Applications of Mathematics

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Applications of Mathematics, Vol. 59 (2014), No. 4, 453–472

Persistent URL: http://dml.cz/dmlcz/143874

ON AN INITIAL INVERSE PROBLEM IN NONLINEAR HEAT EQUATION ASSOCIATED WITH TIME-DEPENDENT COEFFICIENT

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(Received May 24, 2012)

Abstract. In this paper, a nonlinear backward heat problem with time-dependent coefficient in the unbounded domain is investigated. A modified regularization method is established to solve it. New error estimates for the regularized solution are given under some assumptions on the exact solution.

Keywords: nonlinear heat problem; ill-posed problem; Fourier transform; time-dependent coefficient

MSC 2010: 35K05, 35K99, 47J06

1. Introduction

Let T be a positive number. We consider the problem of finding the temperature $u(x,t), (x,t) \in \mathbb{R} \times [0;T]$, such that

$$\begin{cases} \dfrac{\partial u}{\partial t} - a(t)\dfrac{\partial^2 u}{\partial x^2} = f(x,t,u(x,t)), \quad (x,t) \in \mathbb{R} \times (0,T), \\ u(x,T) = \varphi(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $a(t)$, $\varphi(x)$, $f(x,t,z)$ are given functions satisfying conditions specified later. This problem is well-known to be severely ill-posed [7] and regularization methods for it are required. It is called the initial inverse heat problem, backward heat problem, backward Cauchy problem, or final value problem.

As is known, if the initial temperature distribution in a heat conducting body is given, then the temperature distribution at a later time can be determined and the

The research has been supported by Institute for Computational Science and Technology Ho Chi Minh City.

problem is well-posed. This is the direct problem. In geophysical exploration, one is often faced with the problem of determining the temperature distribution in the Earth or any part of the Earth at a time $t_0 > 0$ from the temperature measurement at a time $t_1 > t_0$. This is the backward heat problem. This type of problem is severely ill-posed; i.e., solutions do not always exist, and in the case of existence, they do not depend continuously on the given data. In fact, even with a small noise contaminated physical measurements, the corresponding solutions have large errors. It makes it difficult to do numerical calculations. Due to the severe ill-posedness of the problem, it is impossible to solve the backward heat problem by using classical numerical methods. Hence, regularization strategies are to be employed. In the simplest cases $f = 0$ and $a(t) = 1$, the problem (1.1) becomes

$$\begin{cases} u_t - u_{xx} = 0, & (x,t) \in \mathbb{R} \times (0,T), \\ u(x,T) = \varphi(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Such authors as Lattes and Lions [5], Showalter [12], Clark et al. [2] have approximated the problem (1.2) by the quasi-reversibility method. Tautenhahn and Schröter [13] established an optimal error estimate for (2). Liu in [6] introduced a group preserving scheme. Some papers [1], [4] have approximated (1.1) by truncated methods. A modified quasi-reversibility for problem (1.2) is investigeted by Denche et al. [3], Fu et al. [8]. Optimal filtering method for (1.2) is established by Seidman [11] Stability estimate on the homogeneous backward heat has been studied by Yildiz et al. [16]. Very recently, problem (1.2) was also investigated by Wang [15]. Very recently, the authors [10] regularized problem (1.1) in the homogeneous case $f = 0$ and more generally for some nonhomogeneous f. However, the most of the above mentioned results deal only with the linear case.

Although there are many works on the initial inverse heat problem with constant coefficients, the literature on the nonlinear case of the problem with time-dependent coefficient (namely problem (1.1) is quite scarce. In this paper, we present a modified method in order to regularize problem (1.1). Under some assumptions on the exact solution, we obtain some faster convergence speeds. In a sense, this is an improvement of known results in [9], [10], [14].

This paper is organized as follows. In Section 2, we give some auxiliary definitions. In Section 3, we outline the nonlinear case of the backward heat with time-dependent coefficient.

2. Some auxiliary definitions

We assume that $a\colon\ [0,T]\to\mathbb{R}$ is a continuous function on $[0,T]$ satisfying $a(t)>0$. The function $B(t)$ is defined by

$$B(t)=\int_0^t\frac{1}{a(s)}\,\mathrm{d}s. \tag{2.1}$$

Let $\hat{g}(\xi)$ denote the Fourier transform of a function $g\in L^2(\mathbb{R})$ defined formally as

$$\hat{g}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(x)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x. \tag{2.2}$$

Let $H^1=W^{1,2}$, $H^2=W^{2,2}$ be the Sobolev spaces which are defined by

$$\begin{aligned}H^1(\mathbb{R})&=\{g\in L^2(\mathbb{R}),\ \xi\hat{g}(\xi)\in L^2(\mathbb{R})\},\\ H^2(\mathbb{R})&=\{g\in L^2(\mathbb{R}),\ \xi^2\hat{g}(\xi)\in L^2(\mathbb{R})\}.\end{aligned}$$

We denote by $\|\cdot\|$, $\|\cdot\|_{H^1}$, $\|\cdot\|_{H^2}$ the norms in $L^2(\mathbb{R})$, $H^1(\mathbb{R})$, $H^2(\mathbb{R})$, respectively, namely

$$\begin{aligned}\|g\|^2_{H^1}&=\|g\|^2+\|g_x\|^2=\|(1+\xi^2)^{\frac12}\hat{g}(\xi)\|^2,\\ \|g\|^2_{H^2}&=\|g\|^2+\|g_x\|^2+\|g_{xx}\|^2=\|(1+\xi^2+\xi^4)^{\frac12}\hat{g}(\xi)\|^2.\end{aligned}$$

3. The nonlinear problem with time-dependent coefficient

In this section, we consider the problem (1.1) of finding the temperature $u(x,t)$ with $(x,t)\in\mathbb{R}\times[0;T]$, where $a(t)$ is defined in Section 2, $\varphi\in L^2(\mathbb{R})$ and $f(x,y,z)$ satisfies the conditions of Lemma 3.1. Let us first make clear what a weak solution to problem (1.1) is.

Lemma 3.1. *Let $f\in L^\infty(\mathbb{R}\times[0,T]\times\mathbb{R})$ be a function such that $f(x,y,0)=0$ and*

$$|f(x,t,u)-f(x,t,v)|\leqslant K|u-v| \tag{3.1}$$

for all $(x,t)\in\mathbb{R}\times[0,T]$ and for some constant $K>0$ independent of x, t, u, v. Let $\varphi\in L^2(\mathbb{R})$. Assume that $u\in C([0,T],H^2(\mathbb{R}))\cap C^1([0,T],L^2(\mathbb{R}))$ is a solution of the equation

$$\hat{u}(\xi,t)=\mathrm{e}^{(B(T)-B(t))\xi^2}\hat{\varphi}(\xi)-\int_t^T\mathrm{e}^{-(B(t)-B(s))\xi^2}\hat{f}(\xi,s,u)\,\mathrm{d}s. \tag{3.2}$$

Then $u_t,u_{xx}\in C([0,T],L^2(\mathbb{R}))$.

