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A NEW FAMILY OF COMPOUND LIFETIME DISTRIBUTIONS

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In this paper, we introduce a general family of continuous lifetime distributions by compounding any continuous distribution and the Poisson–Lindley distribution. It is more flexible than several recently introduced lifetime distributions. The failure rate functions of our family can be increasing, decreasing, bathtub shaped and unimodal shaped. Several properties of this family are investigated including shape characteristics of the probability density, moments, order statistics, (reversed) residual lifetime moments, conditional moments and Rényi entropy. The parameters are estimated by the maximum likelihood method and the Fisher's information matrix is determined. Several special cases of this family are studied in some detail. An application to a real data set illustrates the performance of the family of distributions.

Keywords: estimation, failure rate shapes, moments, Poisson–Lindley distribution

Classification: 62E15, 62E20

1. INTRODUCTION

Modeling and analyzing lifetime data are important aspects of statistical research in many applied sciences such as engineering, medicine, economics and so on. Various recent probability distributions discussed modeling of such data by compounding well-known continuous distributions such as the exponential, Weibull, and exponentiated exponential distributions with the power series distribution that includes the Poisson, logarithmic, geometric and binomial distributions as particular cases. The compounding approach gives new distributions that extend well-known families of distributions. At the same time they offer more flexibility for modeling lifetime data. The extensions, sometimes, provide reasonable parametric fits to practical applications as in lifetime and reliability studies. The flexibility of such compound distributions comes in terms of one or more failure rate shapes that may be decreasing or increasing or bathtub shaped or unimodal shaped.

Two prominent compound distributions introduced recently are the Weibull power series distribution due to Morais and Barreto-Souza [12] and the exponentiated exponential binomial distribution due to Bakouch et al. [2]. The former contains as particular cases the exponential power series distribution (Chahkandi and Ganjali [5]), the exponential geometric distribution (Adamidis and Loukas [1]), the exponential Poisson dis-

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tribution (Kus [9]), the exponential logarithmic distribution (Tahmasbi and Rezaei [15]), the Weibull geometric distribution (Barreto-Souza et al. [4]) and the Weibull Poisson distribution (Lu and Shi [10]).

Let X_i, $i = 1, 2, \ldots, N$ be independent and identical failure times of a system and let N be a discrete random variable independent of the X's. Let $X = \min(X_1, X_2, \ldots, X_N)$ denote the time of first failure. If a system is made of N components in series and if $X_1, X_2, \ldots, X_N$ denote the lifetimes of the components then X denotes the lifetime of the system. For example, if a fiber can be thought of as a series system of N units and if $X_1, X_2, \ldots, X_N$ denote the strength of the units then the breaking strength of the fiber can be expressed as $X = \min(X_1, X_2, \ldots, X_N)$. Here, N can be a random variable since the number of units in a fiber may depend on its length, width and other characteristics. Another example is a factory having N machines functioning in series at any time. Here, N may depend on such factors as manpower and economy, so can be considered a random variable.

Several of the known compound distributions (including some of those stated above) do not have the ability to model first failures well. For example, if N has mode greater than one then the exponential geometric and exponential logarithmic distributions are not appropriate for modeling first failures. This is because geometric and logarithmic distributions have mode equal to one. If N is over-dispersed, the exponential Poisson distribution is not appropriate for modeling first failures. This is because the zero-truncated Poisson distribution is always under-dispersed. Hence, there is a need for flexible compound distributions that can model first failures well. A first attempt in this direction was the exponential Poisson–Lindley distribution due to Barreto-Souza and Bakouch [3]. The zero-truncated Poisson–Lindley distribution used here for compounding can have mode greater than or equal to one and can be under-dispersed, equi-dispersed and over-dispersed (Ghitany et al. [6]).

In this paper, we introduce a general family of continuous lifetime distributions by compounding any continuous distribution and the Poisson–Lindley distribution. As we shall see later, this family encompasses various shapes (including monotonically decreasing, monotonically increasing, bathtub shapes and unimodal shapes) for the failure rate function.

The proposed family of distributions applies not just to reliability data. The notion of the "minimum of a random number of random events" occurs in a wide variety of areas not just in reliability. For example, suppose X_i, $i = 1, 2, \ldots, N$ are claims made to an insurance company over a reference period (say a year, a six-month period, a two-year period, etc). Clearly, N denoting the number of claims is a random variable. Then $X = \min(X_1, X_2, \ldots, X_N)$ shall denote the minimum of the insurance claims over a reference period. This variable will be of interest to insurers.

The contents of this paper are organized as follows. In Section 2, we construct the new family of distributions. Shape characteristics of the probability density and failure rate functions of the family are investigated in Section 3. Various statistical properties of the proposed family are explored in Section 4. The properties include moments, order statistics, (reversed) residual lifetime moments, conditional moments and Rényi entropy. Estimation of the parameters by maximum likelihood method is discussed in Section 5. An expression for the associated Fisher information matrix is given. In

Section 6, several special cases of the family are studied in some detail. These include the Weibull Poisson–Lindley, Burr Poisson–Lindley, exponentiated Weibull Poisson–Lindley and Dagum Poisson–Lindley distributions. An application to a real data set is presented in Section 7 to show flexibility of the family of distributions.

2. CONSTRUCTION OF THE FAMILY

Let $X_1, X_2, \ldots, X_N$ be a random sample from a continuous distribution function $G(\cdot)$ on $(0, \infty)$ with some continuous unknown parameters. Let N be a zero-truncated Poisson–Lindley random variable independent of the X's with probability mass function

$$P(N = n) = \frac{\theta^2}{1 + 3\theta + \theta^2} \frac{2 + \theta + n}{(1 + \theta)^n}$$

for $n = 1, 2, \ldots$ and $\theta > 0$. Let $g(x) = dG(x)/dx$ denote the density function, $\overline{G}(x) = 1 - G(x)$ the survival function, and $G^{-1}(\cdot)$ the quantile function.

We define $X = \min(X_1, X_2, \ldots, X_N)$ as the Poisson–Lindley-G random variable. The distribution function of X can be obtained as

$$F(x) = 1 - \frac{\theta^2}{1 + 3\theta + \theta^2} \frac{\overline{G}(x)}{\left[1 + \theta - \overline{G}(x)\right]^2} \left\{1 + \theta + (2 + \theta)\left[1 + \theta - \overline{G}(x)\right]\right\} \tag{1}$$

for $\theta > 0$. Hereafter, a random variable X with distribution function (1) shall be denoted by $X \sim \mathrm{GPL}(\boldsymbol{\alpha}, \theta)$, where $\boldsymbol{\alpha}$ is an unknown vector of parameters of $G(\cdot)$. Note that $F(x) \to G(x)$ as $\theta \to \infty$. So, G is a limiting case of (1).

To generate a random variable $X \sim \mathrm{GPL}(\boldsymbol{\alpha}, \theta)$, we compute $X = F^{-1}(U)$, with $U \sim U(0, 1)$, where

$$F^{-1}(U) = G^{-1}\left[1 - \frac{(1 + \theta)(1 - U)a + \frac{b - \sqrt{\Delta_U}}{2}}{(1 - U)a + (2 + \theta)}\right] \tag{2}$$

with $\Delta_U = [b - 2a(1 + \theta)(U - 1)]^2 + 4a(1 + \theta)^2(U - 1)\ [2 + \theta + a(1 - U)]$, $a = \theta^{-2}\left(1 + 3\theta + \theta^2\right)$ and $b = 3 + 4\theta + \theta^2$.

3. SHAPE CHARACTERISTICS OF THE DENSITY AND FAILURE RATE FUNCTIONS

In this section, we obtain the shape characteristics of the density and the failure rate functions of $X \sim \mathrm{GPL}(\boldsymbol{\alpha}, \theta)$.

The density, survival and failure rate functions of $X \sim \mathrm{GPL}(\boldsymbol{\alpha}, \theta)$ are

$$f(x) = \frac{\theta^2(1 + \theta)^2 g(x)}{1 + 3\theta + \theta^2} \frac{3 + \theta - \overline{G}(x)}{\left[1 + \theta - \overline{G}(x)\right]^3}, \tag{3}$$

$$S(x) = \frac{\theta^2}{1 + 3\theta + \theta^2} \frac{\overline{G}(x)}{\left[1 + \theta - \overline{G}(x)\right]^2} \left\{1 + \theta + (2 + \theta)\left[1 + \theta - \overline{G}(x)\right]\right\}, \tag{4}$$

and

$$h_f(x) = K_1(x) + K_2(x), \tag{5}$$

respectively, where

$$K_1(x) = \frac{2h_g(x)(\theta+1)^2}{1+\theta+(2+\theta)\left[1+\theta-\overline{G}(x)\right]^2}$$

and

$$K_2(x) = \frac{h_g(x)(\theta+1)^2}{1+\theta+(2+\theta)\left[1+\theta-\overline{G}(x)\right]},$$

where $h_g(x) = g(x)/\left[1-G(x)\right]$. We observe that (3) can be expressed as

$$f(x) = \frac{\theta^2(1+\theta)^2}{1+3\theta+\theta^2}\left[g_1(x)+g_2(x)\right],$$

where $g_i(x) = ig(x)\left[1+\theta-\overline{G}(x)\right]^{-(i+1)}$, $i = 1, 2$. Clearly, $f(\cdot)$ is a linear combination of the $g_i(x)$s.

If $g(\cdot)$ is a monotonic decreasing function then $f(\cdot)$ is a decreasing function too. If $g(\cdot)$ is a decreasing function then $\lim_{x\to a^+} f(x) = \left(\theta^3+4\theta^2+5\theta+2\right) g\left(a^+\right) / \left(\theta^3+3\theta^2+\theta\right)$ and $\lim_{x\to\infty} f(x) = 0$, where $a^+ = \min\left(D_g\right)$ and D_g is the domain of the density function g. We also have that $f(x)/g(x) \to 1$ as $\theta \to \infty$.

Remark 3.1. If $g(x)$ is a decreasing continuous density function then $f(x)$ in (3) is also a decreasing density function.

Proof. We have

$$g_i'(x) = ig'(x)\left[1+\theta-\overline{G}(x)\right]^{-(i+1)} - i(i+1)g^2(x)\left[1+\theta-\overline{G}(x)\right]^{-(i+2)}$$

for $i = 1, 2$. From the assumption and the fact that $1+\theta-\overline{G}(x) = \theta + G(x) \geq 0$, we have $g_i'(x) \leq 0$ for all x. Hence,

$$f'(x) = \frac{\theta^2(1+\theta)^2}{1+3\theta+\theta^2}\left[g_1'(x)+g_2'(x)\right] \leq 0.$$

This completes the proof. □

Note: If $g(x)$ is a unimodal density function then $f(x)$ is a decreasing function for x greater than or equal to the mode of g.

