$. Thus $f = \sigma_{w_f}$.
(vi) Combining (iv) and (v), we see that $\underline{E}^{(\sigma)} = \underline{E}^{(\sigma 1)} \cup \underline{E}^{(1)} = \underline{E}^{(1)} \cup \{f : M \to \to M \mid f(a) = a, o_M \text{ for every} a \in M\}$.
(vii) Let ϱ be a congruence of M. Then $\varrho = (N \times N) \cup \mathrm{id}_M$, where N is the block of ϱ with $o_M \in N$. If $f(N) = o$ and $f(a) = a$ for every $a \in M \setminus N$ then $f \in \underline{E}^{(\sigma 1)}$ and $\ker(f) = \varrho$.

5.2 Example. Let M be a nearantichain (i.e., $o_M \in M$, the set $M \setminus \{o_M\}$ has the greatest element, say z_M, and $a < z_M$ for every $a \in M \setminus \{o_M, z_M\}$) and $|M| \geq 4$. Clearly, $0_M \notin M$. Put $N = M \setminus \{o_M\}$. Then N is a subsemilattice of M and $o_N = z_M$.
(i) Let $f \in \underline{E}^{(\sigma)}$ be such that $f(N) \subseteq N$. If ϱ is a congruence of N then $\varrho \cup \mathrm{id}_M$ is a congruence of M, $f(\varrho \cup \mathrm{id}_M) \subseteq \varrho \cup \mathrm{id}_M$, and hence $f(\varrho) \subseteq \varrho$. It follows that $g = f|N$ is a strong endomorphism of N. By 5.1(vi), either $g = \sigma_u|N$ for some $u \in N$ or $g(v) \in \{v, z_M\}$ for every $v \in N$.
(ii) Let f be a transformation of M such that $f(N) \subseteq N$ and $g = f|N$ is an endomorphism of N. Clearly, $f \in \underline{E}$ iff $f(N) \leq f(o)$. Now, assume that $f \in \underline{E}$ and that g is a strong endomorphism of N (see (i)). If $a \nleq g(a)$ for at least one $a \in N$ then $g = \sigma_u|N$ for some $u \in N$ and $u = f(a) = f(a + o_M) = f(o_M)$ by 2.9(ii3). Thus $f = \sigma_u$. If $a \leq g(a)$ for every $a \in N$ then $g(a) \in \{a, z_M\}$. Since $z_M \in g(N)$, we have $z_M \leq f(o_M) \leq o_M$, so that $f(o_M) \in \{z_M, o_M\}$.

The endomorphism $g = f|N$ is strong, and hence 2.9(ii1,2,3) are true for $a, b \in N$. If $a = b = o_M$ then the conditions are true as well. If $a \not\leq f(a)$ for at least one $a \in N$ then $f = \sigma_u$ for some $u \in N$ and there is nothing to check. Consequently, let $f(a) \in \{a, z_M\}$ for every $a \in N$. The condition 2.9(ii1) is clear for $a = o_M$ or $b = o_M$ and the conditions 2.9(ii2,3) are clear for $a = o_M$. We have proved that $f \in \underline{E}^{(\sigma)}$. Clearly, $f \in \underline{E}^{(\sigma 1)}$ iff $f(o_M) = o_M$ (and $f(a) \in \{a, z_M\}$, $a \in N$).

(iii) Let $f \in \underline{E}$ be such that $f(N) \nsubseteq N$. Then $o_M \in f(N)$ and $o_M = f(a)$ for some $a \in N$. Consequently, $f(z_M) = f(z_M + a) = f(z_M) + f(a) = f(z_M) + o_M = o_M$.

Now, assume that $f \in \underline{E}^{(\sigma)}$. By 2.9(ii2), $z_M + f(x) = z_M + f(z_M) = z_M + o_M = o_M$ for every $x \in M$. Consequently, $f = \sigma_{o_M}$.

(iv) $\underline{E}^{(\sigma 1)} = \{ f : M \to M \mid f(o_M) = o_M \}$ and$f(a) \in \{a, z_M\}$ for every $a \in N \}$.

(v) $\underline{E}^{(\sigma)} = \underline{E}^{(1)} \cup \underline{E}^{(\sigma 1)} \cup \{ f : M \to M \mid f(o_M) = z_M$ and $f(a) \in \{a, z_M\}$ for every $a \in$ $\in N \setminus \{\sigma_{z_M}\}\}$.

(vi) Let ϱ be a congruence of M. Let A and B be the blocks of ϱ such that $o_M \in A$ and $z_M \in B$. If C is a block of ϱ such that $|C| \geq 2$ then $C = A$ or $C = B$. If $|A| \geq 2$ then $A = B$ and $\varrho = \ker(f)$ for some $f \in \underline{E}^{(\sigma)}$ (use(v)). If $|A| = 1$ and $|B| \geq 2$ then $\varrho = \ker(g)$ for some $g \in \underline{E}^{(\sigma 1)}$.

5.3 Example. Let $M = \{0, a, o\}$, where $0 < a < o$.

(i) Clearly, the three-element chain M has just four congruences: $\varrho_1 = \mathrm{id}_M$, $\varrho_2 =$ $= \{(0, a), (a, 0)\} \cup \mathrm{id}_M$, $\varrho_3 = \{(a, o), (o, a)\} \cup \mathrm{id}_M$ and $\varrho_4 = M \times M$.

(ii) $\underline{E} = \underline{E}^{(1)} \cup \{\mathrm{id}_M\} \cup \{f_1, f_2, f_3, f_4, f_5, f_6\}$, where $f_1(0) = 0 = f_1(a)$, $f_1(o) = a$, $f_2(0) =$ $= 0 = f_2(a)$, $f_2(o) = o$, $f_3(0) = a = f_3(a)$, $f_3(o) = o$, $f_4(0) = 0$, $f_4(a) = f_4(o)$, $f_5(0) = 0$, $f_5(a) = o = f_5(o)$, $f_6(0) = a$, $f_6(a) = o = f_6(o)$.

Clearly, $\underline{E}^{(\sigma 1)} = \{\mathrm{id}_M, f_3, f_6\}$ and $\underline{E}^{(\sigma)} = \underline{E}^{(1)} \cup \underline{E}^{(\sigma 1)} \cup \{f_1, f_4\}$. Thus $\underline{E} \setminus \underline{E}^{(\sigma)} =$ $= \{f_2, f_5\}$.

References

[1] K. Al-Zoubi, T. Kepka and P. Němec: *Quasitrivial semimodules I,* Acta Univ. Carolinae Math. Phys. **49/1** (2008), 3–16.

[2] K. Al-Zoubi, T. Kepka and P. Němec: *Quasitrivial semimodules II,* Acta Univ. Carolinae Math. Phys. **49/1** (2008), 17–24.

[3] K. Al-Zoubi, T. Kepka and P. Němec: *Quasitrivial semimodules III,* Acta Univ. Carolinae Math. Phys. **50/1** (2009), 3–13.

[4] T. Kepka and P. Němec: *Quasitrivial semimodules IV,* this issue, pp. 3–22.

[5] T. Kepka and P. Němec: *Quasitrivial semimodules V,* this issue, pp. 23–35.

[6] T. Kepka and P. Němec: *Quasitrivial semimodules VI,* this issue, pp. 43–59.