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CONTINUUM SPECTRUM FOR THE LINEARIZED EXTREMAL EIGENVALUE PROBLEM WITH BOUNDARY REACTIONS

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Abstract. We study the semilinear problem with the boundary reaction

$$-\Delta u + u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \lambda f(u) \quad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a smooth bounded domain, $f\colon [0,\infty) \to (0,\infty)$ is a smooth, strictly positive, convex, increasing function which is superlinear at ∞, and $\lambda > 0$ is a parameter. It is known that there exists an extremal parameter $\lambda^* > 0$ such that a classical minimal solution exists for $\lambda < \lambda^*$, and there is no solution for $\lambda > \lambda^*$. Moreover, there is a unique weak solution u^* corresponding to the parameter $\lambda = \lambda^*$. In this paper, we continue to study the spectral properties of u^* and show a phenomenon of continuum spectrum for the corresponding linearized eigenvalue problem.

Keywords: continuum spectrum; extremal solution; boundary reaction

MSC 2010: 35J25, 35J20

1. Introduction

In this paper, we consider the boundary value problem with the boundary reaction

$$-\Delta u + u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = \lambda f(u) \quad \text{on } \partial\Omega \tag{1.1}$$

where $\lambda > 0$ and $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$ is a smooth bounded domain. Throughout the paper, we assume

$$f\colon [0,\infty) \to (0,\infty) \text{ is smooth, convex, increasing, } f(0) > 0, \tag{1.2}$$

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and superlinear at ∞ in the sense that

$$\lim_{t\to\infty}\frac{f(t)}{t}=\infty. \tag{1.3}$$

Then the maximum principle implies that solutions are positive on $\overline{\Omega}$.

It is known that there exists an extremal parameter $\lambda^*\in(0,\infty)$ such that

(i) for every $\lambda\in(0,\lambda^*)$, $(1.1)_\lambda$ has a positive, classical, minimal solution $u_\lambda\in C^2(\overline{\Omega})$ which is strictly stable in the sense that

$$\int_\Omega(|\nabla\varphi|^2+\varphi^2)\,\mathrm{d}x>\lambda\int_{\partial\Omega}f'(u_\lambda)\varphi^2\,\mathrm{d}s_x \tag{1.4}$$

for every $\varphi\in C^1(\overline{\Omega})$, $\varphi\not\equiv 0$,

(ii) for $\lambda=\lambda^*$, the pointwise limit

$$u^*(x)=\lim_{\lambda\uparrow\lambda^*}u_\lambda(x),\quad x\in\overline{\Omega}, \tag{1.5}$$

becomes a weak solution of $(1.1)_{\lambda^*}$,

(iii) for $\lambda>\lambda^*$, there exists no solution of $(1.1)_\lambda$, not even in the weak sense.

Here, we call a function $u=(u_1,u_2)\in L^1(\Omega)\times L^1(\partial\Omega)$ a *weak solution* to $(1.1)_\lambda$ if $f(u_2)\in L^1(\partial\Omega)$ and

$$\int_\Omega(-\Delta\zeta+\zeta)u_1\,\mathrm{d}x=\int_{\partial\Omega}\Bigl(\lambda f(u_2)\zeta-\frac{\partial\zeta}{\partial\nu}u_2\Bigr)\,\mathrm{d}s_x \tag{1.6}$$

holds for any $\zeta\in C^2(\overline{\Omega})$. The statement (ii) says that, under the assumption (1.3), $u^*=(u^*|_\Omega,u^*|_{\partial\Omega})$ is a weak solution in the above sense. We proved in [10] Theorem 11 that $u^*\in W^{1,\gamma}(\Omega)$ for any $\gamma\in[1,N/(N-2))$ when $N\geqslant 3$ (for any $\gamma\in[1,\infty)$ when $N=2$), so $u^*|_{\partial\Omega}\in W^{1-1/\gamma,\gamma}(\partial\Omega)\subset L^{(N-1)\gamma/(N-\gamma)}(\partial\Omega)$ is the usual trace of the $W^{1,\gamma}$ function u^* on $\partial\Omega$. For the facts (ii), (iii), we refer the reader to [10]. In the following, we call u^* the *extremal solution* of (1.1). In [10], the author obtained several properties such as regularity and uniqueness of the extremal solution u^*. This paper is a sequel to [10]. For related elliptic problems with boundary reaction terms, see, e.g., [4], [6], [9]. For a well-studied problem

$$-\Delta u=\lambda f(u)\quad\text{in }\Omega,\qquad u=0\quad\text{on }\partial\Omega$$

where f satisfies (1.2), (1.3), see [1], [2], [3], [5], [7], [8], and the references therein.