P r o o f. By letting $t=T$ in (3.2), we have immediately $\hat{u}(\xi,T)=\hat{\varphi}(\xi)$. Therefore, we get $u(x,T)=\varphi(x)$ in $L^2(\mathbb{R})$.

Multiplying the above equation by $\mathrm{e}^{B(t)\xi^2}$, we obtain

$$\mathrm{e}^{B(t)\xi^2}\hat{u}(\xi,t)=\mathrm{e}^{B(T)\xi^2}\hat{\varphi}(\xi)-\int_t^T \mathrm{e}^{B(s)\xi^2}\hat{f}(\xi,s,u)\,\mathrm{d}s,\quad t\in[0,T].$$

Differentiating the latter equation w.r.t. the time variable t, we get

$$\mathrm{e}^{B(t)\xi^2}\Big(\xi^2\hat{u}(\xi,t)+\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}(\xi,t)\Big)=\mathrm{e}^{B(t)\xi^2}\hat{f}(\xi,t,u),$$

namely

$$\xi^2 b(t)\hat{u}(\xi,t)+\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}(\xi,t)=\hat{f}(\xi,t,u),\quad t\in[0,T].$$

Since $u\in C([0,T],H^2(\mathbb{R}))\cap C^1([0,T],L^2(R))$ we have that $\xi^2\hat{u}(\xi,t)=\hat{u}_{xx}(\xi)$ and $(\mathrm{d}/\mathrm{d}t)\hat{u}(\xi,t)$ belong to $C([0,T],L^2(\mathbb{R}))$. This gives $u_t,u_{xx}\in C([0,T],L^2(\mathbb{R}))$ and (3.6) gives a weak formulation of the solution of problem (1.1). This ends the proof. □

Let

$$\hat{\varphi}(\xi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi(x)\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x$$

be the Fourier transform of the function $\varphi(x)\in L^2(\mathbb{R})$. By a solution of problem (1.1) we understand a function $u(x,t)$ satisfying (1.1) in the classical sense and for every fixed $t\in[0,T]$, the function $u(\cdot,t)\in L^2(\mathbb{R})$. In this class of functions, if the solution of problem (1.1) exists, then it must be unique (see [9]). In general, we have no guarantee that the solution to problem (1.1) exists. We do not know any general condition under which problem (1.1) is solvable. The main goal of this paper is to find a computation method of the exact solution when it exists. Hence, regularization techniques are required. Let $u(x,t)$ be a unique solution of (1.1) (if it exists). Using the Fourier transform technique to problem (1.1) with respect to the variable x, we can get the Fourier transform $\hat{u}(\xi,t)$ of the exact solution $u(x,t)$ of problem (1.1), which is given in (3.2). Since $t<T$, we know from (3.2) that, when $|\xi|$ becomes large, $\mathrm{e}^{B(t)b\xi^2}$ and $\mathrm{e}^{(B(s)-B(t))\xi^2}$ increase rather quickly. Thus, these terms are the unstability cause. Hence, to regularize the problem, we have to replace the terms by some better terms. Our idea is to replace them by $\mathrm{e}^{-(B(t)+m)\xi^2}/(\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2})$ and $\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}/(\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2})$ ($m\geqslant 0$, $k\geqslant 1$), respectively. The main conclusion of this paper is:

Theorem 3.2. *Let $f\colon \mathbb{R}\times[0,T]\times\mathbb{R}\to\mathbb{R}$ be as in Lemma 3.1. Let $\varphi\in L^2(\mathbb{R})$ and $\varphi_\varepsilon\in L^2(\mathbb{R})$ be measured data such that $\|\varphi_\varepsilon-\varphi\|\leqslant\varepsilon$. Suppose that problem (1.1) has a unique solution $u\in C([0,T],L^2(\mathbb{R}))$. Let $m\geqslant 0$ and $k\geqslant 1$ be real numbers such that*

$$\int_{-\infty}^{\infty}|\xi^{2k}\mathrm{e}^{(B(t)+m)\xi^2}\hat{u}(\xi,t)|^2\,\mathrm{d}\xi<\infty. \tag{3.3}$$

Then we construct a regularized solution w_ε such that

$$\begin{aligned}(3.4)\quad &\|u(\cdot,t)-w_\varepsilon(\cdot,t)\|\\ &\qquad\leqslant P(k,m)\varepsilon^{(B(t)+m)/(B(T)+m)}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{(kB(t)-kB(T))/(B(T)+m)},\end{aligned}$$

for all $t\in(0,T]$, where w_ε is the function whose Fourier transform is defined by

$$\begin{aligned}(3.5)\qquad \hat{w}_\varepsilon(\xi,t)=&\ \frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{\varphi}_\varepsilon(\xi)\\ &-\int_t^T\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,w_\varepsilon)\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}(3.6)\qquad A(k,m)&=2H^2(k,m)\sup_{0\leqslant t\leqslant T}\Big(\int_{-\infty}^{\infty}|\xi^k\mathrm{e}^{(B(t)+m)\xi^2}\hat{u}(\xi,t)|^2\,\mathrm{d}\xi\Big),\\ P(k,m)&=\sqrt{A(k,m)}\mathrm{e}^{T^2K^2H^2(k,m)}+\sqrt{2}H(k,m)\mathrm{e}^{T^2K^2H^2(k,m)},\\ H(k,m)&=\min\{1,(kB(T)+km)^k\},\\ (3.7)\qquad I(k,m)&=\frac{(B(T)+m)^k}{k}.\end{aligned}$$

The proof will be provided after Lemmas 3.3 and 3.4.

Lemma 3.3. *For M, ε, $x>0$, $k\geqslant 1$, we have the inequality*

$$\frac{1}{\varepsilon x^k+\mathrm{e}^{-Mx}}\leqslant D(k,M)\varepsilon^{-1}\Big(\ln\Big(\frac{E(k,M)}{\varepsilon}\Big)\Big)^{-k},$$

where $D(k,M)=(kM)^k$, $E(k,M)=M^k/k$.

Proof. Let g be the function defined by $g(x)=(\varepsilon x^k+\mathrm{e}^{-Mx})^{-1}$. The derivative of g is

$$g'(x)=\frac{\varepsilon kx^{k-1}-M\mathrm{e}^{-Mx}}{-(\varepsilon x^k+\mathrm{e}^{-Mx})^2}.$$

The equation $g'(x) = 0$ gives a unique solution x_0 such that $\varepsilon k x_0^{k-1} - M\mathrm{e}^{-Mx_0} = 0$. It means that $x_0^{k-1}\mathrm{e}^{Mx_0} = M/(k\varepsilon)$. Thus the function g achieves its maximum at a unique point $x = x_0$. Hence,

$$g(x) \leqslant (\varepsilon x_0^k + \mathrm{e}^{-Mx_0})^{-1}.$$

Since $\mathrm{e}^{-Mx_0} = (k\varepsilon/M)x_0^{k-1}$, one has

$$g(x) \leqslant (\varepsilon x_0^k + \mathrm{e}^{-Mx_0})^{-1} \leqslant (\varepsilon x_0^k + (k\varepsilon/M)x_0^{k-1})^{-1}.$$

