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(n,d)-INJECTIVE COVERS, n-COHERENT RINGS, AND (n,d)-RINGS

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Abstract. It is known that a ring R is left Noetherian if and only if every left R-module has an injective (pre)cover. We show that (1) if R is a right n-coherent ring, then every right R-module has an (n,d)-injective (pre)cover; (2) if R is a ring such that every $(n,0)$-injective right R-module is n-pure extending, and if every right R-module has an $(n,0)$-injective cover, then R is right n-coherent. As applications of these results, we give some characterizations of (n,d)-rings, von Neumann regular rings and semisimple rings.

Keywords: cover; envelope; n-coherent ring; (n,d)-injective; (n,d)-ring

MSC 2010: 16D50, 16E40, 18G25

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary right R-modules. $\mathrm{Hom}(M,N)$ and $\mathrm{Ext}^m(M,N)$ mean $\mathrm{Hom}_R(M,N)$ and $\mathrm{Ext}^m_R(M,N)$, and $rD(R)$ and $wD(R)$ denote the usual right and weak, respectively, global dimension of a ring R.

Let $\mathscr{F}$ be a class of right R-modules and M a right R-module. Following [7], a homomorphism $\phi\colon F\to M$ with $F\in\mathscr{F}$ is called an *$\mathscr{F}$-precover* of M if for any homomorphism $f\colon F'\to M$ with $F'\in\mathscr{F}$, there is a homomorphism $g\colon F'\to F$ such that $\phi g=f$. Moreover, if the only such g is an automorphism of F when $F'=F$ and $f=\phi$, then the $\mathscr{F}$-precover ϕ is called an *$\mathscr{F}$-cover*. Dually, we have the definitions of an $\mathscr{F}$-preenvelope and an $\mathscr{F}$-envelope. We say that $\mathscr{F}$ is *(pre)covering* or *(pre)enveloping* provided every right R-module has an $\mathscr{F}$-(pre)cover or $\mathscr{F}$-(pre)envelope, respectively.

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Let n and d be non-negative integers. Following [4], we call a right R-module P *n-presented* if it has an n-presentation, that is, there exists an exact sequence of right R-modules

$$F_n \to F_{n-1} \to \ldots \to F_1 \to F_0 \to P \to 0$$

where each F_i is finitely generated free (equivalently projective), $i = 0, 1, \ldots, n$. It is clear that every m-presented R-module is n-presented for $m \geqslant n$. A ring R is called right *n-coherent* [4] provided every n-presented right R-module is $(n+1)$-presented. It is easy to see that R is right 0-coherent or 1-coherent if and only if R is right Noetherian coherent, respectively. Following [4] and [16], R is said to be a right (n, d)-ring if every n-presented right R-module has projective dimension at most d. A right R-module M is called *(n, d)-injective* [16] if $\mathrm{Ext}^{d+1}(N, M) = 0$ for any n-presented right R-module N. The $(1, 0)$-injective modules are also known as *absolutely pure modules* [11] and *FP-injective modules* [13]. For unexplained concepts and notation we refer the reader to [17], [3], [12], and [14].

It is known that a ring R is left Noetherian if and only if every left R-module has an injective (pre)cover (see [8], Theorem 5.4.1). Recently, Katherine Pinzon proved that if R is left coherent, then every left R-module has a $(1, 0)$-injective (pre)cover (see [11], Theorem 2.6 and Corollary 2.7). On the other hand, Mao and Ding [10], Theorem 3.9, proved that the class of (n, d)-injective R-modules is preenveloping for any ring R. It is natural to ask what conditions on R imply that the class of (n, d)-injective modules is precovering and what conditions on R imply that the class of (n, d)-injective modules is covering?

In Section 3, we show that (1) if R is a right n-coherent ring, then every right R-module has an (n, d)-injective (pre)cover; (2) if R is a ring such that every $(n, 0)$-injective right R-module is n-pure extending, and if every right R-module has an $(n, 0)$-injective cover, then R is right n-coherent $(n \geqslant 1)$; (3) R is a right n-coherent ring if and only if every n-pure submodule of an $(n, 1)$-injective right R-module is $(n, 1)$-injective $(n \geqslant 1)$.

