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A DE BRUIJN-ERDŐS THEOREM FOR 1-2 METRIC SPACES

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Abstract. A special case of a combinatorial theorem of De Bruijn and Erdős asserts that every noncollinear set of n points in the plane determines at least n distinct lines. Chen and Chvátal suggested a possible generalization of this assertion in metric spaces with appropriately defined lines. We prove this generalization in all metric spaces where each nonzero distance equals 1 or 2.

Keywords: line in metric space; De Bruijn-Erdős theorem

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It is well known that

(i) *every noncollinear set of n points in the plane determines at least n distinct lines.*

As noted by Erdős [5], theorem (i) is a corollary of the Sylvester-Gallai theorem (asserting that, for every noncollinear set S of finitely many points in the plane, some line goes through precisely two points of S); it is also a special case of a combinatorial theorem proved later by De Bruijn and Erdős [4].

Chen and Chvátal [2] suggested that theorem (i) might be generalized in the framework of metric spaces. In a Euclidean space, line $\overline{uv}$ is characterized as

$$\overline{uv} = \{p\colon\ \operatorname{dist}(p,u)+\operatorname{dist}(u,v)=\operatorname{dist}(p,v) \text{ or } \operatorname{dist}(u,p)+\operatorname{dist}(p,v)=\operatorname{dist}(u,v) \text{ or } \operatorname{dist}(u,v)+\operatorname{dist}(v,p)=\operatorname{dist}(u,p)\},$$

where dist is the Euclidean metric; in an arbitrary metric space (S,dist), the same relation may be taken for the definition of the line. (Unlike in the case of Euclidean

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lines, $x, y \in \overline{uv}$, $x \neq y$ does not imply $u, v \in \overline{xy}$; nevertheless, $x \in \overline{uv}$, $x \neq u$ still implies $v \in \overline{xu}$.) With this definition of lines in metric spaces, Chen and Chvátal asked:

(ii) *True or false? Every metric space on n points, where $n \geqslant 2$, either has at least n distinct lines or else has a line that consists of all n points.*

Let us say that a metric space on n points has the *De Bruijn-Erdős property* if it either has at least n distinct lines or else has a line that consists of all n points: now we may state (ii) by asking whether or not all metric spaces on at least 2 points have the De Bruijn-Erdős property. A survey of results related to this question appears in [1].

By a 1-2 *metric space*, we mean a metric space where each nonzero distance is 1 or 2. Chiniforooshan and Chvátal [3] proved that

(iii) *every* 1-2 *metric space on n points has $\Omega(n^{4/3})$ distinct lines and this bound is tight.*

This result states that all sufficiently large 1-2 metric spaces have a property far stronger than the De Bruijn-Erdős property, but it does not imply that all 1-2 metric spaces on at least 2 points have the De Bruijn-Erdős property. The purpose of the present note is to remove this blemish.

Theorem 1. *All* 1-2 *metric spaces on at least* 2 *points have the De Bruijn-Erdős property.*

The rest of this note is devoted to a proof of Theorem 1. A key notion in the proof, one borrowed from [3], is the notion of *twins* in a 1-2 metric space: these are points u, v such that $\operatorname{dist}(u, v) = 2$ and $\operatorname{dist}(u, w) = \operatorname{dist}(v, w)$ for all points w distinct from both u and v. Use of this notion in counting lines is pointed out in the following claim (also borrowed from [3]), whose proof is straightforward.

Claim 1. If u_1, u_2, u_3, u_4 are four distinct points in a 1-2 metric space, then

- ▷ if $\operatorname{dist}(u_1, u_2) \neq \operatorname{dist}(u_3, u_4)$, then $\overline{u_1u_2} \neq \overline{u_3u_4}$,
- ▷ if $\operatorname{dist}(u_1, u_2) = \operatorname{dist}(u_2, u_3) = 2$, then $\overline{u_1u_2} \neq \overline{u_2u_3}$,
- ▷ if $\operatorname{dist}(u_1, u_2) = \operatorname{dist}(u_2, u_3) = 1$ and u_1, u_3 are not twins, then $\overline{u_1u_2} \neq \overline{u_2u_3}$.

By a *critical* 1-2 *metric space*, we shall mean a smallest counterexample to Theorem 1; in a sequence of claims, we shall gradually prove the nonexistence of a critical 1-2 metric space. We shall say that a line in a metric space is *universal* if, and only if, it consists of all points of the space.

Claim 2. For every pair u, v of twins in a critical 1-2 metric space, there is a third point w in this space such that $\operatorname{dist}(u, w) = \operatorname{dist}(v, w) = 2$ and $\operatorname{dist}(x, y) = 1$ whenever $x \in \{u, v, w\}$, $y \notin \{u, v, w\}$.