By using the inequality $\mathrm{e}^{Mx_0} \geqslant Mx_0$, we get

$$\frac{M}{k\varepsilon} = x_0^{k-1}\mathrm{e}^{Mx_0} \leqslant \frac{1}{M^{k-1}}\mathrm{e}^{(k-1)Mx_0}\mathrm{e}^{Mx_0} = \frac{1}{M^{k-1}}\mathrm{e}^{kMx_0}.$$

This gives $\mathrm{e}^{kMx_0} \geqslant M^k/(k\varepsilon)$, or equivalently $kMx_0 \geqslant \ln(M^k/(k\varepsilon))$. Therefore, $x_0 \geqslant 1/(kM)\ln(M^k/(k\varepsilon))$. Hence, we obtain

$$g(x) \leqslant \frac{1}{\varepsilon x_0^k} \leqslant \frac{(kM)^k}{\varepsilon \ln^k(M^k/(k\varepsilon))}.$$

□

Lemma 3.4. *Let s, t be real numbers such that $0 \leqslant t \leqslant s \leqslant T$. Let $\varepsilon > 0$, $\xi \in \mathbb{R}$, $m \geqslant 0$, $k \geqslant 1$. Then the following estimates hold:*

$$\frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}} \leqslant H(k,m)\varepsilon^{\frac{B(t)-B(T)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(T)}{B(T)+m}}, \tag{3.8}$$

$$\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}} \leqslant H(k,m)\varepsilon^{\frac{B(t)-B(s)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(s)}{B(T)+m}}. \tag{3.9}$$

P r o o f. We have

$$\begin{aligned}
&\frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}} \\
&= \frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{(\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2})^{\frac{B(t)+m}{B(T)+m}}(\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2})^{\frac{B(T)-B(t)}{B(T)+m}}} \\
&\leqslant \frac{1}{(\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2})^{\frac{B(T)-B(t)}{B(T)+m}}} \\
&\leqslant \Big(D(k,M)\varepsilon^{-1}\Big(\ln\Big(\frac{E(k,M)}{\varepsilon}\Big)\Big)^{-k}\Big)^{\frac{B(T)-B(t)}{B(T)+m}} \\
&\leqslant D(k,B(T)+m)^{\frac{B(T)-B(t)}{B(T)+m}}\varepsilon^{\frac{B(t)-B(T)}{B(T)+m}}\Big(\ln\Big(\frac{E(k,B(T)+m)}{\varepsilon}\Big)\Big)^{k\frac{B(t)-B(T)}{B(T)+m}} \\
&\leqslant H(k,m)\varepsilon^{\frac{B(t)-B(T)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(T)}{B(T)+m}}
\end{aligned}$$

where $H(k,m)=\min\{1,D(k,B(T)+m)\}$, $I(k,m)=E(k,B(T+m))$. We also have

$$\begin{aligned}
&\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\\
&=\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{(\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2})^{\frac{B(T)+m-B(s)+B(t)}{B(T)+m}}(\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2})^{\frac{B(s)-B(t)}{B(T)+m}}}\\
&\leqslant\frac{1}{(\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2})^{\frac{B(s)-B(t)}{B(T)+m}}}\\
&\leqslant D(k,B(T)+m)^{\frac{B(s)-B(t)}{B(T)+m}}\varepsilon^{\frac{B(t)-B(s)}{B(T)+m}}\Big(\ln\Big(\frac{E(k,B(T)+m)}{\varepsilon}\Big)\Big)^{k\frac{B(t)-B(s)}{B(T)+m}}\\
&\leqslant H(k,m)\varepsilon^{\frac{B(t)-B(s)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(s)}{B(T)+m}}.
\end{aligned}$$

□

Next, we continue to prove the main theorem.

P r o o f of Theorem 3.2. We divide the proof into three steps.

Step 1. Construct a regularized solution w_ε. We consider the problem

$$\hat{w}_\varepsilon(\xi,t)=\frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{\varphi}_\varepsilon(\xi)-\int_t^T\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,w_\varepsilon)\,\mathrm{d}s, \tag{3.10}$$

or

$$w_\varepsilon(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{\varphi}_\varepsilon(\xi)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_t^T\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,w_\varepsilon)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}s\,\mathrm{d}\xi. \tag{3.11}$$

First, we prove that problem (3.11) has a unique solution w_ε belonging to $C([0,T];L^2(\mathbb{R}))$. Denote

$$G(w)(x,t)=\frac{1}{\sqrt{2\pi}}\psi(x,t)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_t^T\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,w)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}s\,\mathrm{d}\xi$$

for all $w\in C([0,T];L^2(\mathbb{R}))$ and

$$\psi(x,t)=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{\varphi}_\varepsilon(\xi)\mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi.$$

Since $f(x,y,0)=0$, and due to the Lipschitzian property of $f(x,y,w)$ with respect to w, we get $G(w)\in C([0,T];L^2(\mathbb{R}))$ for every $w\in C([0,T];L^2(\mathbb{R}))$. We claim that, for every $w,v\in C([0,T];L^2(\mathbb{R}))$, $n\geqslant 1$, we have

$$\|G^n(w)(\cdot,t)-G^n(v)(\cdot,t)\|^2\leqslant\Big(\frac{K}{\varepsilon}\Big)^{2n}\frac{(T-t)^nC_1^n}{n!}\|\!|w-v|\!\|^2, \tag{3.12}$$

where $C_1=\max\{T,1\}$ and $\|\!|\cdot|\!\|$ is the sup norm in $C([0,T];L^2(\mathbb{R}))$. We shall prove the latter inequality by induction. When $n=1$, we have

$$\begin{aligned}\|G(w)(\cdot,t)-G(v)(\cdot,t)\|^2&=\|\widehat{G}(w)(\cdot,t)-\widehat{G}(v)(\cdot,t)\|^2\\&=\int_{-\infty}^{\infty}\Big|\int_t^T\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}(\hat f(\xi,s,w)-\hat f(\xi,s,v))\,\mathrm{d}s\Big|^2\,\mathrm{d}\xi\\&\leqslant\int_{-\infty}^{\infty}\Big(\int_t^T\Big(\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\Big)^2\,\mathrm{d}s\\&\quad\times\int_t^T|\hat f(\xi,s,w)-\hat f(\xi,s,v)|^2\,\mathrm{d}s\Big)^2\,\mathrm{d}\xi.\end{aligned}\tag{3.13}$$

Notice that if $0<\varepsilon<1$ then it follows from $(B(t)-B(s))/(B(T)+m)>-1$ that

$$\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\leqslant\varepsilon^{(B(t)-B(s))/(B(T)+m)}\leqslant\varepsilon^{-1}.$$