Remark 3.2. If the failure rate function $h_g(x)$ is decreasing, then $h_f(x)$ is also decreasing.

Proof. Note that $h_g'(x) \leq 0$ implies $K_1'(x) \leq 0$ and $K_2'(x) \leq 0$, so the result follows from (5). □

Analytical study of the behavior of probability density and failure rate functions for other shapes (including increasing shapes, bathtub shapes and unimodal shapes) is complicated. We shall discuss them later in Section 6 graphically.

4. STATISTICAL AND RELIABILITY MEASURES

Several statistical and reliability measures of $X \sim \text{GPL}(\boldsymbol{\alpha}, \theta)$ are explored in this section. Expressions are derived for moments, order statistics, (reversed) residual lifetime moments, conditional moments and Rényi entropy. Throughout, we suppose $X_1, \ldots, X_n$ is a random sample from the GPL $(\boldsymbol{\alpha}, \theta)$ distribution and let $X_{1:n}, \ldots, X_{n:n}$ denote the corresponding order statistics in ascending order.

4.1. Moments, mgf and order statistics

Standard calculations show that the moment generating function (mgf) of X defined by $M_X(t) = E\left(e^{tX}\right)$, the rth moment of X defined by $\mu_r' = E\left(X^r\right)$, the density function of the ith order statistic $X_{i:n}$ and the rth moment of the ith order statistic can be expressed as

$$M_X(t) = \frac{\theta^2 (1+\theta)^{-1}}{1+3\theta+\theta^2} \sum_{j=0}^{\infty} \binom{2+j}{2} (1+\theta)^{-j} \left[(3+\theta) I(j) - I(j+1)\right],$$

$$\mu_r' = \frac{\theta^2 (1+\theta)^{-1}}{1+3\theta+\theta^2} K\left(\boldsymbol{\alpha}, \theta, r\right),$$

$$\begin{aligned} f_{i:n}(x) &= \frac{n! \theta^{2(n-i+1)} (1+\theta)^2 g(x) \overline{G}^{n-i}(x)}{(i-1)!(n-i)! \left(1+3\theta+\theta^2\right)^{n-i+1}} \frac{3+\theta-\overline{G}(x)}{\left[1+\theta-\overline{G}(x)\right]^{2(n-i)+3}} \\ &\quad \cdot \left\{1+\theta+(2+\theta)\left[1+\theta-\overline{G}(x)\right]\right\}^{n-i} \\ &\quad \cdot \left\{1 - \frac{\theta^2 \overline{G}(x)}{1+3\theta+\theta^2} \frac{1+\theta+(2+\theta)\left[1+\theta-\overline{G}(x)\right]}{\left[1+\theta-\overline{G}(x)\right]^2}\right\}^{i-1} \end{aligned}$$

and

$$\begin{aligned} E\left(X_{i:n}^r\right) &= r \sum_{k=n-i+1}^{n} \sum_{l=0}^{k} \sum_{j=0}^{\infty} (-1)^{k-n+i+l-1} \binom{k}{l} \binom{k-1}{n-i} \binom{n}{k} \binom{j+2k-1}{2k-1} \\ &\quad \cdot \frac{\theta^{2k} (3+\theta)^{k-l} (2+\theta)^l}{(1+\theta)^{l+k+j} \left(1+3\theta+\theta^2\right)^k} \int_0^{\infty} x^{r-1} \overline{G}^{k+l+j}(x)\, \mathrm{d}x, \end{aligned}$$

where

$$I(a) = \sum_{j=0}^{\infty} \frac{t^j}{j!} M(j, 0, a),$$

where $M(p, r, s) = E\left[X^p G^r(X) \overline{G}^s(X)\right]$ is the probability weighted moment (Hosking [8]) with expectation taken with respect to g. Note that we have used Lemma 1 in the appendix to calculate the rth moment of X.

4.2. Moments of (reversed) residual life and conditional moments

The residual life and the reversed residual life random variables play an important role in reliability theory (Gupta and Gupta [7]) and the moments of such variables are extensively used in actuarial sciences in the analysis of risks. For lifetime distributions, it is also of interest to know the conditional moments $E\left(X^n|X>t\right)$ $n=1,2,\ldots$ which are important in prediction. Standard calculations show that these moments for the GPL$(\boldsymbol{\alpha},\theta)$ distribution can be expressed as

$$\begin{aligned}
\mu_r(t) &= E\left[(X-t)^r|X>t\right] \\
&= \frac{1}{S(t)}\frac{\theta^2}{(1+\theta)\left(1+3\theta+\theta^2\right)}\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}t^{r-i}L\left(\boldsymbol{\alpha},\theta,i,t\right),
\end{aligned}$$

$$\begin{aligned}
m_r(t) &= E\left[(t-X)^r|X\le t\right] \\
&= \frac{1}{F(t)}\sum_{i=0}^{r}\binom{r}{i}t^{r-i}(-1)^i\left[\mu_i'-\frac{\theta^2}{(1+\theta)\left(1+3\theta+\theta^2\right)}L\left(\boldsymbol{\alpha},\theta,i,t\right)\right]
\end{aligned}$$

and

$$E\left(X^n|X>t\right)=\frac{\theta^2}{S(t)}\frac{(1+\theta)^{-1}}{1+3\theta+\theta^2}L\left(\boldsymbol{\alpha},\theta,n,t\right),$$

where $F(\cdot)$ is given by (1) and $S(\cdot)$ is given by (4). We have used Lemma 2 in the appendix for these calculations.

4.3. Rényi entropy

Entropy is regarded as a measure of the randomness of a system and it is widely used in physical sciences. A popular entropy is the Rényi entropy (Rényi [14]) defined by $I_R(\gamma)=(1-\gamma)^{-1}\log\left(\int_0^\infty f^\gamma(x)\,\mathrm{d}x\right)$ for $\gamma>0$ and $\gamma\neq 1$. If $f(\cdot)$ is the density function of the GPL $(\boldsymbol{\alpha},\theta)$ distribution then standard calculations show that the Rényi entropy can be expressed as

$$\begin{aligned}
I_R(\gamma) &= \gamma(1-\gamma)^{-1}\left[2\log\theta-\log\left(1+3\theta+\theta^2\right)-\log(1+\theta)\right] \\
&\quad +(1+\gamma)(1-\gamma)^{-1}\log(3+\theta)+(1-\gamma)^{-1}\left[\log c_\gamma-\log\Gamma(3\gamma)\right],
\end{aligned}$$

where

$$\begin{aligned}
c_\gamma &\equiv c_{\gamma,\boldsymbol{\alpha},\theta} \\
&= \sum_{j=0}^{\infty}\frac{\Gamma(3\gamma+j)}{j!}\left(\frac{3+\theta}{1+\theta}\right)^j\int_0^{(3+\theta)^{-1}} g^{\gamma-1}\left(G^{-1}\left(1-u(3+\theta)\right)\right)u^j(1-u)^\gamma\,\mathrm{d}u
\end{aligned}$$

and $\Gamma(\cdot)$ denotes the gamma function.

5. PARAMETER ESTIMATION

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the GPL$(\boldsymbol{\alpha}, \theta)$ distribution with observed values $x_1, x_2, \ldots, x_n$. Section 5.1 considers maximum likelihood estimation for the parameters $(\boldsymbol{\alpha}, \theta)$ and gives an expression for the associated Fisher's information matrix. Section 5.2 considers maximum likelihood estimation when some data values are censored. Throughout, we let p denote the length of $\boldsymbol{\alpha}$.

5.1. Maximum likelihood estimation

Given the observed values $x_1, x_2, \ldots, x_n$, the loglikelihood function of the parameters $(\boldsymbol{\alpha}, \theta)$ is

$$\begin{aligned} \ell_n \quad \equiv \quad & \ell_n(\boldsymbol{\alpha}, \theta) = 2n \log \theta(1+\theta) + \sum_{i=1}^{n} \log g(x_i) - n \log\left(1 + 3\theta + \theta^2\right) \\ & + \sum_{i=1}^{n} \log\left[3 + \theta - \overline{G}(x_i)\right] - 3 \sum_{i=1}^{n} \log\left[1 + \theta - \overline{G}(x_i)\right]. \end{aligned} \tag{6}$$

The score function associated with the loglikelihood function is

$$\mathbf{U}_n \equiv \mathbf{U}_n(\boldsymbol{\alpha}, \theta) = \left(\frac{\partial \ell_n}{\partial \boldsymbol{\alpha}}, \frac{\partial \ell_n}{\partial \theta}\right)^{\top},$$

where

$$\frac{\partial \ell_n}{\partial \boldsymbol{\alpha}} = \sum_{i=1}^{n} \frac{1}{g(x_i)} \frac{\partial g(x_i)}{\partial \boldsymbol{\alpha}} + \sum_{i=1}^{n} \frac{\partial G(x_i)}{\partial \boldsymbol{\alpha}} \left[\frac{1}{3 + \theta - \overline{G}(x_i)} - \frac{3}{1 + \theta - \overline{G}(x_i)}\right]$$

and

$$\frac{\partial \ell_n}{\partial \theta} = \frac{2n(1+2\theta)}{\theta(1+\theta)} - \frac{n(3+2\theta)}{1+3\theta+\theta^2} + \sum_{i=1}^{n} \left[\frac{1}{3 + \theta - \overline{G}(x_i)} - \frac{3}{1 + \theta - \overline{G}(x_i)}\right].$$