P r o o f. Let S denote the space we are dealing with. Since S is critical, S does not have the De Bruijn-Erdős property and $S \setminus u$ has the De Bruijn-Erdős property. We will derive the existence of w from these two facts.

The assumption that u, v are twins implies that

(a) if x, y are distinct points in $S \setminus \{u, v\}$, then the line $\overline{xy}$ in S contains either both u, v or neither of u, v;

(b) if $w \in S \setminus u$ and $\mathrm{dist}(w, v) = 1$, then the line $\overline{wv}$ in S (and the line $\overline{wu}$ in S) contains both u, v;

(c) if $w \in S \setminus u$ and $\mathrm{dist}(w, v) = 2$, then the line line $\overline{wv}$ in S contains v and not u and the line $\overline{wu}$ in S contains u and not v.

Since S does not have the De Bruijn-Erdős property, we have $\overline{uv} \neq S$; since u and v are twins, it follows that

(d) there is a w in $S \setminus u$ such that $\mathrm{dist}(w, v) = 2$.

From (a), (b), (c), (d), we conclude that

(e) the number of lines in S exceeds the number of lines in $S \setminus u$.

Since S does not have the De Bruijn-Erdős property, the number of lines in S is less than $|S|$, and so (e) implies that the number of lines in $S \setminus u$ is less than $|S \setminus u|$; since $S \setminus u$ has the De Bruijn-Erdős property, it follows that

(f) $S \setminus u$ has a universal line.

Since S does not have the De Bruijn-Erdős property,

(g) S has no universal line.

Facts (a), (f), and (g) together imply that some line $\overline{wv}$ in $S \setminus u$ is universal. Now (b) and (g) together imply that $\mathrm{dist}(w, v) = 2$; since u, v are twins, it follows that $\mathrm{dist}(u, v) = 2$ and $\mathrm{dist}(w, u) = 2$. Since $\overline{wv}$ is a universal line in $S \setminus u$, we have $\mathrm{dist}(w, y) = \mathrm{dist}(v, y) = 1$ whenever $y \notin \{u, v, w\}$; since u, v are twins, it follows that $\mathrm{dist}(u, y) = 1$ whenever $y \notin \{u, v, w\}$. □

Claim 3. No critical 1-2 metric space contains a pair of twins.

P r o o f. Assume the contrary: some critical 1-2 metric space S contains a pair of twins. We will show that S has at least $|S|$ lines, contradicting the assumption that S does not have the De Bruijn-Erdős property. For this purpose, consider the largest set $\{T_1, T_2, \ldots, T_k\}$ of pairwise disjoint three-point subsets of S such that $\mathrm{dist}(u, v) = 2$ whenever u, v are distinct points in the same T_i and such that $\mathrm{dist}(u, x) = 1$ whenever $u \in T_i$, $x \notin T_i$ for some i. Since S contains a pair of twins, Claim 2 guarantees that $k \geqslant 1$; we will derive the existence of $|S|$ lines in S from this fact.

Let $\mathcal{L}_1$ denote the set of all lines $\overline{uv}$ such that u, v are distinct points in the same T_i. If $\overline{uv} \in \mathcal{L}_1$, then $\overline{uv} = S \setminus w$, where $\{u, v, w\} = T_i$ for some i; it follows that

(a) $\mathcal{L}_1$ consists of the $3k$ sets $S \setminus w$ with w ranging through $\bigcup_{i=1}^{k} T_i$.

Next, choose a point r in T_1 and let $\mathcal{L}_2$ denote the set of all lines $\overline{rx}$ such that $x \in S \setminus \bigcup_{i=1}^{k} T_i$. Claim 2 and the maximality of k together guarantee that S contains no pair x, y of twins such that $x, y \in S \setminus \bigcup_{i=1}^{k} T_i$. This fact and Claim 1 together imply that

(b) $|\mathcal{L}_2| = |S| - 3k$.

Finally, note that each line in $\mathcal{L}_2$ includes all points of T_1 and no points of T_2. This observation and (a) together imply that $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$, and so $|\mathcal{L}_1 \cup \mathcal{L}_2| = |S|$ by (a) and (b). □

Each 1-2 metric space can be thought of as a complete graph with each edge uv labeled by $\text{dist}(u, v)$. Given edges uv, xy of this complete graph, let us write $uv \approx xy$ to mean that $\overline{uv} = \overline{xy}$. The following fact is a direct consequence of Claim 1 combined with Claim 3.

Claim 4. Each equivalence class of the equivalence relation $\approx$ in a critical 1-2 metric space is a set of pairwise disjoint edges with identical labels or else a (not necessarily proper) subset of a cycle of length four with alternating labels.

Claim 5. The size of each equivalence class of the equivalence relation $\approx$ in a critical 1-2 metric space on n points is at most $\max\{(n-1)/2, 4\}$.