For $\lambda\in(0,\lambda^*)$, we denote by $\mu_1(\lambda f'(u_\lambda))$ the first eigenvalue of the eigenvalue problem

$$-\Delta\varphi+\varphi=0\quad\text{in }\Omega,\qquad\frac{\partial\varphi}{\partial\nu}=\lambda f'(u_\lambda)\varphi+\mu\varphi\quad\text{on }\partial\Omega.$$

By the variational characterization, we have

$$\mu_1(\lambda f'(u_\lambda)) = \inf_{\varphi\in C^1(\overline{\Omega}),\, \varphi\not\equiv 0} \frac{\int_\Omega(|\nabla\varphi|^2+\varphi^2)\,\mathrm{d}x - \int_{\partial\Omega}\lambda f'(u_\lambda)\varphi^2\,\mathrm{d}s_x}{\int_{\partial\Omega}\varphi^2\,\mathrm{d}s_x}.$$

Note that $\mu_1(\lambda f'(u_\lambda)) > 0$ since the minimal solution u_λ is strictly stable, and decreases as $\lambda \uparrow \lambda^*$. Denote

$$\mu_1^* = \lim_{\lambda\uparrow\lambda^*} \mu_1(\lambda f'(u_\lambda)). \tag{1.7}$$

If u^* is classical, it must hold that $\mu_1^* = 0$ by considering (iii) above. However, if $u^* = (u^*|_\Omega, u^*|_{\partial\Omega}) \notin L^\infty(\Omega)\times L^\infty(\partial\Omega)$, it could happen that μ_1^* is positive. In [10], we proved that even when $\mu_1^* > 0$, there exists a nonnegative weak solution of

$$-\Delta\varphi+\varphi = 0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial\nu} = \lambda^* f'(u^*)\varphi + \mu\varphi \quad \text{on } \partial\Omega \tag{1.8}$$

for $\mu = 0$. This is a phenomenon of the existence of (L^1-)zero eigenvalue for the eigenvalue problem (1.8). The main purpose of this paper is to prove the following result, which is a generalization of the result by Cabré and Martel [3] to our setting, and may be seen as a phenomenon of the existence of (L^1-)continuum spectrum for the eigenvalue problem (1.8).

Theorem 1.1. *Let μ_1^* be defined by* (1.7). *Then for any $\mu \in [0,\mu_1^*]$ there exists a weak solution φ to* (1.8), *$\varphi \in W^{1,q}(\Omega)$ $(1 \leqslant q < N/(N-1))$, $\varphi \geqslant 0$, in the sense that $f'(u^*)\varphi|_{\partial\Omega} \in L^1(\partial\Omega)$ and*

$$\int_\Omega(-\Delta\zeta+\zeta)\varphi\,\mathrm{d}x = \int_{\partial\Omega}\Big\{(\lambda^* f'(u^*)\varphi|_{\partial\Omega} + \mu\varphi|_{\partial\Omega})\zeta - \frac{\partial\zeta}{\partial\nu}\varphi|_{\partial\Omega}\Big\}\,\mathrm{d}s_x$$

for all $\zeta \in C^2(\overline{\Omega})$. Here $\varphi|_{\partial\Omega}$ is the usual trace of $\varphi \in W^{1,q}(\Omega)$.

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. We need the uniqueness theorem from [10], which is an analogue of the result by Y. Martel [8].