The maximum likelihood estimates of $\boldsymbol{\alpha}$ and θ say $\widehat{\boldsymbol{\alpha}}$ and $\widehat{\theta}$ can be obtained by solving numerically the nonlinear system of equations $\mathbf{U}_n = \mathbf{0}$. In practice, the maximum likelihood estimates can be obtained more easily by maximizing (6) with respect to the parameters. The $(p+1) \times (p+1)$ Fisher's information matrix is given by

$$\mathbf{K}_n \equiv \mathbf{K}_n(\boldsymbol{\alpha}, \theta) = n \left[\begin{array}{cc} k_{\boldsymbol{\alpha},\boldsymbol{\alpha}} & k_{\boldsymbol{\alpha},\theta} \\ k_{\boldsymbol{\alpha},\theta} & k_{\theta,\theta} \end{array}\right],$$

where

$$
\begin{aligned}
k_{\theta,\theta} &= \frac{2\left(1+2\theta+2\theta^2\right)}{\theta^2(1+\theta)^2} - \frac{7+6\theta+2\theta^2}{\left(1+3\theta+\theta^2\right)^2} \\
&\quad -3E\left(\frac{1}{\left[1+\theta-\overline{G}(X)\right]^2}\right) + E\left(\frac{1}{\left[3+\theta-\overline{G}(X)\right]^2}\right), \\
k_{\boldsymbol{\alpha},\boldsymbol{\alpha}} &= E\left(\frac{1}{g^2(X)}\left[\frac{\partial g(X)}{\partial \boldsymbol{\alpha}}\right]^2\right) - E\left(\frac{1}{g(X)}\frac{\partial^2 g(X)}{\partial \boldsymbol{\alpha}^2}\right) \\
&\quad -E\left(\frac{\partial^2 G(X)}{\partial \boldsymbol{\alpha}^2}\left[\frac{1}{3+\theta-\overline{G}(X)} - \frac{3}{1+\theta-\overline{G}(X)}\right]\right) \\
&\quad -E\left(\left[\frac{\partial G(X)}{\partial \boldsymbol{\alpha}}\right]^2\left\{\frac{3}{\left[1+\theta-\overline{G}(X)\right]^2} - \frac{1}{\left[3+\theta-\overline{G}(X)\right]^2}\right\}\right), \\
k_{\boldsymbol{\alpha},\theta} &= E\left(\frac{\partial G(X)}{\partial \boldsymbol{\alpha}}\left\{\frac{3}{\left[1+\theta-\overline{G}(X)\right]^2} - \frac{1}{\left[3+\theta-\overline{G}(X)\right]^2}\right\}\right).
\end{aligned}
$$

Tests of hypothesis and confidence intervals for the parameters can be based on the Fisher's information matrix. Namely, they can be based on the fact that the distribution of $\sqrt{n}\left(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha},\widehat{\theta}-\theta\right)$ for large n is approximately $(p+1)$-variate normal with zero means and covariance matrix $\mathbf{K}_n^{-1}=\left(K^{i,j}\right)$ say. So, for example, an approximate $100(1-\beta)$ percent confidence interval for θ is

$$
\left(\widehat{\theta}-z_{\beta/2}\sqrt{\frac{K^{p+1,p+1}}{n}},\widehat{\theta}+z_{\beta/2}\sqrt{\frac{K^{p+1,p+1}}{n}}\right),
$$

where z_a denotes the $100(1-a)$ percentile of a standard normal random variable. Also, an approximate test of $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ with significance level a is to reject the null hypothesis if $\sqrt{n}\left|\widehat{\theta}-\theta_0\right|/\sqrt{K^{p+1,p+1}}>z_{a/2}$.

5.2. Censored maximum likelihood estimation

Often with lifetime data, we encounter censored data. There are different forms of censoring: type I censoring, type II censoring, etc. Here, we consider the general case of multi-censored data: there are n subjects of which

- n_0 are known to have failed at the times $x_1,\ldots,x_{n_0}$;
- n_1 are known to have failed in the interval $[s_{j-1},s_j]$, $j=1,\ldots,n_1$;
- n_2 survived to a time r_j, $j=1,\ldots,n_2$ but not observed any longer.

Note that $n=n_0+n_1+n_2$ and that type I censoring and type II censoring are contained as particular cases of multi-censoring. The loglikelihood function of the parameters $(\boldsymbol{\alpha},\theta)$

for this multi-censoring data is:

$$\begin{aligned}
\ell_n \equiv\ & 2n \log\theta + 2n_0 \log(1+\theta) - n \log\left(1+3\theta+\theta^2\right) + \sum_{i=1}^{n_0} \log g\left(x_i\right) \\
& + \sum_{i=1}^{n_0} \log\left[3+\theta-\overline{G}\left(x_i\right)\right] - 3\sum_{i=1}^{n_0} \log\left[1+\theta-\overline{G}\left(x_i\right)\right] \\
& + \sum_{i=1}^{n_1} \log\left[\frac{\overline{G}\left(s_{i-1}\right)}{\left[1+\theta-\overline{G}\left(s_{i-1}\right)\right]^2}\left\{1+\theta+(2+\theta)\left[1+\theta-\overline{G}\left(s_{i-1}\right)\right]\right\}\right. \\
& \qquad \left. - \frac{\overline{G}\left(s_i\right)}{\left[1+\theta-\overline{G}\left(s_i\right)\right]^2}\left\{1+\theta+(2+\theta)\left[1+\theta-\overline{G}\left(s_i\right)\right]\right\}\right] \\
& + \sum_{i=1}^{n_2} \log\overline{G}\left(r_i\right) + \sum_{i=1}^{n_2} \log\left\{1+\theta+(2+\theta)\left[1+\theta-\overline{G}\left(r_i\right)\right]\right\} \\
& -2\sum_{i=1}^{n_2} \log\left[1+\theta-\overline{G}\left(r_i\right)\right]. \qquad (7)
\end{aligned}$$

The score function associated with the loglikelihood function is

$$\mathbf{U}_n \equiv \left(\frac{\partial \ell_n}{\partial \boldsymbol{\alpha}}, \frac{\partial \ell_n}{\partial \theta}\right)^{\top},$$

where

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \boldsymbol{\alpha}} =\ & \sum_{i=1}^{n_0} \frac{1}{g\left(x_i\right)} \frac{\partial g\left(x_i\right)}{\partial \boldsymbol{\alpha}} + \sum_{i=1}^{n_0} \frac{\partial G\left(x_i\right)}{\partial \boldsymbol{\alpha}}\left[\frac{1}{3+\theta-\overline{G}\left(x_i\right)} - \frac{3}{1+\theta-\overline{G}\left(x_i\right)}\right] \\
& - \sum_{i=1}^{n_1} \frac{\partial G\left(r_i\right)}{\partial \boldsymbol{\alpha}}\left[\frac{\overline{G}\left(s_{i-1}\right)}{\left[1+\theta-\overline{G}\left(s_{i-1}\right)\right]^2}\left\{1+\theta+(2+\theta)\left[1+\theta-\overline{G}\left(s_{i-1}\right)\right]\right\}\right. \\
& \qquad \left. - \frac{\overline{G}\left(s_i\right)}{\left[1+\theta-\overline{G}\left(s_i\right)\right]^2}\left\{1+\theta+(2+\theta)\left[1+\theta-\overline{G}\left(s_i\right)\right]\right\}\right]^{-1} \\
& \qquad \cdot\left\{\frac{(1+\theta)(3+\theta)-2(2+\theta)\overline{G}\left(s_{i-1}\right)}{\left[1+\theta-\overline{G}\left(s_{i-1}\right)\right]^2}\right. \\
& \qquad + \frac{2\overline{G}\left(s_{i-1}\right)\left[(1+\theta)(3+\theta)-(2+\theta)\overline{G}\left(s_{i-1}\right)\right]}{\left[1+\theta-\overline{G}\left(s_{i-1}\right)\right]^3} \\
& \qquad - \frac{(1+\theta)(3+\theta)-2(2+\theta)\overline{G}\left(s_i\right)}{\left[1+\theta-\overline{G}\left(s_i\right)\right]^2} \\
& \qquad \left. - \frac{2\overline{G}\left(s_i\right)\left[(1+\theta)(3+\theta)-(2+\theta)\overline{G}\left(s_i\right)\right]}{\left[1+\theta-\overline{G}\left(s_i\right)\right]^3}\right\}
\end{aligned}$$

$$-\sum_{i=1}^{n_2} \frac{\partial G(r_i)}{\partial \boldsymbol{\alpha}} \left\{ \frac{1}{\overline{G}(r_i)} - \frac{2+\theta}{1+\theta+(2+\theta)\left[1+\theta-\overline{G}(r_i)\right]} \right\}$$
$$-2\sum_{i=1}^{n_2} \frac{\partial G(r_i)}{\partial \boldsymbol{\alpha}} \frac{1}{1+\theta-\overline{G}(r_i)}$$

and

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \theta} &= \frac{2n}{\theta} + \frac{2n_0}{1+\theta} - \frac{n(3+2\theta)}{1+3\theta+\theta^2} + \sum_{i=1}^{n_0} \left[\frac{1}{3+\theta-\overline{G}(x_i)} - \frac{3}{1+\theta-\overline{G}(x_i)} \right] \\
&+ \sum_{i=1}^{n_1} \left[\frac{\overline{G}(s_{i-1})}{\left[1+\theta-\overline{G}(s_{i-1})\right]^2} \left\{1+\theta+(2+\theta)\left[1+\theta-\overline{G}(s_{i-1})\right]\right\} \right. \\
&\qquad \left. - \frac{\overline{G}(s_i)}{\left[1+\theta-\overline{G}(s_i)\right]^2} \left\{1+\theta+(2+\theta)\left[1+\theta-\overline{G}(s_i)\right]\right\} \right]^{-1} \\
&\qquad \cdot \left\{ \frac{\overline{G}(s_{i-1})\left[4+2\theta-\overline{G}(s_{i-1})\right]}{\left[1+\theta-\overline{G}(s_{i-1})\right]^2} \right. \\
&\qquad - \frac{2\left[(1+\theta)(3+\theta)-(2+\theta)\overline{G}(s_{i-1})\right]}{\left[1+\theta-\overline{G}(s_{i-1})\right]^3} \\
&\qquad - \frac{\overline{G}(s_i)\left[4+2\theta-\overline{G}(s_i)\right]}{\left[1+\theta-\overline{G}(s_i)\right]^2} \\
&\qquad \left. + \frac{2\left[(1+\theta)(3+\theta)-(2+\theta)\overline{G}(s_i)\right]}{\left[1+\theta-\overline{G}(s_i)\right]^3} \right\} \\
&+ \sum_{i=1}^{n_2} \frac{4+2\theta-G(r_i)}{(1+\theta)(3+\theta)-(2+\theta)\overline{G}(r_i)} - \sum_{i=1}^{n_2} \frac{1}{1+\theta-\overline{G}(r_i)}.
\end{aligned}$$

The maximum likelihood estimates of $\boldsymbol{\alpha}$ and θ can be obtained by solving numerically the nonlinear system of equations $\mathbf{U}_n = \mathbf{0}$ or alternatively by maximizing (7) with respect to the parameters. The Fisher's information matrix corresponding to (7) is too complicated to be presented here.

6. SPECIAL CASES

In this section, we investigate in detail some special cases of the GPL($\boldsymbol{\alpha}, \theta$) distribution, including the Weibull Poisson–Lindley (WPL) distribution, the Burr Poisson–Lindley (BPL) distribution, the exponentiated Weibull Poisson–Lindley (EWPL) distribution and the Dagum Poisson–Lindley (DPL) distribution. Some mathematical properties as well as plots of the density and failure rate functions are presented for each special case.