In Section 4, as applications of the previous results, we give some characterizations of (n, d)-rings. We show that R is a right (n, d)-ring if and only if R is a right (n, d)-FC ring and the kernel of any (n, d)-injective cover of a right R-module is (n, d)-injective if and only if R is a right (n, d)-FC ring and the cokernel of any (n, d)-injective preenvelope of a right R- module is (n, d)-injective if and only if R is a right (n, d)-FC ring and R is a right $(n, d+m)$-ring for some $m \geqslant 0$ if and only if R is a right (n, d)-FC ring and every right R-module has an $(n, d+m)$-injective cover with the unique mapping property for some $m \geqslant 0$. Some known results are extended or obtained as corollaries. For example, we get that R is von Neumann regular if and only if R is a right FC ring and the kernel of any $(1, 0)$-injective cover of a right R-module is $(1,0)$-injective if and only if R is a right FC ring and every right R-module has a $(1,m)$-injective cover with the unique mapping property for some $m \geqslant 0$ if and only if R is a right FC ring and $wD(R) < \infty$; R is semisimple if and only if R is a QF ring and every right R-module has a $(0,m)$-injective cover with the unique mapping property for some $m \geqslant 0$ if and only if R is a QF ring and $rD(R) < \infty$.

2. Preliminaries

The results listed in this section will be important ingredients in proving our main results.

Proposition 2.1 ([10], Theorem 4.1). *The following assertions are equivalent for a ring R and $n \geqslant 1$:*

(1) *R is right n-coherent.*

(2) *For any short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if A and B are $(n,0)$-injective, then C is $(n,0)$-injective.*

Recall that a short exact sequence $0 \to A \to B \to C \to 0$ is said to be *n-pure* [10], Definition 3.5, if the sequence $\mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \to 0$ is exact for any n-presented R-module M. A submodule $A \subset B$ is called *n-pure* if the sequence $0 \to A \to B \to B/A \to 0$ is n-pure. It is clear that A is 1-pure in B if and only if it is pure, and if A is n-pure in B, then A is m-pure for any $m \geqslant n$.

Proposition 2.2 ([10], Proposition 3.6). *A module M is $(n,0)$-injective if and only if it is an n-pure submodule of an $(n,0)$-injective module N.*

In the following, we assume that n and d are non-negative integers.

Proposition 2.3 ([16], Proposition 3.1). *Let R be a right n-coherent ring. Then every direct limit of (n,d)-injective right R-modules is (n,d)-injective.*

The next proposition says that the class of (n,d)-injective modules is closed under extensions.

Proposition 2.4. *For any short exact sequence $0 \to A \to B \to C \to 0$ of right R-modules, if A and C are (n,d)-injective, then B is (n,d)-injective.*

P r o o f. It is straightforward. □

Proposition 2.5 ([10], Lemma 3.4). *Let R be a right n-coherent ring. Then the class of (n,d)-injective right R-modules is closed under cokernel of monomorphisms.*

Proposition 2.6. *Let R be a ring. Let $\mathscr{F}$ be the class of right R-modules closed under summands and isomorphisms. If $\mathscr{F}$ is precovering, then $\mathscr{F}$ is closed under direct sums.*

Proof. Let $(F_i)_{i\in I}$ be a family of right R-modules such that each $F_i \in \mathscr{F}$. Then we get an $\mathscr{F}$-precover $f\colon F \to \bigoplus F_i$. For each $j \in I$, let $l_j\colon F_j \to \bigoplus F_i$ be the canonical injection. Then there exists a homomorphism $g_j\colon F_j \to F$ such that $l_j = fg_j$. In addition, there is a homomorphism $\varphi\colon \bigoplus F_i \to F$ such that $g_j = \varphi l_j$, and hence $l_j = f\varphi l_j$. So $f\varphi$ is an isomorphism. Thus, $\bigoplus F_i$ is isomorphic to a summand of F by [1], Lemma 5.1, and so $\bigoplus F_i \in \mathscr{F}$. $\square$

3. (n,d)-injective covers and n-coherent rings

To show that over a right n-coherent ring R the class of (n,d)-injective right R-modules is covering, we need the following four lemmas.