Theorem 2.1 ([10], Theorem 14). *Assume* $(1.1)_{\lambda^*}$ *has a weak supersolution* $w = (w_1, w_2) \in L^1(\Omega) \times L^1(\partial\Omega)$, *in the sense that* $f(w_2) \in L^1(\partial\Omega)$ *and*

$$\int_\Omega (-\Delta\zeta + \zeta) w_1 \,\mathrm{d}x \geqslant \int_{\partial\Omega} \Big\{ \lambda^* f(w_2)\zeta - \frac{\partial\zeta}{\partial\nu} w_2 \Big\} \,\mathrm{d}s_x$$

for any $\zeta \in C^2(\overline{\Omega})$, $\zeta \geqslant 0$ *on* $\overline{\Omega}$. *Then* $(w_1, w_2) = (u^*|_\Omega, u^*|_{\partial\Omega})$, *where* u^* *is defined by* (1.5).

The following is Lemma 17 in [10].

Lemma 2.2. *Let* $\{u_n\} \subset C^2(\overline{\Omega})$ *be a sequence of functions such that*

$$-\Delta u_n + u_n = 0 \quad \text{in } \Omega, \qquad \frac{\partial u_n}{\partial\nu} \geqslant 0 \quad \text{on } \partial\Omega.$$

Assume $\|u_n\|_{L^1(\partial\Omega)} \leqslant C$ *for some* $C > 0$ *independent of* n. *Then there exists a subsequence (denoted again by* u_n*) and* $u \in W^{1,q}(\Omega)$ *such that*

$$u_n \rightharpoonup u \quad \text{weakly in } W^{1,q}(\Omega),\ 1 < q < \frac{N}{N-1}.$$

Moreover, for any $1 \leqslant p < (N-1)/(N-2)$ *there exists a constant* $C_p > 0$ *depending only on* p *such that*

$$\|u_n\|_{L^p(\partial\Omega)} \leqslant C_p \|u_n\|_{L^1(\partial\Omega)} \quad \text{for any } n \in \mathbb{N}.$$

Now, we prove Theorem 1.1.

Proof. We follow the argument by X. Cabré and Y. Martel [3].

Step 1. For $n \in \mathbb{N}$, define a sequence of functions f_n as

$$f_n(s) = \begin{cases} f(s) & \text{if } s \leqslant n, \\ f(n) + f'(n)(s-n) & \text{if } s > n, \end{cases}$$

and consider the approximated problem

$$-\Delta u + u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial\nu} = \lambda f_n(u) \quad \text{on } \partial\Omega. \tag{2.1}$$

Denote $\lambda_n^* = \sup\{\lambda > 0\colon (2.1)_\lambda \text{ admits a minimal solution} \in C^2(\overline{\Omega})\}$, and let $u_{n,\lambda} \in C^2(\overline{\Omega})$ be the classical minimal solution to $(2.1)_\lambda$ for $\lambda < \lambda_n^*$. Since $f_n \leqslant f_{n+1} \leqslant f$, we have $u_{n,\lambda} \leqslant u_{n+1,\lambda} \leqslant u_\lambda$ and $\lambda^* \leqslant \lambda_{n+1}^* \leqslant \lambda_n^*$ for any $n \in \mathbb{N}$. Define

$$\mu_1(\lambda f_n'(u_{n,\lambda})) = \inf_{\varphi \in C^1(\overline{\Omega}),\, \varphi \not\equiv 0} \frac{\int_\Omega (|\nabla\varphi|^2 + \varphi^2)\,\mathrm{d}x - \int_{\partial\Omega} \lambda f_n'(u_{n,\lambda})\varphi^2 \,\mathrm{d}s_x}{\int_{\partial\Omega} \varphi^2 \,\mathrm{d}s_x}. \tag{2.2}$$

Note that $\mu_1(\lambda f_n'(u_{n,\lambda}))$ is continuous with respect to λ by (2.2). Take $0 \leqslant \mu \leqslant \mu_1^*$ where μ_1^* is defined by (1.7). Since u_{n,λ_n^*} is classical (which is because f_n is asymptotically linear) and there is no classical solution of $(2.1)_\lambda$ for $\lambda > \lambda_n^*$, the linearized problem around $(\lambda_n^*, u_{n,\lambda_n^*})$ must have zero eigenvalue. Thus

$$\mu_1(\lambda_n^* f_n'(u_{n,\lambda_n^*})) = 0 \leqslant \mu \leqslant \mu_1^* \leqslant \mu_1(\lambda^* f_n'(u_{n,\lambda^*}));$$