6.1. The Weibull Poisson–Lindley distribution

Weibull distribution is the most popular model in reliability and related areas. Its survival function is

$$\overline{G}(x) = e^{-\lambda x^{\beta}} \tag{8}$$

for $x > 0$, where $\lambda > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. The shape parameter can be interpreted as follows: $\beta < 1$ corresponds to the "infant mortality" period of systems when the failure rate decreases; $\beta = 1$ corresponds to the "constant failure" period of systems; $\beta > 1$ corresponds to the "aging process" of systems when the failure rate increases.

Substituting the density, failure rate and quantile functions corresponding to (8) into (3), (5) and (2), we obtain the density, failure rate and quantile functions of the WPL distribution. The exponential Poisson–Lindley distribution due to Baretto-Souza and Bakouch [3] is the particular case of this distribution for $\beta = 1$.

Since the density function $g(x)$ and failure rate function $h_g(x)$ of the Weibull distribution are decreasing functions for $\beta \leq 1$, $f(x)$ and $h_f(x)$ of the WPL distribution are decreasing functions too for $\beta \leq 1$ (by Remarks 1 and 2). Figures 1 and 2 illustrate the shapes of the density and failure rate functions of the WPL distribution for different values of β, λ, θ. They show that the failure rate function of the WPL distribution can be decreasing, increasing, unimodal shaped and unimodal shaped followed by a bathtub shape. Also, we note that $\lim_{x \to 0^+} f(x) = \infty$ for $\beta \leq 1$ and $\lim_{x \to 0^+} f(x) = 0$ for $\beta > 1$. Moreover, the density function of the WPL distribution is decreasing for $\beta > 1$ and $x \geq \left(\frac{\beta - 1}{\lambda\beta}\right)^{\frac{1}{\beta}}$.

Standard calculations show that the mgf, the rth moment, the density function of the i order statistic and the rth moment of the ith order statistic of the WPL distribution are

$$\begin{aligned} M_X(t) &= \frac{\theta^2(1+\theta)^{-1}}{1+3\theta+\theta^2} \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{t^n \lambda^{-\frac{n}{\beta}} \Gamma(n/\beta+1)}{2n!(1+\theta)^j} \\ &\quad \cdot \left\{ \frac{(3+\theta)(j+2)^{n/\beta+1} - (j+1)^{n/\beta+1}}{\left[(j+1)(j+2)\right]^{n/\beta}} \right\}, \end{aligned}$$

$$\mu_r' = E\left(X^r\right) = \frac{\theta^2(1+\theta)^{-1}}{1+3\theta+\theta^2} K(\beta, \lambda, \theta, r),$$

$$\begin{aligned} f_{i:n}(x) &= \frac{\beta\lambda n!\theta^{2(n-i+1)}(1+\theta)^2 x^{\beta-1} e^{-\lambda(n-i+1)x^{\beta}}}{(i-1)!(n-i)!\left(1+3\theta+\theta^2\right)^{n-i+1}} \frac{3+\theta-e^{-\lambda x^{\beta}}}{\left(1+\theta-e^{-\lambda x^{\beta}}\right)^{2(n-i)+3}} \\ &\quad \cdot \left[1+\theta+(2+\theta)\left(1+\theta-e^{-\lambda x^{\beta}}\right)\right]^{n-i} \\ &\quad \cdot \left[1 - \frac{\theta^2 e^{-\lambda x^{\beta}}}{1+3\theta+\theta^2} \frac{1+\theta+(2+\theta)\left(1+\theta-e^{-\lambda x^{\beta}}\right)}{\left(1+\theta-e^{-\lambda x^{\beta}}\right)^2}\right]^{i-1}, \end{aligned}$$

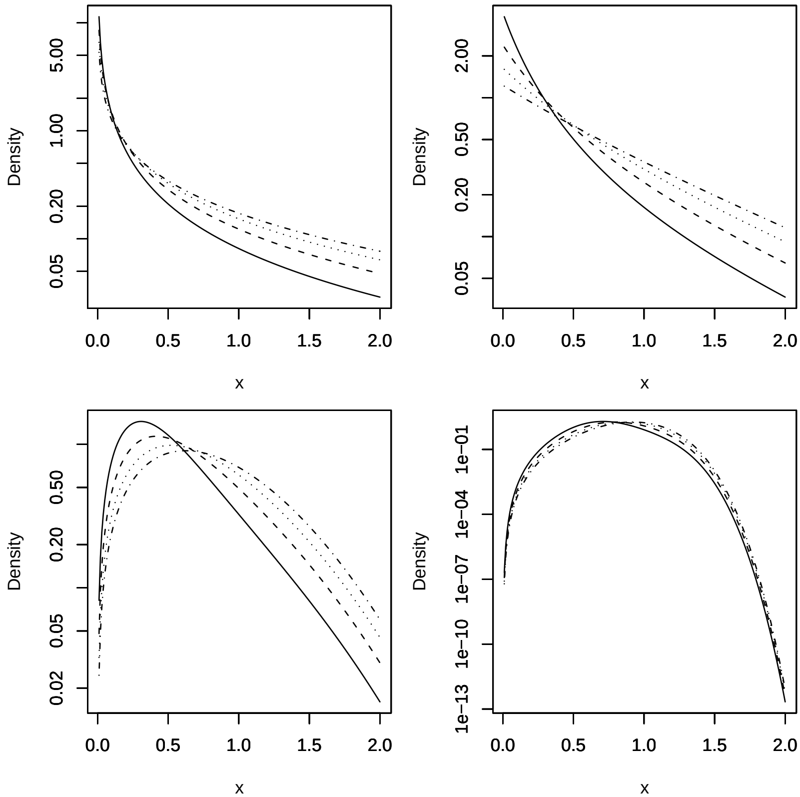

Fig. 1. Plots of the density function of the WPL distribution for $\beta = 0.5$, $\lambda = 1$ (top left), $\beta = 1$, $\lambda = 1$ (top right), $\beta = 2$, $\lambda = 1$ (bottom left), and $\beta = 5$, $\lambda = 1$ (bottom right). The four curves in each plot correspond to $\theta = 0.5$ (solid curve), $\theta = 1$ (curve of dashes), $\theta = 2$ (curve of dots) and $\theta = 5$ (curve of dots and dashes). The y axes are in log scale.

and

$$\begin{aligned} E\left(X_{i:n}^{r}\right) &= \frac{r\Gamma\left(\frac{r}{\beta}\right)}{\beta\lambda^{\frac{r}{\beta}}} \sum_{k=n-i+1}^{n} \sum_{l=0}^{k} \sum_{j=0}^{\infty} (-1)^{k-n+i+l-1} \binom{k}{l} \binom{k-1}{n-i} \\ &\cdot \binom{n}{k} \binom{j+2k-1}{2k-1} \frac{\theta^{2k}(3+\theta)^{k-l}(2+\theta)^{l}}{(1+\theta)^{l+k+j}\left(1+3\theta+\theta^{2}\right)^{k}(k+l+j)^{\frac{r}{\beta}}}, \end{aligned}$$

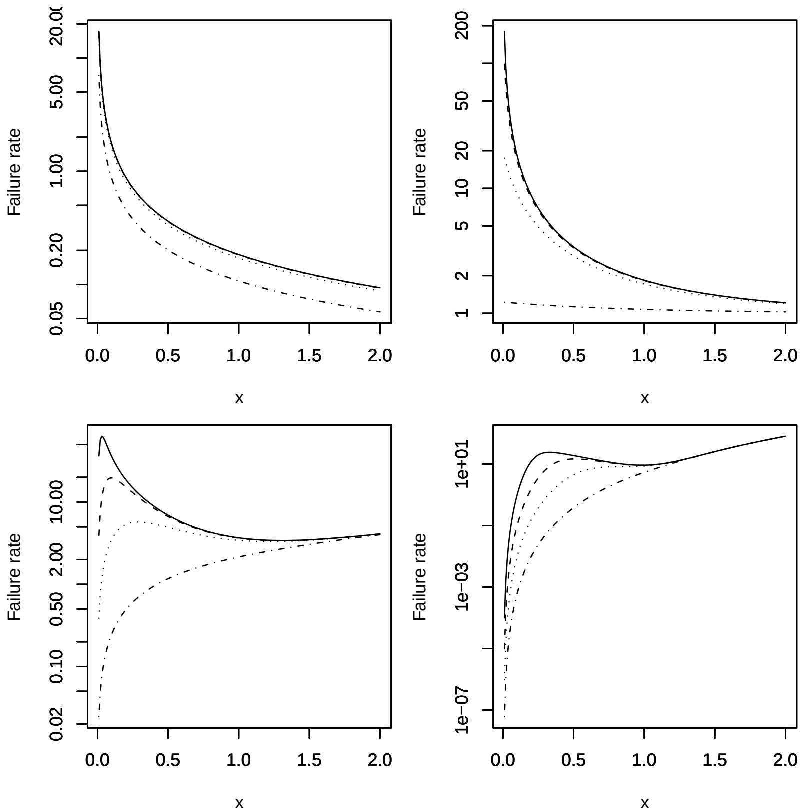

Fig. 2. Plots of the failure rate function of the WPL distribution for $\beta = 0.1$, $\lambda = 1$ (top left), $\beta = 1$, $\lambda = 1$ (top right), $\beta = 2$, $\lambda = 1$ (bottom left), and $\beta = 5$, $\lambda = 1$ (bottom right). The four curves in each plot correspond to $\theta = 0.001$ (solid curve), $\theta = 0.01$ (curve of dashes), $\theta = 0.1$ (curve of dots) and $\theta = 5$ (curve of dots and dashes). The y axes are in log scale.

where

$$K(\beta, \lambda, \theta, c) = \lambda^{-\frac{c}{\beta}} \Gamma\left(\frac{c}{\beta} + 1\right) \sum_{j=0}^{\infty} \frac{(1+\theta)^{-j}}{2} \left\{ \frac{(3+\theta)(j+2)^{\frac{c}{\beta}+1} - (j+1)^{\frac{c}{\beta}+1}}{[(j+1)(j+2)]^{\frac{c}{\beta}}} \right\}.$$

Moments of residual life, moments of reversed residual life and conditional moments of the WPL distribution are

$$\mu_r(t) = \frac{1}{S(t)} \frac{\theta^2}{(1+\theta)\left(1+3\theta+\theta^2\right)} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} t^{r-i} L(\beta, \lambda, \theta, i, t),$$

$$m_r(t) = \frac{1}{F(t)} \sum_{i=0}^{r} \binom{r}{i} t^{r-i} (-1)^i \left[\mu_i' - \frac{\theta^2}{(1+\theta)\left(1+3\theta+\theta^2\right)} L(\beta, \lambda, \theta, i, t) \right],$$

and

$$E\left(X^n | X > t\right) = \frac{\theta^2}{S(t)} \frac{(1+\theta)^{-1}}{1+3\theta+\theta^2} L(\beta, \lambda, \theta, n, t),$$

where

$$\begin{aligned} L(\beta, \lambda, \theta, c, t) &= \lambda^{-\frac{c}{\beta}} \sum_{j=0}^{\infty} \binom{j+2}{2} (1+\theta)^{-j} \\ &\cdot \left[(3+\theta)(j+1)^{-\left(\frac{c}{\beta}+1\right)} \Gamma\left(\frac{c}{\beta}+1, \lambda(j+1)t^\beta\right) \right. \\ &\left. -(j+2)^{-\left(\frac{c}{\beta}+1\right)} \Gamma\left(\frac{c}{\beta}+1, \lambda(j+2)t^\beta\right) \right] \end{aligned}$$

and $\Gamma(\cdot, \cdot)$ denotes the complementary incomplete gamma function.

