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Generalized Luzin Sets

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Many modifications of the classical notions of a Luzin and a Sierpiński sets are investigated in literature. The famous result by Rothberger says that the existence of both classical Luzin and Sierpiński sets implies the continuum hypothesis. We present a generalization of Rothberger's result for modified notion of a Luzin set.

Main result of the paper is Theorem 6, that is a generalization of a classical result by F. Rothberger [6] and a strengthening of a result by J. Cichoń [3]. Note that all results needed for a proof of Theorem 6 are well known with the exception of the simple observation contained in Lemma 5.

We work in **ZFC** set theory. We assume that all considered σ-ideals $\mathcal{I}$ of subsets of a Polish space X are such that $\bigcup \mathcal{I} = X$. The cardinal characteristics $\mathrm{add}(\mathcal{I})$, $\mathrm{non}(\mathcal{I})$, $\mathrm{cof}(\mathcal{I})$ and $\mathrm{cov}(\mathcal{I})$ of a σ-ideal $\mathcal{I}$ are defined e.g. in [1] or [2]. Two σ-ideals $\mathcal{I}$ and $\mathcal{J}$ of subsets of X are said to be **orthogonal**, if there are sets $A \in \mathcal{I}$ and $B \in \mathcal{J}$ such that $X = A \cup B$. If X is a Polish group, i.e. a topological group with Polish topology, then an ideal $\mathcal{I}$ is **shift invariant** if $a + A \in \mathcal{I}$ for any $a \in X$ and any $A \in \mathcal{I}$. We shall be interested in ideals with a Borel basis, i.e. in ideals $\mathcal{I}$ with the following property: for any $A \in \mathcal{I}$, there exists a Borel set $B \in \mathcal{I}$ such that $A \subseteq B$.

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We shall mainly deal with the σ-ideal of meager subsets $\mathcal{M}(X)$ of a Polish space X and with the σ-ideal of measure zero subsets $\mathcal{N}(X)$ of X for some Borel measure on X. Both of them have a Borel basis. It is well known that those two ideals are orthogonal.

The Baire space ${}^{\omega}\omega$ is pre-ordered by the eventual domination relation

$$f \leq^* g \equiv (\exists n_0)(\forall n \geq n_0)\, f(n) \leq g(n).$$

The bounding number $\mathfrak{b}$ is the smallest cardinality of an unbounded subset of ${}^{\omega}\omega$ and the dominating number $\mathfrak{d}$ is the smallest cardinality of a dominating (= cofinal) subset of ${}^{\omega}\omega$, see [1]. Main relationships between cardinal characteristics of $\mathcal{M}(X)$, $\mathcal{N}(X)$, $\mathfrak{b}$ and $\mathfrak{d}$ are expressed by so called Cichoń diagram, see [1] or [2].

The Baire space ${}^{\omega}\omega$ is homeomorphic with the Polish group ${}^{\omega}\mathbb{Z}$ with coordinate wise addition on the group of integers $\mathbb{Z}$. Thus, we can consider ${}^{\omega}\omega$ as a Polish group. We denote by $\mathcal{K}_\sigma$ the σ-ideal of subsets of ${}^{\omega}\omega$ generated by compact subsets of ${}^{\omega}\omega$. By definition, $\mathcal{K}_\sigma$ has a Borel basis. The well-known result by F. Rothberger (see e.g. [2] or [1]) says that

$$\operatorname{cov}(\mathcal{K}_\sigma) = \operatorname{cof}(\mathcal{K}_\sigma) = \mathfrak{d}. \tag{1}$$

Another classical result by F. Rothberger [6] can be easily generalized. A proof is a simple modification of Rothberger's proof.

Theorem 1 *If $\mathcal{I}$ and $\mathcal{J}$ are orthogonal shift invariant σ-ideals on a Polish group X, then*

$$\operatorname{cov}(\mathcal{J}) \leq \operatorname{non}(\mathcal{I}). \tag{2}$$

Proof is easy. Assume that $A \in \mathcal{I}$ and $B \in \mathcal{J}$ are such that $X = A \cup B$. If $C \subseteq X$ is such that $C + B = \bigcup_{x \in C}(x + B) \neq X$, then $C \in \mathcal{I}$. Actually, there exists a $y \in X$ such that $y \notin C + B$. Then $(y - C) \cap B = \emptyset$ and hence $(y - C) \subseteq A$. Thus $C \in \mathcal{I}$.