here we have used the fact that $f_n' \leqslant f'$ and $u_{n,\lambda} \leqslant u_\lambda$, which implies $\mu_1(\lambda f'(u_\lambda)) \leqslant \mu_1(\lambda f_n'(u_{n,\lambda}))$. By the Intermediate Value Theorem, there exists $\lambda_n \in [\lambda^*, \lambda_n^*]$ such that

$$\mu_1(\lambda_n f_n'(u_{n,\lambda_n})) = \mu,$$

which in turn implies there exists $\varphi_n > 0$ with $\int_{\partial\Omega} \varphi_n \, \mathrm{d}s_x = 1$ such that

$$-\Delta\varphi_n + \varphi_n = 0 \quad \text{in } \Omega, \qquad \frac{\partial\varphi_n}{\partial\nu} = \lambda_n f_n'(u_{n,\lambda_n})\varphi_n + \mu\varphi_n \quad \text{on } \partial\Omega. \tag{2.3}$$

Recall also that u_{n,λ_n} satisfies

$$-\Delta u_{n,\lambda_n} + u_{n,\lambda_n} = 0 \quad \text{in } \Omega, \qquad \frac{\partial u_{n,\lambda_n}}{\partial\nu} = \lambda_n f_n(u_{n,\lambda_n}) \quad \text{on } \partial\Omega. \tag{2.4}$$

We claim there exists $n_0 \in \mathbb{N}$ such that

$$\|u_{n,\lambda_n}\|_{L^1(\partial\Omega)} \leqslant C \quad \text{for any } n \geqslant n_0. \tag{2.5}$$

Indeed, let ψ_1 be the first eigenfunction of the Steklov type eigenvalue problem

$$-\Delta\psi_1 + \psi_1 = 0 \quad \text{in } \Omega, \qquad \frac{\partial\psi_1}{\partial\nu} = \kappa_1\psi_1 \quad \text{on } \partial\Omega \tag{2.6}$$

with the first eigenvalue κ_1, which is normalized as $\int_{\partial\Omega} \psi_1 \, \mathrm{d}s_x = 1$. Multiplying (2.4) by ψ_1 and using Jensen's inequality for f_n, we obtain

$$\begin{aligned}\kappa_1 \int_{\partial\Omega} \psi_1 u_{n,\lambda_n} \, \mathrm{d}s_x &= \lambda_n \int_{\partial\Omega} f_n(u_{n,\lambda_n})\psi_1 \, \mathrm{d}s_x \\ &\geqslant \lambda_n f_n\Big(\int_{\partial\Omega} \psi_1 u_{n,\lambda_n} \, \mathrm{d}s_x\Big) \geqslant \lambda^* f_n\Big(\int_{\partial\Omega} \psi_1 u_{n,\lambda_n} \, \mathrm{d}s_x\Big).\end{aligned}$$

Put $a_n = \int_{\partial\Omega} \psi_1 u_{n,\lambda_n} \, \mathrm{d}s_x$. Then we have

$$a_n \geqslant \frac{\lambda^*}{\kappa_1} f_n(a_n). \tag{2.7}$$

Assume by contradiction that $f_n(a_n) = f'(n)(a_n - n) + f(n)$ for some $n \in \mathbb{N}$ sufficiently large. Then, since $a_n > n$ and $f(n) > (\kappa_1/\lambda^*)n$, $f'(n) > (\kappa_1/\lambda^*)$ for n sufficiently large by (1.2) and (1.3), we have, by (2.7),

$$\begin{aligned} a_n &\geqslant \frac{\lambda^*}{\kappa_1} f_n(a_n) = \frac{\lambda^*}{\kappa_1}\{f'(n)(a_n - n) + f(n)\} \\ &> a_n - n + n = a_n, \end{aligned}$$

which is a contradiction. Thus we conclude there exists $n_0 \in \mathbb{N}$ such that $f_n(a_n) = f(a_n)$ for any $n \geqslant n_0$. Again, this and (2.7) imply $a_n \geqslant (\lambda^*/\kappa_1) f(a_n)$ for any $n \geqslant n_0$. Now, by the assumption on f, we have $C > 0$ such that $f(s) \geqslant (2\kappa_1/\lambda^*)s - C$ holds for any $s > 0$. From this and the former estimate, we have $a_n \leqslant (\lambda^*/\kappa_1)C$ for $n \geqslant n_0$. This implies the claim (2.5).