Finally, the Rényi entropy of the WPL distribution is

$$\begin{aligned} I_R(\gamma) &= \gamma(1-\gamma)^{-1} \left[2\log\theta - \log\left(1+3\theta+\theta^2\right) - \log(1+\theta)\right] \\ &+ (1+\gamma)(1-\gamma)^{-1} \log(3+\theta) + (1-\gamma)^{-1} \left[\log c_\gamma - \log\Gamma(3\gamma)\right], \end{aligned}$$

where

$$c_\gamma \equiv c_{\gamma,\beta,\lambda,\theta} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{\gamma}{k} \frac{(-1)^k \Gamma(3\gamma+j)}{j!} \frac{(3+\theta)^{-(\gamma+k)} \Gamma\left(\gamma - \frac{\gamma-1}{\beta}\right)}{(1+\theta)^j (k+\gamma+j)^{\gamma - \frac{\gamma-1}{\beta}}}.$$

6.2. The Burr Poisson–Lindley distribution

The Burr distribution is specified by the survival function

$$\overline{G}(x) = \left(1 + x^\lambda\right)^{-\beta} \tag{9}$$

for $x > 0$, where both $\beta > 0$ and $\lambda > 0$ are shape parameters. This is a heavy tailed distribution. The upper tails of the distribution become heavier with decreasing values of both parameters. The failure rate function is a monotonic decreasing function for $\lambda \le 1$ and is unimodal for $\lambda > 1$.

Substituting the density, failure rate and quantile functions corresponding to (9) into (3), (5) and (2), we obtain the density, failure rate and quantile functions of the BPL distribution.

Since the density function of the Burr distribution is decreasing for $\lambda \le 1$, the density function of the BPL distribution is decreasing too for $\lambda \le 1$. Since the density function of the Burr distribution is unimodal for $\lambda > 1$, the density function of the BPL distribution is decreasing for $\lambda > 1$ and $x > \left(\frac{\lambda-1}{\beta\lambda+1}\right)^{\frac{1}{\lambda}}$. Also, since the failure rate function of the Burr distribution is decreasing for $\lambda \leq 1$, that of the BPL distribution is decreasing too for $\lambda \leq 1$ (by Remark 2). Figures 3 and 4 illustrate the shapes of the density and failure rate functions of the BPL distribution for different values of β, λ, and θ. The figures show that the failure rate function can be decreasing, increasing, unimodal shaped and unimodal shaped followed by a bathtub shape and then followed by a decreasing shape.

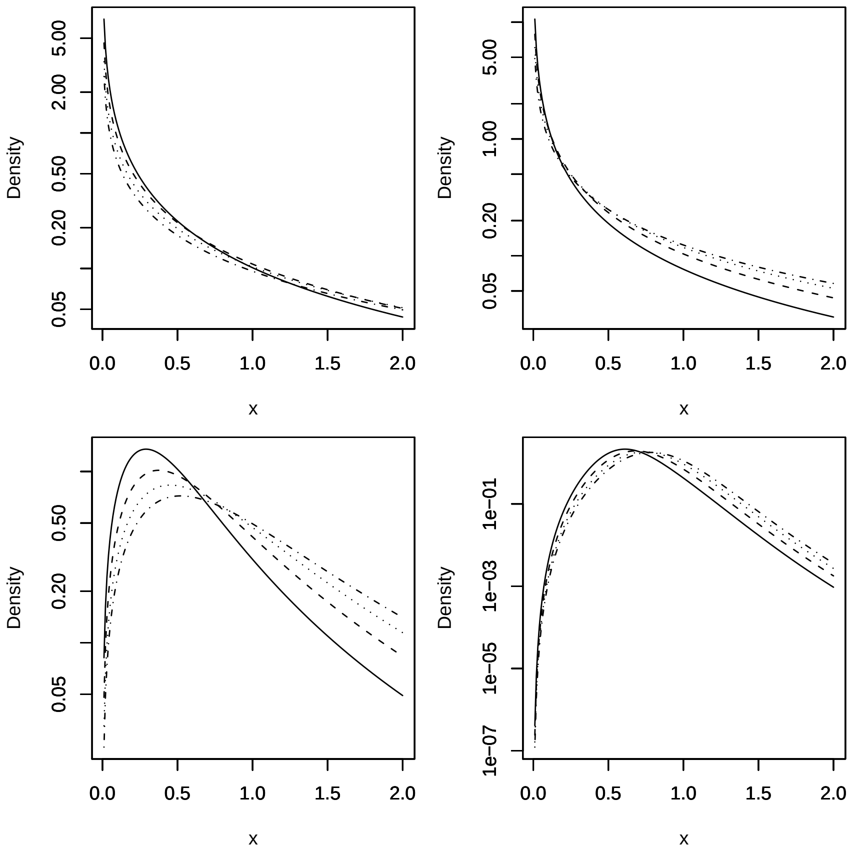

Fig. 3. Plots of the density function of the BPL distribution for $\beta = 0.5$, $\lambda = 0.5$ (top left), $\beta = 1$, $\lambda = 0.5$ (top right), $\beta = 1$, $\lambda = 2$ (bottom left), and $\beta = 2$, $\lambda = 5$ (bottom right). The four curves in each plot correspond to $\theta = 0.5$ (solid curve), $\theta = 1$ (curve of dashes), $\theta = 2$ (curve of dots) and $\theta = 5$ (curve of dots and dashes). The y axes are in log scale.

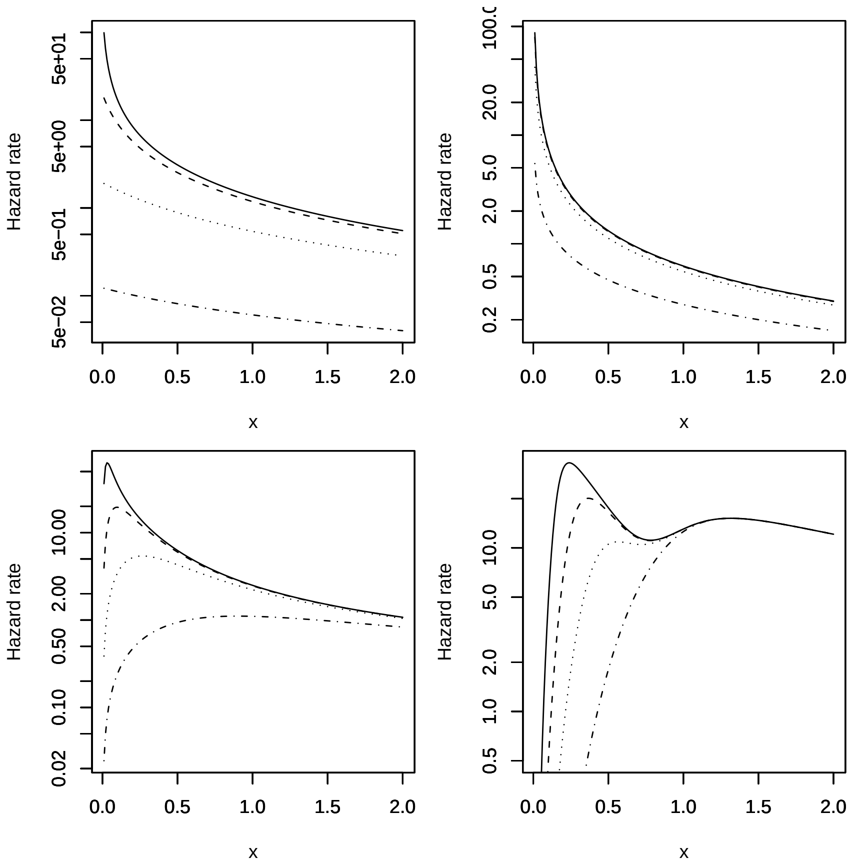

Fig. 4. Plots of the failure rate function of the BPL distribution for $\beta = 0.1$, $\lambda = 1$ (top left), $\beta = 1$, $\lambda = 0.5$ (top right), $\beta = 1$, $\lambda = 2$ (bottom left), and $\beta = 5$, $\lambda = 5$ (bottom right). The four curves in each plot correspond to $\theta = 0.001$ (solid curve), $\theta = 0.01$ (curve of dashes), $\theta = 0.1$ (curve of dots) and $\theta = 5$ (curve of dots and dashes). The y axes are in log scale.