This gives

$$\int_t^T\Big(\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\Big)^2\,\mathrm{d}s\leqslant\int_t^T\frac{1}{\varepsilon^2}\,\mathrm{d}s=\frac{1}{\varepsilon^2}(T-t).\tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\begin{aligned}&\|G(w)(\cdot,t)-G(v)(\cdot,t)\|^2\\&\quad\leqslant\frac{1}{\varepsilon^2}(T-t)\int_t^T\|\hat f(\cdot,s,w(\cdot,s))-\hat f(\cdot,s,v(\cdot,s))\|^2\,\mathrm{d}s\\&\quad=\frac{1}{\varepsilon^2}(T-t)\int_t^T\|f(\cdot,s,w(\cdot,s))-f(\cdot,s,v(\cdot,s))\|^2\,\mathrm{d}s\\&\quad=\frac{K^2}{\varepsilon^2}(T-t)\int_t^T\|w(\cdot,s)-v(\cdot,s)\|^2\,\mathrm{d}s\leqslant C_1\frac{K^2}{\varepsilon^2}(T-t)\|\!|w-v|\!\|^2.\end{aligned}$$

Therefore, (3.12) holds for $n=1$. Suppose that (3.12) holds for $n=p$. We prove that (3.12) holds for $n=p+1$. We have

$$
\begin{aligned}
\|G^{p+1}(w)(\cdot,t)-G^{p+1}(v)(\cdot,t)\|^2 &= \|\widehat{G}(G^p(w))(\cdot,t)-\widehat{G}(G^p(v))(\cdot,t)\|^2 \\
&= \int_{-\infty}^{\infty}\left|\int_t^T \frac{\mathrm{e}^{(s-t-T-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(T+m)\xi^2}}(\hat{f}(\xi,s,G^p(w))-\hat{f}(\xi,s,G^p(v)))\,\mathrm{d}s\right|^2\mathrm{d}\xi \\
&\leqslant \int_{-\infty}^{\infty}\biggl(\int_t^T\left(\frac{\mathrm{e}^{(s-t-T-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(T+m)\xi^2}}\right)^2\mathrm{d}s \\
&\quad\times\int_t^T|\hat{f}(\xi,s,G^p(w))-\hat{f}(\xi,s,G^p(v))|^2\,\mathrm{d}s\biggr)^2\mathrm{d}\xi.
\end{aligned}
$$

By using (3.14), we have

$$
\begin{aligned}
&\|G^{p+1}(w)(\cdot,t)-G^{p+1}(v)(\cdot,t)\|^2 \\
&\qquad\leqslant \frac{1}{\varepsilon^2}(T-t)\int_t^T\|f(\cdot,s,G^p(w)(\cdot,s))-f(\cdot,s,G^p(v)(\cdot,s))\|^2\,\mathrm{d}s \\
&\qquad\leqslant \frac{K^2}{\varepsilon^2}(T-t)\int_t^T\|G^p(w)(\cdot,s)-G^p(v)(\cdot,s)\|^2\,\mathrm{d}s.
\end{aligned}
$$

If follows from

$$
\|G^p(w)(\cdot,t)-G^p(v)(\cdot,t)\|^2\leqslant\left(\frac{K}{\varepsilon}\right)^{2p}\frac{(T-t)^pC_1^p}{p!}\|w-v\|^2
$$

that

$$
\begin{aligned}
&\|G^{p+1}(w)(\cdot,t)-G^{p+1}(v)(\cdot,t)\|^2 \\
&\qquad\leqslant \frac{K^2}{\varepsilon^2}(T-t)\left(\frac{K}{\varepsilon}\right)^{2p}\int_t^T\frac{(T-s)^p}{p!}\,\mathrm{d}s\;C_1^p\|w-v\|^2 \\
&\qquad\leqslant \left(\frac{K}{\varepsilon}\right)^{2(p+1)}\frac{(T-t)^{(p+1)}C_1^{(p+1)}}{(p+1)!}\|w-v\|^2.
\end{aligned}
$$

Therefore, by the induction principle, we have for every m

$$
\|G^m(w)-G^m(v)\|\leqslant\left(\frac{K}{\varepsilon}\right)^m\frac{T^{m/2}}{\sqrt{m!}}C_1^m\|w-v\|
$$

for every $w,v\in C([0,T];L^2(\mathbb{R}))$. We consider $G\colon C([0,T];L^2(\mathbb{R}))\to C([0,T];L^2(\mathbb{R}))$. Since

$$
\lim_{m\to\infty}\left(\frac{K}{\varepsilon}\right)^m\frac{T^{m/2}C_1^m}{\sqrt{m!}}=0,
$$

there exists a positive integer number m_0 such that G^{m_0} is a contraction. It follows that $G^{m_0}(w) = w$ has a unique solution $w_\varepsilon \in C([0,T]; L^2(\mathbb{R}))$.

We claim that $G(w_\varepsilon) = w_\varepsilon$. In fact, one has $G(G^{m_0}(w_\varepsilon)) = G(w_\varepsilon)$. Hence, $G^{m_0}(G(w_\varepsilon)) = G(w_\varepsilon)$. By the uniqueness of the fixed point of G^{m_0}, one has $G(w_\varepsilon) = w_\varepsilon$, i.e., the equation $G(w) = w$ has a unique solution w_ε in $C([0,T]; L^2(\mathbb{R}))$. The main purpose of this paper is to estimate the error $\|w_\varepsilon - u\|$. To this end, we proceed by the next two steps.

Step 2. Let u_ε be the solution of the problem (3.11) corresponding to the final value φ. We shall estimate the error $\|w_\varepsilon - u_\varepsilon\|$.

From the formulas for w_ε and u_ε, we have

$$\hat{w}_\varepsilon(\xi,t) = \frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}}\hat{\varphi}_\varepsilon(\xi) - \int_t^T \frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,w_\varepsilon)\,\mathrm{d}s, \tag{3.15}$$

and

$$\hat{u}_\varepsilon(\xi,t) = \frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}}\hat{\varphi}(\xi) - \int_t^T \frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,u_\varepsilon)\,\mathrm{d}s. \tag{3.16}$$

Using the Parseval equality and the inequality $(a+b)^2 \leqslant 2a^2 + 2b^2$, we get

$$\begin{aligned} \|w_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,t)\|^2 &= \|\hat{w}_\varepsilon(\cdot,t) - \hat{u}_\varepsilon(\cdot,t)\|^2 \\ &\leqslant 2\int_{-\infty}^{\infty}\Big|\frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}}(\hat{\varphi}_\varepsilon(\xi) - \hat{\varphi}(\xi))\Big|^2\,\mathrm{d}\xi \\ &\quad + 2\int_{-\infty}^{\infty}\Big|\int_t^T \frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}}(\hat{f}(\xi,s,w_\varepsilon) - \hat{f}(\xi,s,u_\varepsilon))\,\mathrm{d}s\Big|^2\,\mathrm{d}\xi \\ &\leqslant J_1 + J_2. \end{aligned} \tag{3.17}$$

The term (3.17) can be estimated as follows.