Step 2. By (2.5), we have $\|u_{n,\lambda_n}\|_{L^1(\partial\Omega)} \leqslant C$ for some C independent of n. Also recall that $\|\varphi_n\|_{L^1(\partial\Omega)} = 1$ for a solution φ_n of (2.3). Thus we can apply Lemma 2.2 and the trace Sobolev embedding to obtain $w, \varphi \in L^1(\Omega)$, $\varphi \geqslant 0$ a.e. satisfying

$$\begin{aligned} &u_{n,\lambda_n} \rightharpoonup w, \quad \varphi_n \rightharpoonup \varphi \quad \text{weakly in } W^{1,q}(\Omega), \\ &u_{n,\lambda_n} \to w, \quad \varphi_n \to \varphi \quad \text{strongly in } L^p(\partial\Omega) \text{ and a.e. on } \partial\Omega \end{aligned} \tag{2.8}$$

for any $1 < q < N/(N-1)$ and $1 \leqslant p < (N-1)/(N-2)$. Since $\int_{\partial\Omega} \varphi \, \mathrm{d}s_x = 1$, we see $\varphi \not\equiv 0$ on $\partial\Omega$.

In the following, we prove that $\lambda_n \downarrow \lambda^*$ as $n \to \infty$ and $w = u^*$. We will show that $w \in W^{1,q}(\Omega)$ is a weak supersolution in the sense of Theorem 2.1. Then the conclusion is obtained by Theorem 2.1. To prove that w is a weak supersolution, put $\bar\lambda = \inf_{n\in\mathbb{N}} \lambda_n$. Since $\lambda_n \geqslant \lambda^*$, we have $\bar\lambda \geqslant \lambda^*$. We observe that

$$\begin{aligned} \int_\Omega (-\Delta\zeta + \zeta) u_{n,\lambda_n} \, \mathrm{d}x &= \lambda_n \int_{\partial\Omega} f_n(u_{n,\lambda_n})\zeta \, \mathrm{d}s_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} u_{n,\lambda_n} \, \mathrm{d}s_x \\ &\geqslant \bar\lambda \int_{\partial\Omega} f_n(u_{n,\lambda_n})\zeta \, \mathrm{d}s_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} u_{n,\lambda_n} \, \mathrm{d}s_x \end{aligned}$$

holds for all $\zeta \in C^2(\overline{\Omega})$, $\zeta \geqslant 0$. Using the fact that $u_{n,\lambda_n} \to w$ in $L^1(\Omega)$ or $L^1(\partial\Omega)$, respectively, and Fatou's lemma, we have

$$\begin{aligned} \int_\Omega (-\Delta\zeta + \zeta) w \, \mathrm{d}x &\geqslant \bar\lambda \int_{\partial\Omega} f(w)\zeta \, \mathrm{d}s_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} w \, \mathrm{d}s_x \\ &\geqslant \lambda^* \int_{\partial\Omega} f(w)\zeta \, \mathrm{d}s_x - \int_{\partial\Omega} \frac{\partial\zeta}{\partial\nu} w \, \mathrm{d}s_x, \quad \forall \zeta \in C^2(\overline{\Omega}),\ \zeta \geqslant 0. \end{aligned}$$

This implies also $f(w) \in L^1(\partial\Omega)$ if we take $\zeta \equiv 1$. Thus, we conclude that w is a weak supersolution to $(1.1)_{\lambda^*}$