Standard calculations show that the mgf, the rth moment, the density function of the i order statistic and the rth moment of the ith order statistic of the BPL distribution are

$$M_X(t) = \frac{\theta^2(1+\theta)^{-1}}{1+3\theta+\theta^2} \sum_{j=0}^{\infty} \binom{2+j}{2} (1+\theta)^{-j} \left[(3+\theta)I(j) - I(j+1)\right],$$

$$\mu_r' = \frac{\theta^2(1+\theta)^{-1}}{1+3\theta+\theta^2} K(\beta, \lambda, \theta, r),$$

$$\begin{aligned} f_{i:n}(x) &= \frac{\beta\lambda n!\theta^{2(n-i+1)}(1+\theta)^2 x^{\lambda-1}(1+x^\lambda)^{-[\beta(n-i+1)+1]}}{(i-1)!(n-i)!\left(1+3\theta+\theta^2\right)^{n-i+1}} \\ &\quad \cdot \frac{3+\theta-(1+x^\lambda)^{-\beta}}{[1+\theta-(1+x^\lambda)^{-\beta}]^{2(n-i)+3}} \\ &\quad \cdot \left\{1+\theta+(2+\theta)\left[1+\theta-(1+x^\lambda)^{-\beta}\right]\right\}^{n-i} \\ &\quad \cdot \left\{1-\frac{\theta^2(1+x^\lambda)^{-\beta}}{1+3\theta+\theta^2}\frac{1+\theta+(2+\theta)\left[1+\theta-(1+x^\lambda)^{-\beta}\right]}{[1+\theta-(1+x^\lambda)^{-\beta}]^2}\right\}^{i-1}, \end{aligned}$$

and

$$\begin{aligned} E\left(X_{i:n}^r\right) &= \frac{r}{\lambda}\sum_{k=n-i+1}^{n}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{k-n+i+l-1}\binom{k}{l}\binom{k-1}{n-i}\binom{n}{k}\binom{j+2k-1}{2k-1} \\ &\quad \cdot \frac{\theta^{2k}(3+\theta)^{k-l}(2+\theta)^l}{(1+\theta)^{l+k+j}\left(1+3\theta+\theta^2\right)^k}\mathbf{B}\left((k+l+j)\beta-\frac{r}{\lambda},\frac{r}{\lambda}\right), \end{aligned}$$

where

$$I(a)=\beta\lambda\int_0^\infty x^{\lambda-1}(1+x^\lambda)^{-(\beta+\beta a+1)}e^{tx}\,\mathrm{d}x,$$

$$K(\beta,\lambda,\theta,c)=\beta\sum_{j=0}^{\infty}\binom{j+2}{2}(1+\theta)^{-j}\left[(3+\theta)I(\beta,\lambda,c,j)-I(\beta,\lambda,c,j+1)\right],$$

$$I(\beta,\lambda,c,b)=\beta\mathbf{B}\left(\beta(1+b)-\frac{c}{\lambda},\frac{c}{\lambda}+1\right)=\frac{\beta\Gamma\left(\beta(1+b)-\frac{c}{\lambda}\right)\Gamma\left(\frac{c}{\lambda}+1\right)}{\Gamma\left(\beta(1+b)+1\right)}$$

and $\mathbf{B}(\cdot,\cdot)$ denotes the beta function. Note that the expression for $E\left(X_{i:n}^r\right)$ holds only for those i such that $r<(n-i+1)\lambda\beta$.

Finally, the Rényi entropy of the BPL distribution is

$$\begin{aligned} I_R(\gamma) &= \gamma(1-\gamma)^{-1}\left[2\log\theta-\log\left(1+3\theta+\theta^2\right)-\log(1+\theta)\right] \\ &\quad +(1+\gamma)(1-\gamma)^{-1}\log(3+\theta)+(1-\gamma)^{-1}\left[\log c_\gamma-\log\Gamma(3\gamma)\right], \end{aligned}$$

where

$$\begin{aligned} c_\gamma &\equiv \sum_{j=0}^{\infty}\frac{\Gamma(3\gamma+j)}{j!}\left(\frac{3+\theta}{1+\theta}\right)^j(\lambda\beta)^{\gamma-1}(3+\theta)^{\frac{(\gamma-1)(\beta+1)}{\beta}} \\ &\quad \cdot\int_0^{(3+\theta)^{-1}}u^{\frac{(\gamma-1)(\beta+1)}{\beta}+j}(1-u)^\gamma\left\{[u(3+\theta)]^{-\frac{1}{\beta}}-1\right\}^{\frac{(\lambda-1)(\gamma-1)}{\lambda}}\,\mathrm{d}u. \end{aligned}$$

6.3. Other special cases

Here, we discuss briefly two more special cases of the GPL$(\boldsymbol{\alpha},\theta)$ distribution.

6.3.1. Exponentiated Weibull Poisson–Lindley distribution

The survival function of the exponentiated Weibull (EW) distribution (Mudholkar and Srivastava [13]) is

$$\overline{G}(x) = 1 - \left(1 - e^{-x^{\lambda}}\right)^{\beta} \tag{10}$$

for $x > 0$, where both $\lambda > 0$ and $\beta > 0$ are shape parameters. These parameters control the failure rates in different ways: $c < 1$ and $c\alpha > 1$ correspond to bathtub shaped failure rates with a unique change point; $c > 1$ and $c\alpha < 1$ correspond to unimodal failure rates with a unique change point; $c \leq 1$ and $c\alpha \leq 1$ correspond to monotonically

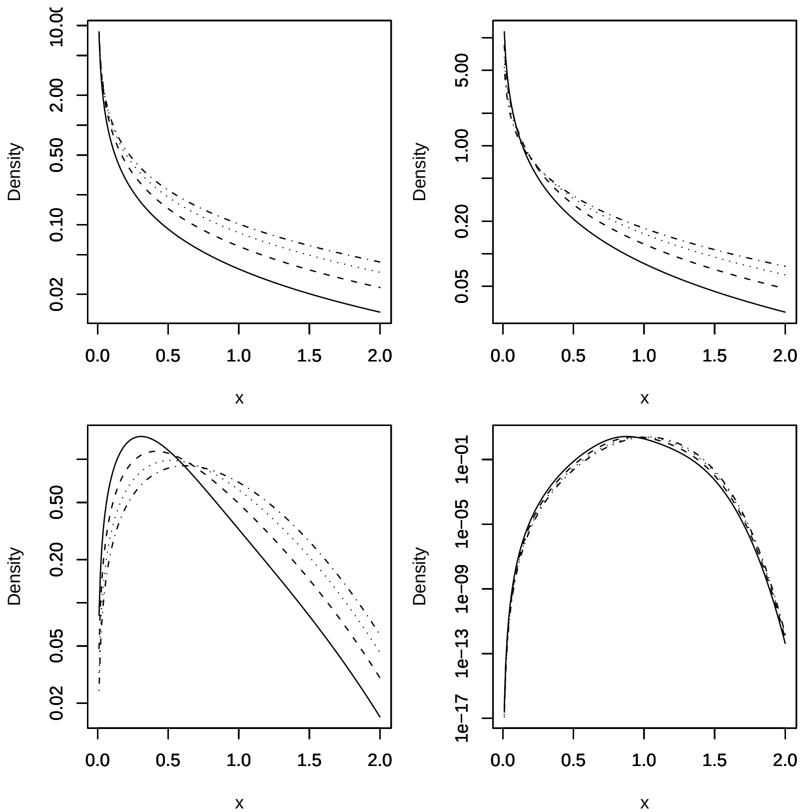

Fig. 5. Plots of the density function of the EWPL distribution for $\beta = 0.5$, $\lambda = 0.5$ (top left), $\beta = 1$, $\lambda = 0.5$ (top right), $\beta = 1$, $\lambda = 2$ (bottom left), and $\beta = 2$, $\lambda = 5$ (bottom right). The four curves in each plot correspond to $\theta = 0.5$ (solid curve), $\theta = 1$ (curve of dashes), $\theta = 2$ (curve of dots) and $\theta = 5$ (curve of dots and dashes). The y axes are in log scale.

increasing failure rates; $c \geq 1$ and $c\alpha \geq 1$ correspond to monotonically decreasing failure rates; $c = \alpha = 1$ corresponds to constant failure rates.

Substituting the density, failure rate and quantile functions corresponding to (10) into (3), (5) and (2), we obtain the density, failure rate and quantile functions of the EWPL distribution.

Figures 5 and 6 illustrate the shapes of the density and failure rate functions of the EWPL distribution for different values of β, λ and θ. The failure rate function can be decreasing, increasing, unimodal shaped followed by an increasing shape, unimodal shaped followed by a bathtub shape, and bathtub shaped.

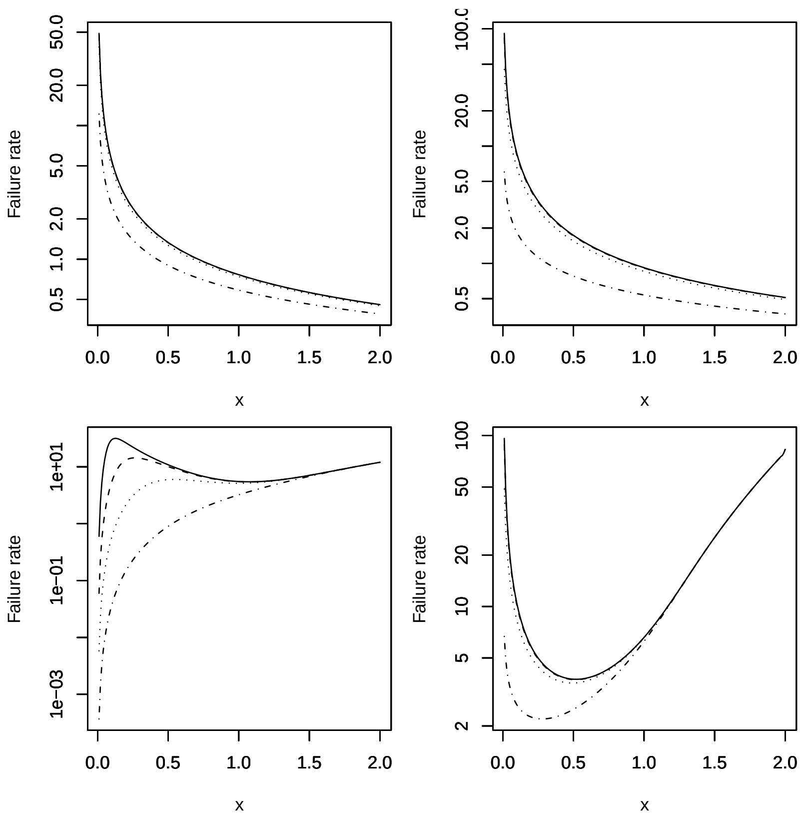

Fig. 6. Plots of the failure rate function of the EWPL distribution for $\beta = 0.5$, $\lambda = 0.5$ (top left), $\beta = 1$, $\lambda = 0.5$ (top right), $\beta = 1$, $\lambda = 3$ (bottom left), and $\beta = 0.1$, $\lambda = 5$ (bottom right). The four curves in each plot correspond to $\theta = 0.001$ (solid curve), $\theta = 0.01$ (curve of dashes), $\theta = 0.1$ (curve of dots) and $\theta = 5$ (curve of dots and dashes). The y axes are in log scale.

6.3.2. Dagum Poisson–Lindley distribution

The survival function of the Dagum (inverse Burr type XII) distribution is

$$\overline{G}(x) = 1 - \left(1 + x^{-\lambda}\right)^{-\beta} \tag{11}$$

for $x > 0$, where both $\lambda > 0$ and $\beta > 0$ are shape parameters. This too is a heavy tailed distribution. The upper tails of the distribution become heavier with decreasing values of both parameters. The failure rate function is a monotonic decreasing function for $\lambda \leq 1$ and is unimodal for $\lambda > 1$.