Proof. This is a direct corollary of Claim 4 combined with the observation that an equivalence class of $n/2$ pairwise disjoint edges defines a universal line. □

Claim 6. Every critical 1-2 metric space has at most 7 points.

Proof. Consider an arbitrary critical 1-2 metric space and let n denote the number of its points. Since this space does not have the De Bruijn-Erdős property, it has fewer than n lines, and so its equivalence relation $\approx$ partitions the $n(n-1)/2$ edges of its complete graph into at most $n-1$ classes. Since the largest of these classes has size at least $n/2$, Claim 5 implies that $n/2 \leqslant \max\{(n-1)/2, 4\}$, and so $n \leqslant 8$. If $n = 8$, then the 28 edges of the complete graph are partitioned into 7 equivalence classes of size 4. Now Claim 4 and the absence of a universal line together imply that each of these equivalence classes is a cycle of length four. But this is impossible, since the edge set of the complete graph on eight vertices cannot be partitioned into cycles: each vertex of this graph has an odd degree. □

Claim 7. No critical 1-2 metric space has 7 points.

Proof. Consider an arbitrary critical 1-2 metric space on 7 points. Since this space does not have the De Bruijn-Erdős property, it has fewer than 7 lines, and so its equivalence relation $\approx$ partitions the 21 edges of its complete graph into at most

6 classes. By Claim 5, each of these classes has size at most 4, and so at least three of them have size precisely 4; by Claim 4, each of these three classes is a cycle of length four. Let G_1, G_2, G_3 denote these three subgraphs of the complete graph on seven vertices.

Since G_1, G_2, G_3 are pairwise edge-disjoint, every two of them share at most two vertices; since their union has only seven vertices, some two of them share at least two vertices; we may assume (after a permutation of subscripts if necessary) that G_1 and G_2 share precisely two vertices. Let us name these two vertices u, v. Since G_1 and G_2 are edge-disjoint, we may assume (after a switch of subscripts if necessary) that vertices u, v are adjacent in G_1 and nonadjacent in G_2.

Next, we may name w, x the remaining two vertices in G_1 in such a way that the four edges of G_1 are uv, vw, wx, ux; we may name y, z the remaining two vertices in G_2 in such a way that the four edges of G_2 are uy, uz, vz, vy. Since the labels on the edges of G_2 alternate, we may assume (after switching y and z if necessary) that $\mathrm{dist}(u, y) = 1$, $\mathrm{dist}(u, z) = 2$, $\mathrm{dist}(v, z) = 1$, $\mathrm{dist}(v, y) = 2$. Since $\overline{uy} = \overline{vy}$, we have $u \in \overline{vy}$; since $\mathrm{dist}(v, y) = 2$, it follows that $\mathrm{dist}(u, v) = 1$. In turn, since the labels on the edges of G_1 alternate, we have $\mathrm{dist}(v, w) = 2$, $\mathrm{dist}(w, x) = 1$, $\mathrm{dist}(u, x) = 2$.

Now $\mathrm{dist}(y, u) + \mathrm{dist}(u, v) = \mathrm{dist}(y, v)$, and so $y \in \overline{uv}$; since $uv \approx vw$, it follows that $y \in \overline{vw}$. But this is impossible, since $\mathrm{dist}(v, w) = 2$ and $\mathrm{dist}(v, y) = 2$. □

Claim 8. Every critical 1-2 metric space on 5 or 6 points contains pairwise distinct points u, v, w, x, y such that

$$\mathrm{dist}(u, w) = \mathrm{dist}(u, x) = \mathrm{dist}(v, w) = \mathrm{dist}(v, x) = 1,$$
$$\mathrm{dist}(u, v) = \mathrm{dist}(w, x) = 2,$$
$$\mathrm{dist}(u, y) \neq \mathrm{dist}(v, y), \quad \mathrm{dist}(w, y) \neq \mathrm{dist}(x, y).$$

P r o o f. Consider an arbitrary critical 1-2 metric space on n points such that $n = 5$ or $n = 6$. Since this space does not have the De Bruijn-Erdős property, it has fewer than n lines, and so its equivalence relation $\approx$ partitions the $n(n-1)/2$ edges of its complete graph into at most $n - 1$ classes. Since the largest of these classes has size at least 3, Claim 4 and the absence of a universal line together imply that there are points u, v, w, x such that

$$\mathrm{dist}(u, v) = 2,\ \mathrm{dist}(v, w) = 1,\ \mathrm{dist}(w, x) = 2 \text{ and } \overline{uv} = \overline{vw} = \overline{wx}$$

or else

$$\mathrm{dist}(v, w) = 1,\ \mathrm{dist}(w, x) = 2,\ \mathrm{dist}(u, x) = 1 \text{ and } \overline{vw} = \overline{wx} = \overline{ux}.$$

In both cases, the equality of the three lines implies that

$$\mathrm{dist}(u,w) = \mathrm{dist}(u,x) = \mathrm{dist}(v,w) = \mathrm{dist}(v,x) = 1,$$
$$\mathrm{dist}(u,v) = \mathrm{dist}(w,x) = 2.$$

Since w, x are not twins, there is a point y distinct from both of them and such that $\mathrm{dist}(w,y) \neq \mathrm{dist}(x,y)$; we will complete the proof by showing that $\mathrm{dist}(u,y) \neq \mathrm{dist}(v,y)$.