Step 3. Let φ_n, φ be as in Step 2. We claim that

$$\lambda_n f_n'(u_{n,\lambda_n})\varphi_n \to \lambda^* f'(u^*)\varphi \quad \text{strongly in } L^1(\partial\Omega) \tag{2.9}$$

as $n \to \infty$. For the proof, we invoke Vitali's Convergence Theorem. First, by (2.8), we see

$$\lambda_n f_n'(u_{n,\lambda_n}(x))\varphi_n(x) \to \lambda^* f'(u^*(x))\varphi(x) \quad \text{a.e. } x \in \partial\Omega$$

for a subsequence. Next, we prove the uniformly absolute continuity property of the sequence $\{\lambda_n f_n'(u_{n,\lambda_n})\varphi_n\}_{n\in\mathbb{N}}$. For that purpose, let $A \subset \partial\Omega$ be measurable and $\varepsilon > 0$ be given arbitrary. Since f_n is convex, we have

$$f_n\Big(\frac{\chi_A(x)}{\varepsilon}\Big) \geqslant f_n(u_{n,\lambda_n}(x)) + f_n'(u_{n,\lambda_n}(x))\Big(\frac{\chi_A(x)}{\varepsilon} - u_{n,\lambda_n}(x)\Big) \tag{2.10}$$

a.e. $x \in \partial\Omega$; here χ_A is the characteristic function of A. By (2.3) and (2.4), we have

$$\begin{aligned} \lambda_n \int_{\partial\Omega} f_n(u_{n,\lambda_n})\varphi_n \,\mathrm{d}s_x &= \lambda_n \int_{\partial\Omega} f_n'(u_{n,\lambda_n})u_{n,\lambda_n}\varphi_n \,\mathrm{d}s_x + \mu \int_{\partial\Omega} u_{n,\lambda_n}\varphi_n \,\mathrm{d}s_x \\ &\geqslant \lambda_n \int_{\partial\Omega} f_n'(u_{n,\lambda_n})u_{n,\lambda_n}\varphi_n \,\mathrm{d}s_x. \end{aligned} \tag{2.11}$$

Also an easy consideration shows that

$$\Big\{f_n\Big(\frac{\chi_A(x)}{\varepsilon}\Big) - f(0)\Big\}\varphi_n(x) \leqslant f\Big(\frac{1}{\varepsilon}\Big)\varphi_n(x)\chi_A(x) \quad \text{a.e. on } \partial\Omega. \tag{2.12}$$

Thus by (2.10), (2.11) and (2.12), we have

$$\begin{aligned} &\int_{\partial\Omega} f_n'(u_{n,\lambda_n})\frac{\chi_A}{\varepsilon}\varphi_n \,\mathrm{d}s_x \\ &\quad\leqslant \int_{\partial\Omega} f_n\Big(\frac{\chi_A}{\varepsilon}\Big)\varphi_n \,\mathrm{d}s_x + \int_{\partial\Omega} f_n'(u_{n,\lambda_n})u_{n,\lambda_n}\varphi_n \,\mathrm{d}s_x - \int_{\partial\Omega} f_n(u_{n,\lambda_n})\varphi_n \,\mathrm{d}s_x \\ &\quad\leqslant \int_{\partial\Omega} f_n\Big(\frac{\chi_A}{\varepsilon}\Big)\varphi_n \,\mathrm{d}s_x = \int_{\partial\Omega} \Big\{f_n\Big(\frac{\chi_A}{\varepsilon}\Big) - f(0)\Big\}\varphi_n \,\mathrm{d}s_x + \int_{\partial\Omega} f(0)\varphi_n \,\mathrm{d}s_x \\ &\quad\leqslant \int_{\partial\Omega} f\Big(\frac{1}{\varepsilon}\Big)\varphi_n\chi_A \,\mathrm{d}s_x + f(0) \leqslant f\Big(\frac{1}{\varepsilon}\Big)|A|^{1/p'}\|\varphi_n\|_{L^p(\partial\Omega)} + f(0) \\ &\quad\leqslant Cf\Big(\frac{1}{\varepsilon}\Big)|A|^{1/p'} + f(0) \end{aligned} \tag{2.13}$$

for any $1 \leqslant p < (N-1)/(N-2)$, where $|A|$ denotes the $(N-1)$ dimensional Hausdorff measure of $A \subset \partial\Omega$ and $p' = p/(p-1)$. In (2.13) we have used $\|\varphi_n\|_{L^p(\partial\Omega)} \leqslant C$ for some $C > 0$ independent of n by (2.8). Define