$$\begin{aligned} J_1 &= 2\int_{-\infty}^{\infty}\Big|\frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k} + \mathrm{e}^{-(B(T)+m)\xi^2}}(\hat{\varphi}_\varepsilon(\xi) - \hat{\varphi}(\xi))\Big|^2\,\mathrm{d}\xi \\ &\leqslant 2H^2(k,m)\varepsilon^{\frac{2B(t)-2B(T)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(T)}{B(T)+m}}\|\hat{\varphi}_\varepsilon - \hat{\varphi}\|^2 \\ &\leqslant 2H^2(k,m)\varepsilon^{\frac{2B(t)-2B(T)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(T)}{B(T)+m}}\|\varphi_\varepsilon - \varphi\|^2, \end{aligned} \tag{3.18}$$

and

$$J_2 = 2\int_{-\infty}^{\infty}\left|\int_t^T \frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon+\mathrm{e}^{-(B(T)+m)\xi^2}}(\hat{f}(\xi,s,w_\varepsilon)-\hat{f}(\xi,s,u_\varepsilon))\,\mathrm{d}s\right|^2\mathrm{d}\xi \tag{3.19}$$

$$\begin{aligned}
&\leqslant 2(T-t)\int_{-\infty}^{\infty}\int_t^T H^2(k,m)\varepsilon^{\frac{2B(t)-2B(s)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(s)}{B(T)+m}}\\
&\quad\times|\hat{f}(\xi,s,w_\varepsilon)-\hat{f}(\xi,s,u_\varepsilon)|^2\,\mathrm{d}s\,\mathrm{d}\xi\\
&\leqslant 2(T-t)K^2H^2(k,m)\int_t^T \varepsilon^{\frac{2B(t)-2B(s)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(s)}{B(T)+m}}\\
&\quad\times\|w_\varepsilon(\cdot,s)-u_\varepsilon(\cdot,s)\|^2\,\mathrm{d}s.
\end{aligned}$$

Combining (3.17), (3.18), and (3.19), we have

$$\begin{aligned}
&\|w_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t)\|^2\\
&\leqslant 2H^2(k,m)\varepsilon^{\frac{2B(t)-2B(T)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(T)}{B(T)+m}}\|\varphi_\varepsilon-\varphi\|^2\\
&\quad+2(T-t)K^2H^2(k,m)\int_t^T \varepsilon^{\frac{2B(t)-2B(s)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(s)}{B(T)+m}}\\
&\qquad\times\|w_\varepsilon(\cdot,s)-u_\varepsilon(\cdot,s)\|^2\,\mathrm{d}s.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\varepsilon^{\frac{-2B(t)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{-2kB(t)}{B(T)+m}}\|w_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t)\|^2\\
&\leqslant 2H^2(k,m)\varepsilon^{\frac{-2B(T)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{-2kB(T)}{B(T)+m}}\|\varphi_\varepsilon-\varphi\|^2\\
&\quad+2K^2H^2(k,m)(T-t)\int_t^T \varepsilon^{\frac{-2B(s)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{-2kB(s)}{B(T)+m}}\\
&\qquad\times\|w_\varepsilon(\cdot,s)-u_\varepsilon(\cdot,s)\|^2\,\mathrm{d}s.
\end{aligned}$$

By using the Gronwall inequality, we obtain

$$\begin{aligned}
&\varepsilon^{\frac{-2B(t)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{-2kB(t)}{B(T)+m}}\|w_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t)\|^2\\
&\leqslant 2\mathrm{e}^{2K^2H^2(k,m)(T-t)^2}\varepsilon^{\frac{-2B(T)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{-2kB(T)}{B(T)+m}}\|\varphi_\varepsilon-\varphi\|^2.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
&\|w_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t)\| \\
&\quad\leqslant \sqrt{2}H(k,m)\mathrm{e}^{K^2H^2(k,m)(T-t)^2}\varepsilon^{\frac{B(t)-B(T)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(T)}{B(T)+m}}\|\varphi_\varepsilon-\varphi\| \\
&\quad\leqslant \sqrt{2}H(k,m)\mathrm{e}^{K^2H^2(k,m)(T-t)^2}\varepsilon^{\frac{B(t)-B(T)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(T)}{B(T)+m}}\varepsilon \\
&\quad= \sqrt{2}H(k,m)\mathrm{e}^{K^2H^2(k,m)(T-t)^2}\varepsilon^{\frac{B(t)+m}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(T)}{B(T)+m}}.
\end{aligned}$$

Step 3. Let u be the exact solution of problem (1.1) corresponding to the final value φ. We shall estimate the error $\|u_\varepsilon-u\|$.

Let u_ε be the function defined in step 2. We recall the Fourier transform of u and u_ε from (3.2) and (3.16):

$$\hat{u}(\xi,t)=\mathrm{e}^{(B(T)-B(t))\xi^2}\hat{\varphi}(\xi)-\int_t^T\mathrm{e}^{-(B(t)-B(s))\xi^2}\hat{f}(\xi,s,u)\,\mathrm{d}s \tag{3.20}$$

and

$$\begin{aligned}
\hat{u}_\varepsilon(\xi,t)=&\frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{\varphi}(\xi) \\
&-\int_t^T\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,u_\varepsilon)\,\mathrm{d}s.
\end{aligned}\tag{3.21}$$

By direct calculation, from (3.20) and (3.21) we get

$$\begin{aligned}
&\hat{u}(\xi,t)-\hat{u}_\varepsilon(\xi,t) \\
&\quad=\Big(\mathrm{e}^{(B(T)-B(t))\xi^2}-\frac{\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\Big)\hat{\varphi}(\xi) \\
&\qquad-\int_t^T\mathrm{e}^{-(B(t)-B(s))\xi^2}\hat{f}(\xi,s,u)\,\mathrm{d}s+\int_t^T\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,u_\varepsilon)\,\mathrm{d}s \\
&\quad=\frac{\varepsilon\mathrm{e}^{(B(T)-B(t))\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{\varphi}(\xi)-\int_t^T\frac{\mathrm{e}^{(B(s)-B(t))\xi^2}\varepsilon}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,u)\,\mathrm{d}s \\
&\qquad+\int_t^T\frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}(\hat{f}(\xi,s,u)-\hat{f}(\xi,s,u_\varepsilon))\,\mathrm{d}s.
\end{aligned}$$

Using the Parseval equality, we obtain

$$
\begin{aligned}
\|u(\cdot,t)-u_\varepsilon(\cdot,t)\|^2 &= \int_{-\infty}^{\infty} |\hat{u}(\xi,t)-\hat{u}_\varepsilon(\xi,t)|^2 \,\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} \bigg| \frac{\varepsilon\xi^{2k}\mathrm{e}^{(B(T)-B(t))\xi^2}}{\varepsilon\xi^{2k}-\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{\varphi}(\xi) - \int_t^T \frac{\mathrm{e}^{(B(s)-B(t))\xi^2}\varepsilon}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{f}(\xi,s,u)\,\mathrm{d}s \\
&\qquad + \int_t^T \frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}(\hat{f}(\xi,s,u)-\hat{f}(\xi,s,u_\varepsilon))\,\mathrm{d}s\bigg|^2 \,\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} \bigg| \frac{\varepsilon\xi^{2k}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\hat{u}(\xi,t) \\
&\qquad - \int_t^T \frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}(\hat{f}(\xi,s,u)-\hat{f}(\xi,s,u_\varepsilon))\,\mathrm{d}s\bigg|^2 \,\mathrm{d}\xi \\
&\leqslant 2\int_{-\infty}^{\infty} \bigg| \frac{\varepsilon\mathrm{e}^{-(B(t)+m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}\xi^{2k}\mathrm{e}^{(B(t)+m)\xi^2}\hat{u}(\xi,t)\bigg|^2 \,\mathrm{d}\xi \\
&\quad + 2\int_{-\infty}^{\infty} \bigg| \int_t^T \frac{\mathrm{e}^{(B(s)-B(t)-B(T)-m)\xi^2}}{\varepsilon\xi^{2k}+\mathrm{e}^{-(B(T)+m)\xi^2}}|\hat{f}(\xi,s,u)-\hat{f}(\xi,s,u_\varepsilon)|\,\mathrm{d}s\bigg|^2 \,\mathrm{d}\xi.
\end{aligned}
$$