To do this, assume the contrary: $\mathrm{dist}(u,y) = \mathrm{dist}(v,y)$. Since $y \notin \overline{wx}$ and $\overline{vw} = \overline{wx}$, we have $y \notin \overline{vw}$, and so $\mathrm{dist}(v,y) = \mathrm{dist}(w,y)$. Now $\mathrm{dist}(u,y) \neq \mathrm{dist}(x,y)$, and so $y \in \overline{ux}$; since $y \notin \overline{wx}$, we cannot have $\overline{vw} = \overline{wx} = \overline{ux}$, and so we must have $\overline{uv} = \overline{vw} = \overline{wx}$. In particular, $y \notin \overline{uv}$; since $\mathrm{dist}(u,y) = \mathrm{dist}(v,y)$, we conclude that

$$\mathrm{dist}(u,y) = \mathrm{dist}(v,y) = \mathrm{dist}(w,y) = 2, \quad \mathrm{dist}(x,y) = 1.$$

Since u, v are not twins, there is a point z distinct from both of them and such that $\mathrm{dist}(u,z) \neq \mathrm{dist}(v,z)$; it follows that $\mathrm{dist}(x,z)$ is distinct from one of $\mathrm{dist}(u,z)$, $\mathrm{dist}(v,z)$, and so z belongs to one of the lines $\overline{ux}, \overline{vx}$. But then this line is universal, a contradiction. □

Claim 9. No critical 1-2 metric space has 5 or 6 points.

P r o o f. Consider an arbitrary critical 1-2 metric space on n points such that $n = 5$ or $n = 6$ and let u, v, w, x, y be as in Claim 8. We may assume (after a cyclic shift of u, w, v, x if necessary) that

$$\mathrm{dist}(u,w) = \mathrm{dist}(u,x) = \mathrm{dist}(v,w) = \mathrm{dist}(v,x) = 1,$$
$$\mathrm{dist}(u,v) = \mathrm{dist}(w,x) = 2,$$
$$\mathrm{dist}(u,y) = \mathrm{dist}(w,y) = 1, \quad \mathrm{dist}(v,y) = \mathrm{dist}(x,y) = 2.$$

Since

$$\overline{ux} \supseteq \{u,v,w,x,y\} \text{ and } \overline{vw} \supseteq \{u,v,w,x,y\},$$

absence of a universal line implies that $n = 6$ and that the sixth point of our space lies outside the lines $\overline{ux}$ and $\overline{vw}$. Let z denote this sixth point. Since $z \notin \overline{ux}$, $z \notin \overline{vw}$, we have $\mathrm{dist}(u,z) = \mathrm{dist}(x,z)$, $\mathrm{dist}(v,z) = \mathrm{dist}(w,z)$, and so symmetry allows us to distinguish three cases:

- ▷ $\mathrm{dist}(u,z) = \mathrm{dist}(x,z) = 1$, $\mathrm{dist}(v,z) = \mathrm{dist}(w,z) = 1$,
- ▷ $\mathrm{dist}(u,z) = \mathrm{dist}(x,z) = 1$, $\mathrm{dist}(v,z) = \mathrm{dist}(w,z) = 2$,
- ▷ $\mathrm{dist}(u,z) = \mathrm{dist}(x,z) = 2$, $\mathrm{dist}(v,z) = \mathrm{dist}(w,z) = 2$.

Each of these three cases comprises two metric spaces, one with $\mathrm{dist}(y,z) = 1$ and the other with $\mathrm{dist}(y,z) = 2$. Altogether, there are six metric spaces on six points to inspect; each of them has at least six lines. □

Claim 10. Every metric space on 2, 3, or 4 points has the De Bruijn-Erdős property.

Proof. Consider an arbitrary critical 1-2 metric space on n points with $2 \leqslant n \leqslant 4$. If each of its lines has precisely 2 points or if one of its lines has precisely n points, then this space has the De Bruijn-Erdős property; otherwise one of its lines has precisely 3 points and $n = 4$. Let T denote the 3-point line and let w denote the fourth point of the space. If there are distinct x, y in T such that $\overline{wx} = \overline{wy}$, then $\overline{xy}$ is a universal line; else the three lines $\overline{wx}$ with x ranging through T are pairwise distinct 2-point lines. □

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