$$\delta(\varepsilon) = \Big(\frac{f(0)}{f(1/\varepsilon)C}\Big)^{p'}.$$

Then for any $\varepsilon > 0$ we obtain $\int_A f_n'(u_{n,\lambda_n})\varphi_n \,\mathrm{d}s_x \leqslant 2f(0)\varepsilon$ if $A \subset \partial\Omega$ satisfies that $|A| < \delta(\varepsilon)$ by (2.13). This implies the uniform absolute continuity of the sequence $\{\lambda_n f_n'(u_{n,\lambda_n})\varphi_n\}_{n\in\mathbb{N}}$. Also for any $\varepsilon > 0$, if we take $E \subset \partial\Omega$ such that $|\partial\Omega\setminus E| < \delta(\varepsilon)$ where $\delta(\varepsilon)$ is as above, we obtain that $\int_{\partial\Omega\setminus E} \lambda_n f_n'(u_{n,\lambda_n})\varphi_n \,\mathrm{d}s_x \leqslant C\varepsilon$. This implies the uniform integrability of $\{\lambda_n f_n'(u_{n,\lambda_n})\varphi_n\}_{n\in\mathbb{N}}$. Therefore, Vitali's Convergence Theorem ensures the claim (2.9).

By (2.9), we pass to the limit $n \to \infty$ in the weak formulation of (2.3):

$$\int_\Omega (-\Delta\zeta + \zeta)\varphi_n \,\mathrm{d}x = \int_{\partial\Omega} (\lambda_n f_n'(u_{n,\lambda_n}) + \mu)\varphi_n\zeta - \frac{\partial\zeta}{\partial\nu}\varphi_n \,\mathrm{d}s_x, \quad \forall\zeta \in C^2(\overline{\Omega}),$$

and conclude that φ is a weak solution of

$$-\Delta\varphi + \varphi = 0 \quad \text{in } \Omega, \quad \frac{\partial\varphi}{\partial\nu} = \lambda^* f'(u^*)\varphi + \mu\varphi \quad \text{on } \partial\Omega.$$

Recall $\varphi \in W^{1,q}(\Omega)$ for any $1 \leqslant q < N/(N-1)$. The proof of Theorem 1.1 is completed. □

References

[1] *H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa*: Blow up for $u_t - \Delta u = g(u)$ revisited. Adv. Differ. Equ. *1* (1996), 73–90.

[2] *H. Brezis, J. L. Vázquez*: Blow-up solutions of some nonlinear elliptic problems. Rev. Mat. Univ. Complutense Madr. *10* (1997), 443–469.

[3] *X. Cabré, Y. Martel*: Weak eigenfunctions for the linearization of extremal elliptic problems. J. Funct. Anal. *156* (1998), 30–56.

[4] *M. Chipot, I. Shafrir, M. Fila*: On the solutions to some elliptic equations with nonlinear Neumann boundary conditions. Adv. Differ. Equ. *1* (1996), 91–110.

[5] *J. Dávila*: Singular solutions of semi-linear elliptic problems. Handbook of Differential Equations: Stationary Partial Differential Equations (M. Chipot, ed.). Elsevier, Amsterdam, 2008, pp. 83–176.

[6] *J. Dávila, L. Dupaigne, M. Montenegro*: The extremal solution of a boundary reaction problem. Commun. Pure Appl. Anal. *7* (2008), 795–817.

[7] *L. Dupaigne*: Stable Solutions of Elliptic Partial Differential Equations. Chapman & Hall Monographs and Surveys in Pure and Applied Mathematics 143, CRC Press, Boca Raton, 2011.

[8] *Y. Martel*: Uniqueness of weak extremal solutions of nonlinear elliptic problems. Houston J. Math. *23* (1997), 161–168.

[9] *P. Quittner, W. Reichel*: Very weak solutions to elliptic equations with nonlinear Neumann boundary conditions. Calc. Var. Partial Differ. Equ. *32* (2008), 429–452.

[10] *F. Takahashi*: Extremal solutions to Liouville-Gelfand type elliptic problems with nonlinear Neumann boundary conditions. Commun. Contemp. Math. (2014), 27 pages, DOI:10.1142/S0219199714500163.

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