It follows from Lemma 3.1 that

$$
\begin{aligned}
&\|u(\cdot,t)-u_\varepsilon(\cdot,t)\|^2 \\
&\quad \leqslant 2H^2(k,m)\varepsilon^{\frac{2B(t)+2m}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(T)}{B(T)+m}} \\
&\qquad \times \int_{-\infty}^{\infty} |\xi^{2k}\mathrm{e}^{(B(t)+m)\xi^2}\hat{u}(\xi,t)|^2\,\mathrm{d}\xi + 2H^2(k,m) \\
&\qquad \times \int_{-\infty}^{\infty} \bigg| \int_t^T \varepsilon^{\frac{B(t)-B(s)}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(s)}{B(T)+m}} |\hat{f}(\xi,s,u)-\hat{f}(\xi,s,u_\varepsilon)|\,\mathrm{d}s\bigg|^2 \,\mathrm{d}\xi \\
&\quad = 2\widetilde{A_1} + 2\widetilde{A_2},
\end{aligned}
$$

where the term $\widetilde{A_1}$ is equal to

$$
\widetilde{A_1} = \varepsilon^{\frac{2B(t)+2m}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(T)}{B(T)+m}} \int_{-\infty}^{\infty} |\xi^{2k}\mathrm{e}^{(B(t)+m)\xi^2}\hat{u}(\xi,t)|^2\,\mathrm{d}\xi.
$$

We estimate $\widetilde{A_2}$ as follows:

$$\begin{aligned}
\widetilde{A_2} &= \int_{-\infty}^{\infty} \bigg| \int_t^T \varepsilon^{\frac{B(t)-B(s)}{B(T)+m}} \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(s)}{B(T)+m}} |\hat{f}(\xi,s,u)-\hat{f}(\xi,s,u_\varepsilon)|\,\mathrm{d}s\bigg|^2 \,\mathrm{d}\xi \\
&\leqslant \varepsilon^{\frac{2B(t)+2m}{B(T)+m}} \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(T)}{B(T)+m}} (T-t) \int_{-\infty}^{\infty}\int_t^T \varepsilon^{\frac{-2B(s)-2m}{B(T)+m}} \\
&\quad \times \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(T)-kB(s)}{B(T)+m}} |\hat{f}(\xi,s,u)-\hat{f}(\xi,s,u_\varepsilon)\,\mathrm{d}s|^2\,\mathrm{d}\xi \\
&= \varepsilon^{\frac{2B(t)+2m}{B(T)+m}} \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(T)}{B(T)+m}} (T-t) \int_t^T \varepsilon^{\frac{-2B(s)-2m}{B(T)+m}} \\
&\quad \times \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(T)-2kB(s)}{B(T)+m}} \|f(\cdot,s,u(\cdot,s))-f(\cdot,s,u_\varepsilon(\cdot,s))\|^2\,\mathrm{d}s \\
&\leqslant \varepsilon^{\frac{2B(t)+2m}{B(T)+m}} \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(T)-2kB(s)}{B(T)+m}} K^2(T-t) \\
&\quad \times \int_t^T \varepsilon^{\frac{-2B(s)-2m}{B(T)+m}} \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(T)-2kB(s)}{B(T)+m}} \|u(\cdot,s)-u_\varepsilon(\cdot,s)\|^2\,\mathrm{d}s.
\end{aligned}$$

This implies

$$\begin{aligned}
&\|u(\cdot,t)-u_\varepsilon(\cdot,t)\|^2 \\
&\quad \leqslant 2H^2(k,m)\varepsilon^{\frac{2B(t)+2m}{B(T)+m}} \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(t)-2kB(T)}{B(T)+m}} \\
&\qquad \times \bigg[\int_{-\infty}^{\infty} |\xi^{2k}\mathrm{e}^{(B(t)+m)\xi^2}\hat{u}(\xi,t)|^2\,\mathrm{d}\xi + 2K^2H^2(k,m)(T-t)\int_t^T \varepsilon^{\frac{-2B(s)-2m}{B(T)+m}} \\
&\qquad \times \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(T)-2kB(s)}{B(T)+m}} \|u(\cdot,s)-u_\varepsilon(\cdot,s)\|^2\,\mathrm{d}s\bigg].
\end{aligned}$$

Thus

$$\begin{aligned}
&\varepsilon^{\frac{-2B(t)-2m}{B(T)+m}} \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(T)-2kB(t)}{B(T)+m}} \|u(\cdot,t)-u_\varepsilon(\cdot,t)\|^2 \\
&\quad \leqslant A(k,m) + 2K^2H^2(k,m)T\int_t^T \varepsilon^{\frac{-2B(s)-2m}{B(T)+m}} \\
&\qquad \times \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(T)-2kB(s)}{B(T)+m}} \|u(\cdot,s)-u_\varepsilon(\cdot,s)\|^2\,\mathrm{d}s,
\end{aligned}$$

where $A(k,m)$ is defined in (3.6). Applying the Gronwall inequality, we obtain

$$\begin{aligned}
&\varepsilon^{\frac{-2B(t)-2m}{B(T)+m}} \Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{2kB(T)-2kB(t)}{B(T)+m}} \|u(\cdot,t)-u_\varepsilon(\cdot,t)\|^2 \\
&\qquad \leqslant A(k,m)\mathrm{e}^{2K^2H^2(k,m)T(T-t)}.
\end{aligned}$$

Hence,

$$\begin{aligned} (3.22)\qquad &\|u(\cdot,t)-u_\varepsilon(\cdot,t)\| \\ &\leqslant \sqrt{A(k,m)}\mathrm{e}^{K^2H^2(k,m)T(T-t)}\varepsilon^{\frac{B(t)+m}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(T)}{B(T)+m}}. \end{aligned}$$