Lemma 3.1. *Let R be a right n-coherent ring. Then every m-pure submodule of an (n,d)-injective right R-module is (n,d)-injective, for any non-negative integer m.*

Proof. Let N be an m-pure submodule of an (n,d)-injective right R-module M, and P an n-presented right R-module. Since R is right n-coherent, P has an $(m+d)$-presentation

$$F_{m+d} \to F_{m+d-1} \to \ldots \to F_1 \to F_0 \to P \to 0.$$

Let $K = \ker(F_{d-1} \to F_{d-2})$, then K is m-presented. Since M is (n,d)-injective, $\operatorname{Ext}^1(K,M) \cong \operatorname{Ext}^{d+1}(P,M) = 0$. In addition, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces a long exact sequence

$$0 \to \operatorname{Hom}(K,N) \to \operatorname{Hom}(K,M) \to \operatorname{Hom}(K,M/N) \to \operatorname{Ext}^1(K,N) \to \operatorname{Ext}^1(K,M) = 0.$$

Note that N is an m-pure submodule of M. So $\operatorname{Hom}(K,M) \to \operatorname{Hom}(K,M/N) \to 0$ is exact and hence $\operatorname{Ext}^{d+1}(P,N) \cong \operatorname{Ext}^1(K,N) = 0$, that is, N is (n,d)-injective. $\square$

A deep result of Robort El Bashir [2], Theorem 5, is that given a ring R and given a cardinal λ, there is a cardinal κ such that if $\operatorname{Card} M \geqslant \kappa$ and $\operatorname{Card} M/L \leqslant \lambda$ then L contains a nonzero pure submodule of M.

Lemma 3.2. *Let R be a right n-coherent ring, and M an (n,d)-injective right R-module. Given a cardinal λ, there is a cardinal κ such that if* $\operatorname{Card} M \geqslant \kappa$ *and* $\operatorname{Card} M/L \leqslant \lambda$ *then L contains a nonzero (n,d)-injective submodule of M.*

P r o o f. This follows from Lemma 3.1 and [2], Theorem 5. $\square$

The proof of the following lemma is similar to that of [11], Lemma 2.5.

Lemma 3.3. *Let R be right n-coherent and let* $\operatorname{Card} M = \lambda$ *for a right R-module M. There is a cardinal κ such that any homomorphism $E \to M$ with E (n,d)-injective has a factorization $E \to E' \to M$ with $E'(n,d)$-injective and* $\operatorname{Card} E' < \kappa$.

P r o o f. For any homomorphism $E \to M$ with E (n,d)-injective, by Lemma 3.2, we get a cardinal κ such that if $\operatorname{Card} E \geqslant \kappa$ and $\operatorname{Card} E/L \leqslant \lambda$ then L contains a nonzero (n,d)-injective submodule of E. If $\operatorname{Card} E < \kappa$, let $E' = E$ and we are done. So assume $\operatorname{Card} E \geqslant \kappa$. We can choose a submodule $A \subset E$ maximal with respect to the two properties that A is (n,d)-injective and that $A \subset \ker(E \to M)$. Let $E' = E/A$. Then it is easy to see that the homomorphism $E \to M$ has a factorization $E \to E' \to M$. Note that R is right n-coherent. So by Proposition 2.5, the exactness of the sequence $0 \to A \to E \to E' \to 0$ implies that E' is (n,d)-injective. Next we argue that $\operatorname{Card} E' < \kappa$. Suppose $\operatorname{Card} E' \geqslant \kappa$. Let $K = \ker(E' \to M)$. Clearly, $\operatorname{Card} E'/K \leqslant \operatorname{Card} M = \lambda$. Again by Lemma 3.2, there is a nonzero (n,d)-injective submodule B/A of E/A contained in K, and so $B \subset \ker(E \to M)$. Considering the exact sequence $0 \to A \to B \to B/A \to 0$, we see that B is also (n,d)-injective by Proposition 2.4. This contradicts the choice of A. Hence, $E \to M$ has a factorization $E \to E' \to M$ with E' (n,d)-injective and $\operatorname{Card} E' < \kappa$. $\square$