Substituting the density, failure rate and quantile functions corresponding to (11) into (3), (5) and (2), we obtain the density, failure rate and quantile functions of the DPL distribution.

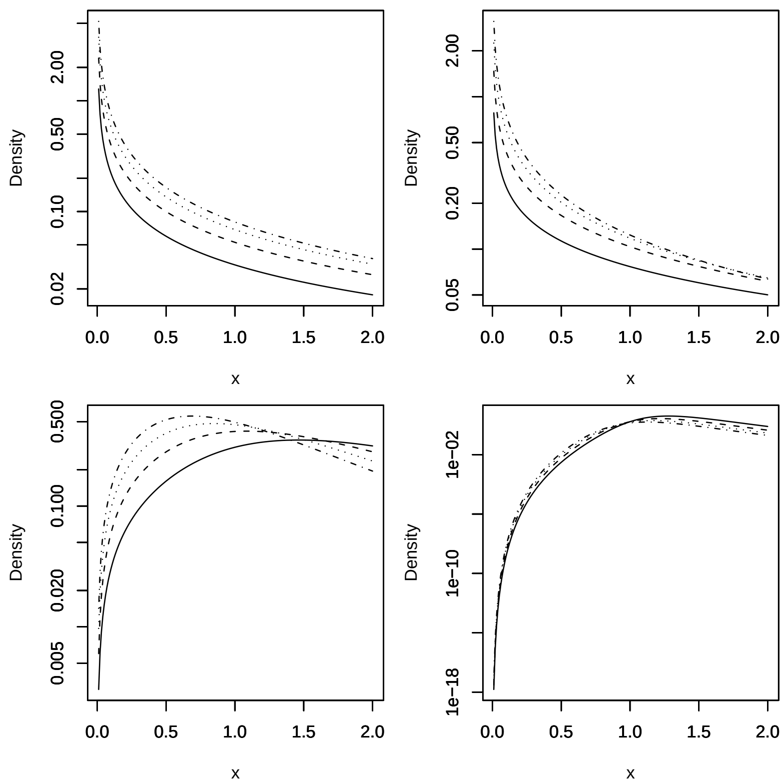

Fig. 7. Plots of the density function of the DPL distribution for $\beta = 0.5$, $\lambda = 0.5$ (top left), $\beta = 1$, $\lambda = 0.5$ (top right), $\beta = 1$, $\lambda = 2$ (bottom left), and $\beta = 2$, $\lambda = 5$ (bottom right). The four curves in each plot correspond to $\theta = 0.5$ (solid curve), $\theta = 1$ (curve of dashes), $\theta = 2$ (curve of dots) and $\theta = 5$ (curve of dots and dashes). The y axes are in log scale.

Figures 7 and 8 illustrate the shapes of the density and failure rate functions of the DPL distribution for different values of β, λ, and θ. We see that the failure rate function can be decreasing, increasing and unimodal shaped. Since the density function of the Dagum distribution is a decreasing function for $\beta\lambda \leq 1$, the density function of the DPL distribution is also a decreasing function for $\beta\lambda \leq 1$ by Remark 1. Since the density function of the Dagum distribution is unimodal for $\beta\lambda > 1$, the density function of the DPL distribution is decreasing for $\beta\lambda > 1$ and $x > \left(\frac{1+\lambda}{\beta\lambda-1}\right)^{-\frac{1}{\lambda}}$.

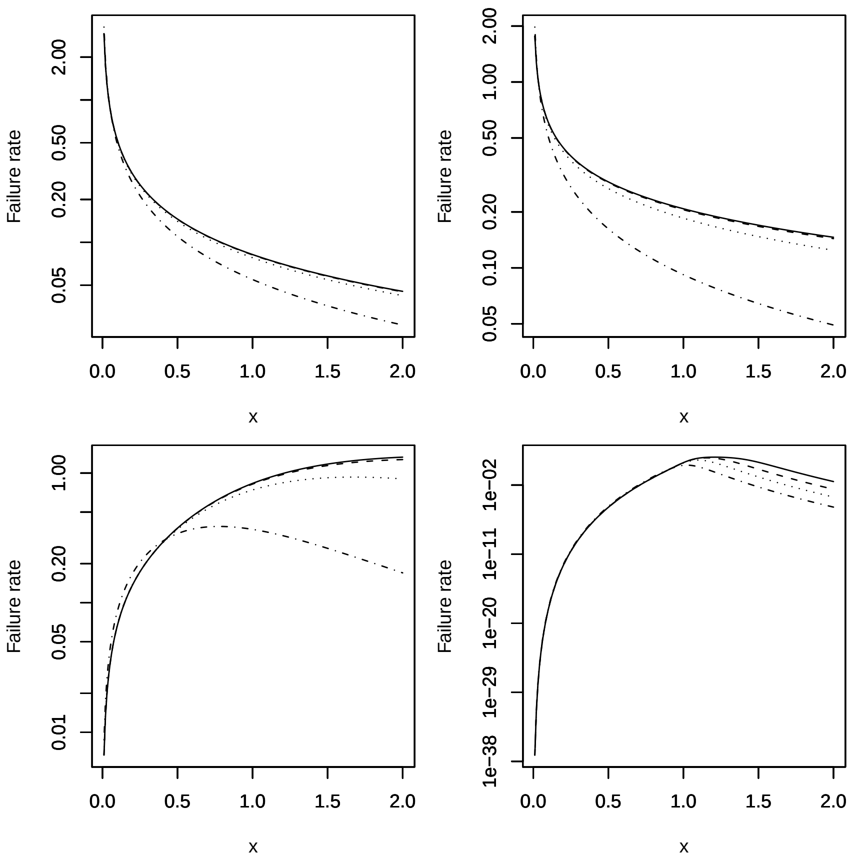

Fig. 8. Plots of the failure rate function of the DPL distribution for $\beta = 0.5$, $\lambda = 0.5$ (top left), $\beta = 1$, $\lambda = 0.5$ (top right), $\beta = 1$, $\lambda = 2$ (bottom left), and $\beta = 1$, $\lambda = 20$ (bottom right). The four curves in each plot correspond to $\theta = 0.001$ (solid curve), $\theta = 0.01$ (curve of dashes), $\theta = 0.1$ (curve of dots) and $\theta = 5$ (curve of dots and dashes). The y axes are in log scale.

7. APPLICATION TO REAL DATA

In this section, we illustrate the flexibility of the GPL distributions. We use the Danish fire insurance claims data downloaded from

http://www.macs.hw.ac.uk/~mcneil/ftp/DanishData.txt

This data set gives insurance claims (exceeding one million DKK) due to fire from the 3rd of January 1980 to the 31 of December of 1990. This data must be considered censored because of the following explanation: "The full Danish data comprise 2492 losses and can be considered as being essentially all Danish fire losses over one million Danish Krone (DKK) from 1980 to 1990 plus a number of smaller losses below one million DKK. We restrict our attention to the 2156 losses exceeding one million ..." (page 121, McNeil [11]).

Following the motivation described in Section 1, we computed the minimum insurance claim for every six month period from the 3rd of January 1980 to the 31 of December of 1990. This resulted in the values 1.464129, 1.449488, 1.314548, 1.310616, 1.189061, 1.189061, 1.112347, 1.112347, 1.046911, 1.047120, 1.000000, 1.000000, 1.000000, 1.002893, 1.011132, 1.004638, 1.027507, 1.020408, 1.003387, 1.003387, 1.006601, 1.005776, 1.005776. The sample size is $n = 23$. According to the motivation described in Section 1, GPL distributions can be considered as possible models for these data values.

We fitted the Weibull and WPL distributions mentioned in Section 6.1, the Burr and BPL distributions mentioned in Section 6.2, the EW and EWPL distributions mentioned in Section 6.3.1, and the Dagum and DPL distributions mentioned in Section 6.3.2. In addition, we fitted the following distributions: the Weibull Poisson (WP) distribution (Lu and Shi [10]) with the density function

$$f(x) = \frac{\lambda \beta \alpha x^{\beta - 1} e^{-\alpha - \lambda x^{\beta} + \alpha \exp\left(-\lambda x^{\beta}\right)}}{1 - e^{-\alpha}}$$

for $x > 0$, $\alpha > 0$, $\beta > 0$ and $\lambda > 0$; the lognormal distribution with the density function

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma x} \exp\left[-\frac{(\log x - \mu)^2}{2\sigma^2}\right]$$

for $x > 0$, $-\infty < \mu < \infty$ and $\sigma > 0$; the loglogistic distribution with the density function

$$f(x) = \frac{\beta \alpha^{\beta} x^{\beta - 1}}{\left(x^{\beta} + \alpha^{\beta}\right)^2}$$

for $x > 0$, $\alpha > 0$ and $\beta > 0$.

Each distribution was fitted by the method of maximum likelihood. Censoring was accounted for in the maximum likelihood estimation, see Section 5.2. The maximum likelihood estimates, the standard errors computed by inverting observed information matrices, loglikelihood values, values of Akaike information criterion (AIC), values of Bayesian information criterion (BIC) and p-values based on the one-sample Kolmogorov Smirnov statistics are reported in Table 1.

Distribution	Estimates (se)	$-\log L$	AIC	BIC	p-value
Weibull	$\widehat{\beta} = 6.969(1.016)$, $\widehat{\lambda} = 0.337(0.106)$	-8.833	-13.666	-11.395	0.009
WP	$\widehat{\beta} = 8.875(1.291)$, $\widehat{\lambda} = 0.101(0.056)$, $\widehat{\alpha} = 3.329(1.715)$	-10.765	-15.530	-12.123	0.010
WPL	$\widehat{\beta} = 11.520(1.856)$, $\widehat{\lambda} = 0.022(0.023)$, $\widehat{\theta} = 0.116(0.117)$	-12.613	-19.226	-15.819	0.020
Burr	$\widehat{\beta} = 0.006(0.005)$, $\widehat{\lambda} = 1861.673(294.152)$	-28.596	-53.193	-50.922	0.051
BPL	$\widehat{\beta} = 0.006(0.007)$, $\widehat{\lambda} = 1861.673(542.945)$, $\widehat{\theta} = 0.760(0.165)$	-31.769	-57.538	-54.132	0.062
EW	$\widehat{\beta} = 3.233(0.679)$, $\widehat{\lambda} = 4.695(0.528)$	-11.599	-19.199	-16.928	0.014
EWPL	$\widehat{\beta} = 7.588(1.707)$, $\widehat{\lambda} = 3.850(0.788)$, $\widehat{\theta} = 0.200(0.152)$	-15.821	-25.642	-22.236	0.036
Dagum	$\widehat{\beta} = 2.356(0.500)$, $\widehat{\lambda} = 14.809(2.669)$	-16.707	-29.414	-27.143	0.044
DPL	$\widehat{\beta} = 4.293(1.089)$, $\widehat{\lambda} = 11.791(3.096)$, $\widehat{\theta} = 0.409(0.340)$	-18.593	-31.187	-27.780	0.045
Lognormal	$\widehat{\mu} = 0.089(0.025)$, $\widehat{\sigma} = 0.121(0.018)$	-13.832	-23.665	-21.394	0.024
Loglogistic	$\widehat{\alpha} = 1.068(0.025)$, $\widehat{\beta} = 15.395(2.789)$	-14.309	-24.618	-22.347	0.024

Tab. 1. Estimates of the parameters, standard errors, loglikelihood values, AIC values, BIC values and p-values based on Kolmogorov Smirnov statistics.