By the results of Step 2 and Step 3, we get the following estimate by using the triangle inequality:

$$\begin{aligned} \|w_\varepsilon(\cdot,t)-u(\cdot,t)\| &\leqslant \|w_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t)\| + \|u_\varepsilon(\cdot,t)-u(\cdot,t)\| \\ &\leqslant \sqrt{2}H(k,m)\mathrm{e}^{K^2H^2(k,m)(T-t)^2}\varepsilon^{\frac{B(t)+m}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(T)}{B(T)+m}} \\ &\quad + \sqrt{A(k,m)}\mathrm{e}^{K^2H^2(k,m)T(T-t)}\varepsilon^{\frac{B(t)+m}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(T)}{B(T)+m}} \\ &\leqslant P(k,m)\varepsilon^{\frac{B(t)+m}{B(T)+m}}\Big(\ln\Big(\frac{I(k,m)}{\varepsilon}\Big)\Big)^{\frac{kB(t)-kB(T)}{B(T)+m}} \end{aligned}$$

for all $t\in[0,T]$. This completes the proof of Theorem. □

Remark 3.5. (1) a) In Tautenhahn and Schröter [13], the authors regularized the homogeneous problem ($f=0$) and showed that the best possible estimate of the worst case error is given by

$$\|u(\cdot,t)-u^\beta(\cdot,t)\| \leqslant 2E^{1-t/T}\varepsilon^{t/T},$$

where E is a positive constant such that

$$\|u(\cdot,0)\| \leqslant E. \tag{3.23}$$

If $a(t)=1$ and $f=k=m=0$ then we have

$$\int_{-\infty}^{\infty} |\mathrm{e}^{(t+m)\xi^2}\hat{u}(\xi,t)|^2\,\mathrm{d}\xi = \|u(\cdot,0)\|^2.$$

Then the condition (3.3) is similar to (3.23) and it may be natural and acceptable. Moreover, in this case, our result (3.4) is of the same order as the results of Tautenhahn. Thus, in this case, our method is of optimal order.

b) If $a(t)=1$ and $f=f(x,t,u)$, we refer the readers to [9]. In [9], the authors established a regularized solution u^ε and, under a strong smoothness assumption on the exact solution u, they obtained the error estimate

$$\|u(\cdot,t)-u^\varepsilon(\cdot,t)\|^2 \leqslant M\varepsilon^{t/T}, \tag{3.24}$$

where M is a constant dependent on u. The right-hand side of (3.24) is not close to zero if $t=0$ for any nonzero epsilon. The convergence of the approximate solution is very slow when t is in a neighborhood of zero. This is an open point of the paper [9]. If $m=0$ then the error (3.4) is similar to the results obtained by Trong and Quan [9]. To improve this, we should choose $m>0$ to attain the error of Hölder type.

(2) If $m=0$, then with $k>0$ we get the logarithmic order

$$\Big(\ln\Big(\frac{I(k,0)}{\varepsilon}\Big)\Big)^{(kB(t)-kB(T))/B(T)}.$$

(3) In Theorem 1, if we select $m>0$ then under a strong assumption of u, we get the error which is of order $\varepsilon^{m/(B(T)+m)}$. This error estimate is much better than the logarithmic order estimates obtained in some previous results [8], [4].

Remark 3.6. We now compare our result with those in [15] in the case $f(x,t,u)=0$ and $a(t)=1$. If $\|u(\cdot,0)\|_{H^s}\leqslant E$ then there exists a positive F such that

$$\|w_\varepsilon(\cdot,t)-u(\cdot,t)\|\leqslant F\varepsilon^{t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^{s/2}. \tag{3.25}$$

Under the same condition $\|u(\cdot,0)\|_{H^s}\leqslant E$, the error (3.25) is of the same order as that of Wang in [15] (see Remark 4.6, Theorem 4.4).

To prove Remark 3.6, we need the following lemma.

Lemma 3.7. *Let $s>0$, $X\geqslant 0$, $m>0$. Then for all $0\leqslant t\leqslant T$ and $0<\varepsilon<1$, we have*

$$\frac{\varepsilon \mathrm{e}^{-tX}}{(1+X)^s(\varepsilon+\mathrm{e}^{-(T+m)X})}\leqslant C(k,s,m)\varepsilon^{t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^s,$$

where $C(k,s,m)=(s(T+m)/m)^s\mathrm{e}^{1-s}$.

Proof. We have two cases.

Case 1: $X\in[0,1/(T+m)]$. It is clear that

$$\frac{\varepsilon \mathrm{e}^{-tX}}{(1+X)^s(\varepsilon+\mathrm{e}^{-(T+m)X})}\leqslant\frac{\varepsilon}{(1+X)^s\mathrm{e}^{-(T+m)X}}\leqslant\varepsilon\mathrm{e}^{(T+m)X}\leqslant\mathrm{e}\varepsilon.$$

From the inequality $\varepsilon\leqslant(s/\mathrm{e})^s(1/\ln(1/\varepsilon))^s$, we get

$$\frac{\varepsilon \mathrm{e}^{-tX}}{(1+X)^k(\varepsilon+\mathrm{e}^{-(T+m)X})}\leqslant s^s\mathrm{e}^{1-s}\Big(\frac{1}{\ln(1/\varepsilon)}\Big)^s. \tag{3.26}$$

Case 2: $X > 1/(T+m)$. Set $\mathrm{e}^{-(T+m)X} = \varepsilon Y$. Then we obtain

$$\begin{aligned}
(3.27)\qquad \frac{\varepsilon \mathrm{e}^{-tX}}{(1+X)^s(\varepsilon + \mathrm{e}^{-(T+m)X})} &= \frac{\varepsilon(\varepsilon Y)^{\frac{t}{T+m}}}{\varepsilon + \varepsilon Y}\Big(\frac{T+m}{T+m-\ln(\varepsilon Y)}\Big)^s \\
&= \varepsilon^{\frac{t}{T+m}}\frac{Y^{\frac{t}{T+m}}}{1+Y}\Big(\frac{T+m}{T+m-\ln(\varepsilon Y)}\Big)^s \\
&= \varepsilon^{\frac{t}{T+m}}\frac{Y^{\frac{t}{T+m}}}{1+Y}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^s\Big(\frac{-\ln(\varepsilon)}{T+m-\ln(\varepsilon Y)}\Big)^s \\
&= \varepsilon^{\frac{t}{T+m}}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^s\frac{Y^{\frac{t}{T+m}}}{1+Y}\Big(\frac{-\ln(\varepsilon)}{T+m-\ln(\varepsilon Y)}\Big)^s.
\end{aligned}$$

We continue to estimate the term $(Y^{t/(T+m)}/(1+Y))(-\ln(\varepsilon)/(T+m-\ln(\varepsilon Y)))^s$.