Let $C \subseteq X$ be such that $|C| = \operatorname{non}(\mathcal{I})$ and $C \notin \mathcal{I}$. Then $\bigcup_{x \in C}(x + B) = X$ and therefore $\operatorname{cov}(\mathcal{J}) \leq \operatorname{non}(\mathcal{I})$. □

If $\mathcal{I}$ is a σ-ideal of subsets of a Polish space X and κ is an uncountable regular cardinal $\leq \mathfrak{c}$, then a set $L \subseteq X$ is said to be a κ-$\mathcal{I}$-**Luzin set**, if $|L| \geq \kappa$ and $|A \cap L| < \kappa$ for any $A \in \mathcal{I}$. J. Cichoń [3] calls such a set $(|L|, \kappa)$-Luzin set for $\mathcal{I}$. A κ-$\mathcal{M}(X)$-Luzin set is simply called a κ-Luzin set and a κ-$\mathcal{N}(X)$-Luzin set is called a κ-Sierpiński set. By the definition, if there exists a κ-$\mathcal{I}$-Luzin set L, then

$$\operatorname{non}(\mathcal{I}) \leq \kappa \leq |L|. \tag{3}$$

Note also that a κ-$\mathcal{J}$-**Luzin set** is a κ-$\mathcal{I}$-**Luzin set** as well, provided that $\mathcal{I} \subseteq \mathcal{J}$. Since $\mathcal{K}_\sigma \subseteq \mathcal{M}({}^{\omega}\omega)$ we obtain that every κ-Luzin subset of ${}^{\omega}\omega$ is a κ-$\mathcal{K}_\sigma$-Luzin set.

P. Mahlo [5] assuming the continuum hypothesis has constructed a $\aleph_1$-Luzin set. Independently N. N. Luzin [4] obtained the same result. Then W. Sierpiński [7] constructed a $\aleph_1$-Sierpiński set again assuming $2^{\aleph_0} = \aleph_1$. The next construction is essentially that of N. N. Luzin [4] and we consider it as a folklore result.[1]

[1] Note that J. Cichoń [3] in Proposition 4.6 asks an additional property of the ideal $\mathcal{I}$.

Theorem 2 *Let $\mathscr{I}$ be a σ-ideal of subsets of a Polish space X, κ being an uncountable regular cardinal. If $\kappa = \mathrm{cov}(\mathscr{I}) = \mathrm{cof}(\mathscr{I})$, then there exists a κ-$\mathscr{I}$-Luzin set.*

Proof. Let $\{B_\xi : \xi < \kappa\}$ be a base of the ideal $\mathscr{I}$. Since $\kappa = \mathrm{cov}(\mathscr{I})$, for any $\xi < \kappa$ we have $|X \setminus \bigcup_{\eta<\xi} B_\eta| \geq \kappa$. Thus, there exists an $x_\xi \in X \setminus \bigcup_{\eta<\xi} B_\eta$ such that $x_\xi \neq x_\eta$ for every $\eta < \xi$. Set $L = \{x_\xi : \xi < \kappa\}$. Then $|L| = \kappa$.

If $A \in \mathscr{I}$, then $A \subseteq B_\xi$ for some $\xi < \kappa$ and therefore $L \cap A \subseteq \{x_\eta : \eta \leq \xi\}$. Thus $|L \cap A| < \kappa$. □

Hence, by (1) we obtain

Corollary 3 *There exists a $\mathfrak{d}$-$\mathscr{K}_\sigma$-Luzin set.*

On the other hand by a transfinite induction, one can easily construct an increasing unbounded sequence $\{f_\xi\}_{\xi<\mathfrak{b}}$ of elements of ${}^\omega\omega$.

Theorem 4 *The set $L = \{f_\xi : \xi < \mathfrak{b}\}$ is a $\mathfrak{b}$-$\mathscr{K}_\sigma$-Luzin set.*

Proof. It is well known that a compact subset of ${}^\omega\omega$ is strictly bounded by a function $f \in {}^\omega\omega$. Therefore, a σ-compact subset of ${}^\omega\omega$ is eventually bounded. Hence, if $K \subseteq {}^\omega\omega$ is σ-compact, then $|K \cap L| < \mathfrak{b}$. □

Thus, it is consistent with **ZFC** (in any model in which $\mathfrak{b} < \mathfrak{d}$), that there are κ-$\mathscr{I}$-Luzin sets for different κ's. Hence, from the existence of a κ-$\mathscr{I}$-Luzin set one cannot conclude in **ZFC** that $\kappa = \mathrm{cov}(\mathscr{I}) = \mathrm{cof}(\mathscr{I})$.