For computing the Kolmogorov Smirnov statistics we do not know the true value of the parameters. We replace them by the maximum likelihood estimates, so the standard asymptotes of the Kolmogorov Smirnov statistics may not apply. For this reason, we used the following scheme for computing the p-values:

1. simulate ten thousand samples of size $n = 23$ from the fitted distribution (one of Weibull, WPL, Burr, BPL, EW, EWPL, Dagum, DPL, WP, lognormal, loglogistic distributions);
2. refit the fitted distribution to each of the ten thousand samples;
3. use the fitted estimates to compute the Kolmogorov Smirnov statistic for each of the ten thousand samples;
4. compute the empirical distribution function of the Kolmogorov Smirnov statistic;
5. use the empirical distribution function to read off the p-value of the observed Kolmogorov Smirnov statistic.

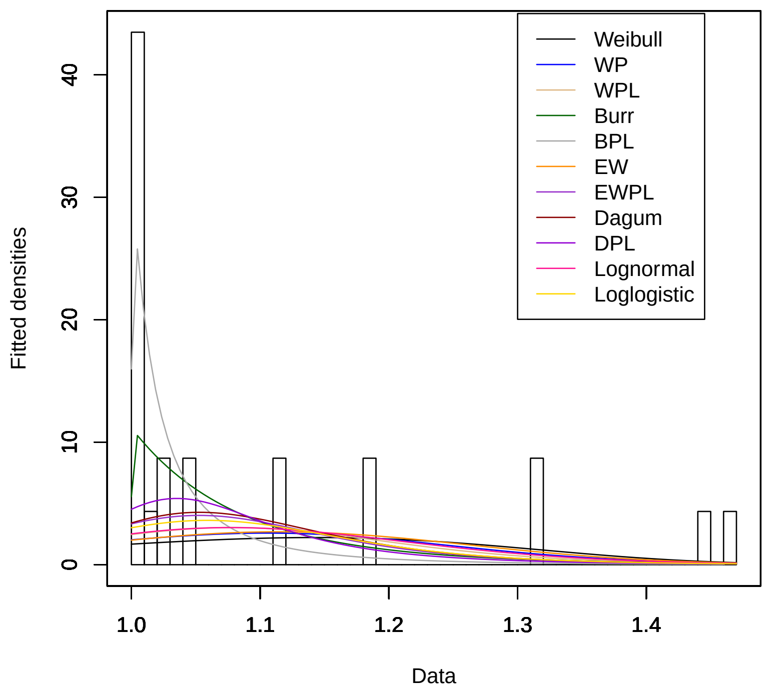

Fig. 9. Histogram of the data set and the fitted density functions of the Weibull, WP, WPL, Burr, BPL, EW, EWPL, Dagum, DPL, lognormal and loglogistic distributions.

We see that the BPL distribution has the largest loglikelihood value, the smallest AIC value, the smallest BIC value and the largest p-value. The Burr distribution has the second largest loglikelihood value, the second smallest AIC value, the second smallest BIC value and the second largest p-value.

The remaining distributions do not appear to provide acceptable p-values. The Weibull distribution has the smallest loglikelihood value, the largest AIC value, the largest BIC value and the smallest p-value. The WP distribution has the second smallest loglikelihood value, the second largest AIC value, the second largest BIC value and the second smallest p-value.

The flexibility of GPL distributions over the corresponding G distributions can be verified by means of likelihood ratio tests. Comparing the loglikelihood values of the Weibull distribution ($\log L = 8.833$) and the WPL distribution ($\log L = 12.613$), we see that the latter provides a significantly better fit. Comparing the loglikelihood values of

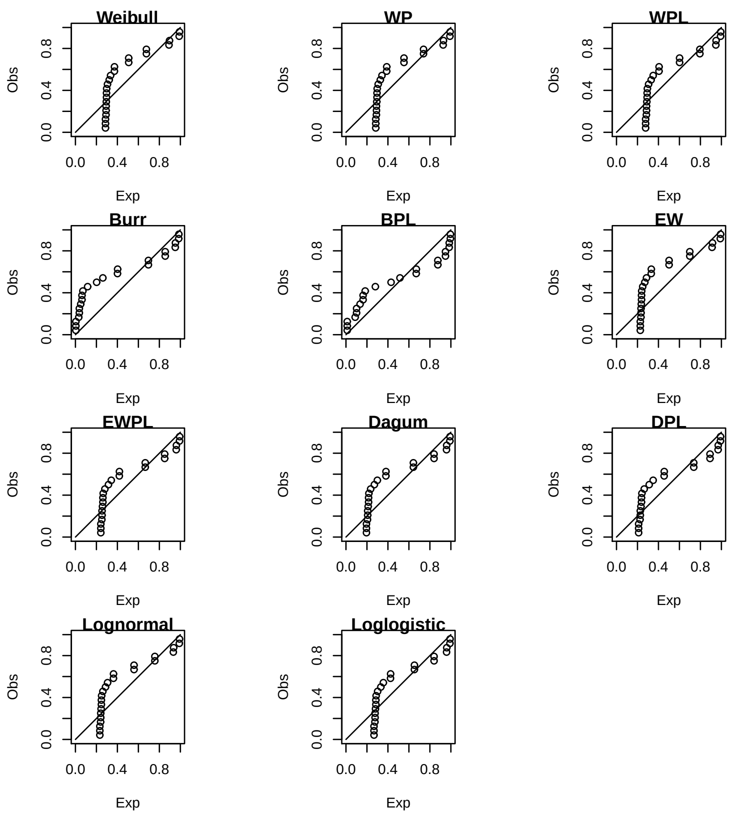

Fig. 10. Probability plots for the fits of the Weibull, WP, WPL, Burr, BPL, EW, EWPL, Dagum, DPL, lognormal and loglogistic distributions.

the Burr distribution ($\log L = 28.596$) and the BPL distribution ($\log L = 31.769$), we see that the latter provides a significantly better fit. Comparing the loglikelihood values of the EW distribution ($\log L = 11.599$) and the EWPL distribution ($\log L = 15.821$), we see that the latter provides a significantly better fit. Comparing the loglikelihood values of the Dagum distribution ($\log L = 16.707$) and the DPL distribution ($\log L = 18.593$), we see that the fits are not significantly different but only marginally.

Based on the loglikelihood values, the AIC values, the BIC values and the p-values, we can say that the BPL distribution provides the best fit. This is confirmed by the density plots and probability plots shown in Figures 9 and 10. We see from these figures that only the Burr and BPL distributions capture the lower tail of the data well.

APPENDIX

This appendix introduces two lemmas.

Lemma 1. Let

$$K\left(\boldsymbol{\alpha},\theta,c\right)=\int_0^\infty \frac{x^c g(x)\left[3+\theta-\overline{G}(x)\right]}{\left[1-(1+\theta)^{-1}\overline{G}(x)\right]^3}\,\mathrm{d}x.$$

Then, we have

$$K\left(\boldsymbol{\alpha},\theta,c\right)=\sum_{j=0}^\infty \binom{j+2}{2}(1+\theta)^{-j}\left[(3+\theta)I\left(\boldsymbol{\alpha},c,j\right)-I\left(\boldsymbol{\alpha},c,j+1\right)\right],$$

where

$$I\left(\boldsymbol{\alpha},c,b\right)=\int_0^\infty x^c g(x)\overline{G}(x)^b\,\mathrm{d}x=M(c,0,b).$$

P r o o f. The result follows by writing

$$\begin{aligned}K\left(\boldsymbol{\alpha},\theta,c\right) &= (3+\theta)\sum_{j=0}^\infty \binom{j+2}{2}(1+\theta)^{-j}\int_0^\infty x^c g(x)\overline{G}^j(x)\,\mathrm{d}x \\ &\quad -\sum_{j=0}^\infty \binom{j+2}{2}(1+\theta)^{-j}\int_0^\infty x^c g(x)\overline{G}^{j+1}(x)\,\mathrm{d}x.\end{aligned}$$

The proof is complete. □

Lemma 2. Let

$$L\left(\boldsymbol{\alpha},\theta,c,t\right)=\int_t^\infty \frac{x^c g(x)\left[3+\theta-\overline{G}(x)\right]}{\left[1-(1+\theta)^{-1}\overline{G}(x)\right]^3}\,\mathrm{d}x.$$

Then, we have

$$L(\boldsymbol{\alpha},\theta,c,t)=\sum_{j=0}^{\infty}\binom{j+2}{2}(1+\theta)^{-j}\left[(3+\theta)J(\boldsymbol{\alpha},c,j,t)-J(\boldsymbol{\alpha},c,j+1,t)\right],$$

where

$$J(\boldsymbol{\alpha},c,b,t)=\int_t^{\infty}x^c g(x)\overline{G}^b(x)\,\mathrm{d}x=\int_{G(t)}^1\left[G^{-1}(u)\right]^c(1-u)^b\,\mathrm{d}u.$$

Proof. The result follows by writing

$$\begin{aligned} L(\boldsymbol{\alpha},\theta,c,t) &= (3+\theta)\sum_{j=0}^{\infty}\binom{j+2}{2}(1+\theta)^{-j}\int_t^{\infty}x^c g(x)\overline{G}^j(x)\,\mathrm{d}x \\ &\quad -\sum_{j=0}^{\infty}\binom{j+2}{2}(1+\theta)^{-j}\int_t^{\infty}x^c g(x)\overline{G}^{j+1}(x)\,\mathrm{d}x. \end{aligned}$$

The proof is complete. □

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