If $0 < Y \leqslant 1$ then $0 < -\ln(\varepsilon) < -\ln(\varepsilon Y)$, thus

$$(3.28)\qquad \frac{Y^{t/(T+m)}}{1+Y}\Big(\frac{-\ln(\varepsilon)}{T+m-\ln(\varepsilon Y)}\Big)^s < 1,$$

while if $Y > 1$ then $\ln Y > 0$ and $\ln(\varepsilon Y) = -(T+m)X < -1$ due to the assumption $X \in (1/(T+m), \infty)$. Therefore, $\ln Y(1+\ln(\varepsilon Y)) \leqslant 0$. This implies that

$$0 < \frac{-\ln\varepsilon}{T+m-\ln(\varepsilon Y)} < \frac{-\ln\varepsilon}{-\ln(\varepsilon Y)} < 1 + \ln Y.$$

Hence, in this case, we get

$$\begin{aligned}
\frac{Y^{t/(T+m)}}{1+Y}\Big(\frac{-\ln(\varepsilon)}{T+m-\ln(\varepsilon Y)}\Big)^s &< (1+\ln Y)^s\frac{Y^{t/(T+m)}}{Y} \\
&< (1+\ln Y)^s Y^{t/(T+m)-1}.
\end{aligned}$$

Set $g(x) = (1+\ln x)^s x^{t/(T+m)-1}$ for $x > \mathrm{e}^{-1}$. Taking the derivative of this function, we get

$$g'(x) = (1+\ln x)^{s-1}x^{t/(T+m)-2}\Big(s - \Big(1 - \frac{t}{T+m}\Big)(1+\ln x)\Big).$$

The function g has maximum at the point x_0 such that $g'(x_0) = 0$. This implies that $x_0 = \mathrm{e}^{((s-1)(T+m)+t)/(T+m-t)}$. Therefore,

$$(3.29)\qquad \sup_{x\geqslant 1}(1+\ln x)^s x^{t/(T+m)-1} \leqslant g(x_0) = \Big(\frac{k(T+m)}{T+m-t}\Big)^s \mathrm{e}^{1-k-t/(T+m)}.$$

Since $g(x_0) > g(1) = 1$ and due to (3.28), (3.29), we have

$$\frac{Y^{t/(T+m)}}{1+Y}\Big(\frac{-\ln(\varepsilon)}{T+m-\ln(\varepsilon Y)}\Big)^s \leqslant \Big(\frac{s(T+m)}{T+m-t}\Big)^s \mathrm{e}^{1-s-t/(T+m)} \leqslant \Big(\frac{s(T+m)}{m}\Big)^s \mathrm{e}^{1-s}.$$

From (3.27), we get

$$\frac{\varepsilon \mathrm{e}^{-tX}}{(1+X)(\varepsilon+\mathrm{e}^{-(T+m)X})} \leqslant \Big(\frac{s(T+m)}{m}\Big)^s \mathrm{e}^{1-s}\varepsilon^{t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^s \leqslant C(k,s,m)\varepsilon^{t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^s. \tag{3.30}$$

where $C(k,s,m) = (s(T+m)/m)^s\mathrm{e}^{1-s}$. □

P r o o f of Remark 2. Since (3.2) and (3.16), we obtain

$$\hat{u}(\xi,t) - \hat{u}_\varepsilon(\xi,t) = \Big(\mathrm{e}^{(T-t)\xi^2} - \frac{\mathrm{e}^{-(t+m)\xi^2}}{\varepsilon+\mathrm{e}^{-(T+m)\xi^2}}\Big)\hat{\varphi}(\xi) = \frac{\varepsilon \mathrm{e}^{(T-t)\xi^2}}{\varepsilon+\mathrm{e}^{-(T+m)\xi^2}}\hat{\varphi}(\xi).$$

It follows from $\mathrm{e}^{T\xi^2}\hat{\varphi}(\xi) = \hat{u}(\xi,0)$ that

$$\hat{u}(\xi,t) - \hat{u}_\varepsilon(\xi,t) = \frac{\varepsilon \mathrm{e}^{-t\xi^2}}{\varepsilon+\mathrm{e}^{-(T+m)\xi^2}}\hat{u}(\xi,0).$$

Using Lemma 3, we obtain

$$\begin{aligned}\|u(\cdot,t) - u_\varepsilon(\cdot,t)\|^2 &= \|\hat{u}(\xi,t) - \hat{u}_\varepsilon(\xi,t)\|^2 \\ &= \int_{-\infty}^{\infty}\Big(\frac{\varepsilon \mathrm{e}^{-t\xi^2}}{\varepsilon+\mathrm{e}^{-(T+m)\xi^2}}\hat{u}(\xi,0)\Big)^2\,\mathrm{d}\xi \\ &= \int_{-\infty}^{\infty}\Big(\frac{\varepsilon \mathrm{e}^{-t\xi^2}}{(1+\xi^2)^{s/2}(\varepsilon+\mathrm{e}^{-(T+m)\xi^2})}(1+\xi^2)^{s/2}\hat{u}(\xi,0)\Big)^2\,\mathrm{d}\xi \\ &\leqslant C^2(k,s,m)\varepsilon^{2t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^s\int_{-\infty}^{\infty}(1+\xi^2)^s|\hat{u}(\xi,0)|^2\,\mathrm{d}\xi \\ &\leqslant C^2(k,s,m)\varepsilon^{2t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^s\|u(\cdot,0)\|^2_{H^s}.\end{aligned}$$

Hence,

$$\|u(\cdot,t) - u_\varepsilon(\cdot,t)\| \leqslant C(k,s,m)\varepsilon^{t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^{s/2}\|u(\cdot,0)\|_{H^s} \leqslant EC(k,s,m)\varepsilon^{t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^{s/2}.$$

By the results of Step 2 and Step 3, we get the following estimate by using the triangle inequality:

$$\begin{aligned}\|w_\varepsilon(\cdot,t)-u(\cdot,t)\| &\leqslant \|w_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t)\|+\|u_\varepsilon(\cdot,t)-u(\cdot,t)\|\\ &\leqslant \sqrt{2}\mathrm{e}^{K^2(T-t)^2}\varepsilon^{(t+m)/(T+m)}+EC(k,s,m)\varepsilon^{t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^{s/2}.\end{aligned}$$

Since ε is small enough, there exists a positive constant D such that

$$\varepsilon^{m/(T+m)}\leqslant D\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^{s/2}.$$

Then, we conclude that

$$\|w_\varepsilon(\cdot,t)-u(\cdot,t)\|\leqslant\big(\sqrt{2}\mathrm{e}^{K^2(T-t)^2}D+EC(k,s,m)\big)\varepsilon^{t/(T+m)}\Big(\frac{T+m}{\ln(1/\varepsilon)}\Big)^{s/2}.$$

This completes the proof of Remark (3.6). □

4. Conclusion

We have considered a regularization problem for a nonlinear backward heat equations with time-dependent coefficient, namely Problem (1.1). We have also established an error estimate of Hölder type for all $t\in[0,T]$. This estimate improves some results of several earlier works. In the future, we will consider the regularized problem for the problem

$$(4.1)\qquad \begin{cases}\dfrac{\partial u}{\partial t}-\dfrac{\partial}{\partial x}\Big(a(x,t)\dfrac{\partial u}{\partial x}\Big)=f(x,t,u(x,t)), & (x,t)\in\mathbb{R}\times(0,T),\\ u(x,T)=\varphi(x), \quad x\in\mathbb{R},\end{cases}$$

where $a(x,t)$ is a function dependent on both the variables x,t.

Acknowledgement. This project was supported by Institute for Computational Science and Technology Ho Chi Minh City under project named *"Inverse parabolic equation and application to groundwater polution source" (PI: Nguyen Huy Tuan)*. The author would like to thank the anonymous referees for their valuable suggestions and comments leading to the improvement of the manuscript.

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