Lemma 5 *Let $\mathscr{I}$ be a σ-ideal of subsets of a Polish space X, κ being an uncountable regular cardinal. If there exists a κ-$\mathscr{I}$-Luzin set L, then*

$$|L| \leq \mathrm{cov}(\mathscr{I}). \tag{4}$$

Proof. Let L be a κ-$\mathscr{I}$-Luzin set. By definition $|L| \geq \kappa$. Assume that the family $\{A_\xi : \xi < \mathrm{cov}(\mathscr{I})\} \subseteq \mathscr{I}$ witnesses $\mathrm{cov}(\mathscr{I})$, i.e. $X = \bigcup_{\xi<\mathrm{cov}(\mathscr{I})} A_\xi$. Then

$$L = \bigcup_{\xi<\mathrm{cov}(\mathscr{I})} (L \cap A_\xi),$$

where $|L \cap A_\xi| < \kappa$ for every $\xi < \mathrm{cov}(\mathscr{I})$. Therefore $|L| \leq \mathrm{cov}(\mathscr{I})$. □

Theorem 6 *Assume that $\mathscr{I}$ and $\mathscr{J}$ are orthogonal shift invariant σ-ideals on a Polish group X and κ, λ are uncountable regular cardinals not greater than $\mathfrak{c}$. If L is a κ-$\mathscr{I}$-Luzin set and S is a λ-$\mathscr{J}$-Luzin set, then*

$$\kappa = \lambda = \mathrm{non}(\mathscr{I}) = \mathrm{non}(\mathscr{J}) = \mathrm{cov}(\mathscr{I}) = \mathrm{cov}(\mathscr{J}) = |L| = |S|.$$

Proof. By (3), by Lemma 5 and by Rothberger's Theorem 1 we have

$$\mathrm{non}(\mathscr{I}) \leq \kappa \leq |L| \leq \mathrm{cov}(\mathscr{I}) \leq \mathrm{non}(\mathscr{J}),\ \mathrm{non}(\mathscr{J}) \leq \lambda \leq |S| \leq \mathrm{cov}(\mathscr{J}) \leq \mathrm{non}(\mathscr{I}).$$

J. Cichoń [3] proves Theorem 6 assuming that $\kappa = |L|$ and $\lambda = |S|$. Moreover, by Corollary 2.13 of [3], under assumptions of Theorem 6 one obtains $\kappa = \lambda$. However in [3] there is no conclusion without the assumption that the size of L and S is κ and

λ, respectively. Let us recall that $\mathrm{L}(\mathscr{I}, \mu, \kappa)$ introduced in [3] means that there exists a κ-$\mathscr{I}$-Luzin set of cardinality μ. Thus, in terminology of [3], Theorem 6 says: if $\mathrm{L}(\mathscr{I}, \mu, \kappa)$ and $\mathrm{L}(\mathscr{J}, \nu, \lambda)$, then

$$\kappa = \lambda = \mathrm{non}(\mathscr{I}) = \mathrm{non}(\mathscr{J}) = \mathrm{cov}(\mathscr{I}) = \mathrm{cov}(\mathscr{J}) = \mu = \nu.$$

Corollary 7 *Assume that κ, λ are uncountable regular cardinals not greater than $\mathfrak{c}$. If L is a κ-Luzin set and S is a λ-Sierpiński set, then*

$$\kappa = \lambda = \mathrm{non}(\mathscr{M}) = \mathrm{non}(\mathscr{N}) = \mathrm{cov}(\mathscr{M}) = \mathrm{cov}(\mathscr{N}) = |L| = |S|.$$

Together with Corrolary 3 and Theorem 4 we obtain

Corollary 8 *If $\mathfrak{b} < \mathfrak{d}$ and $\mathscr{I}$ is a shift invariant σ-ideal on ${}^{\omega}\omega$ orthogonal to $\mathscr{K}_\sigma$, then there exists no κ-$\mathscr{I}$-Luzin set for any uncountable regular cardinal κ.*

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