Lemma 3.4 ([11], Lemma 2.4). *Let $\mathscr{F}$ be a class of R-modules that is closed under direct sums. If $\mathscr{X} \subset \mathscr{F}$, for some set $\mathscr{X}$, is such that any homomorphism $F \to M$ with $F \in \mathscr{F}$ can be factored $F \to X \to M$ for some $X \in \mathscr{X}$, then M has an $\mathscr{F}$-precover.*

Theorem 3.5. *Let R be a right n-coherent ring. Then every right R-module has an (n,d)-injective precover.*

P r o o f. Let M be any right R-module with $\operatorname{Card} M = \lambda$. Then by Lemma 3.3, there is a cardinal κ such that any homomorphism $E \to M$ with E (n,d)-injective has a factorization $E \to E' \to M$ with E' (n,d)-injective and $\operatorname{Card} E' < \kappa$. Let $\mathbf{X}$ be any set with $\operatorname{Card} \mathbf{X} = \kappa$. Let $\mathbf{A}$ be all (n,d)-injective right R-modules such that $\mathbf{A} \subset \mathbf{X}$ (as sets). Hence, replacing E' by an isomorphic copy we may assume $E' \subset \mathbf{X}$ (as a set), and so we can apply Lemma 3.4. Thus, the conclusion follows. $\square$

Theorem 3.6. *Let R be a right n-coherent ring, then every right R-module has an (n,d)-injective cover.*

Proof. By Theorem 3.5, we get that every right R-module has an (n,d)-injective precover. But for a right n-coherent ring R, the class of (n,d)-injective right R-modules is closed under well ordered inductive limits by Proposition 2.3. Hence, the result follows from [8], Corollary 5.2.7. $\square$

Corollary 3.7 ([8], Theorem 5.4.1)**.** *R is right Noetherian if and only if every right R-module has an injective precover if and only if every right R-module has an injective cover.*

Proof. This follows from Theorem 3.5, Theorem 3.6, Proposition 2.6, and the fact that R is right Noetherian if and only if the class of injective right R-modules is closed under direct sums. $\square$

By Theorem 3.5, Theorem 3.6, and Corollary 3.7, we have

Corollary 3.8. *R is right Noetherian if and only if every right R-module has a $(0,d)$-injective precover if and only if every right R-module has a $(0,d)$-injective cover for any non-negative integer d.*

Corollary 3.9 ([11], Theorem 2.6)**.** *If R is a right coherent ring, then every right R-module has an absolutely pure precover.*

An (n,d)-injective (pre)cover is not necessarily an epimorphism. It is known that if R is right coherent or Noetherian and R_R (as a right R-module) is FP-injective or injective, respectively, then every right R-module has an epimorphic FP-injective or injective cover. In general, we have

Corollary 3.10. *The following are equivalent for a right n-coherent ring R:*

(1) *Every right R-module has an (n,d)-injective (pre)cover which is an epimorphism;*
(2) *R_R has an (n,d)-injective (pre)cover which is an epimorphism;*
(3) *R_R is (n,d)-injective;*
(4) *every (n,d)-injective (pre)cover of a right R-module is an epimorphism.*

Proof. Obvious. $\square$

Definition 3.11. Let $n \geqslant 1$. An *n-pure* monomorphism is a monomorphism $A \to B$ whose image is an n-pure submodule of B. A module B is called *n-pure extending* if for any n-pure submodule $A \subset B$, any n-pure monomorphism $A \to B$ can be extended to $B \to B$.

We note that the class of n-pure extending modules contains all *quasi-injective* modules [15], *pure injective* modules [9] and simple modules, and a homomorphism is an 1-pure monomorphism if and only if it is a *pure monomorphism* [9].

Lemma 3.12. *Let $n \geqslant 1$. Let B be an n-pure extending right R-module and $A \subset B$ an n-pure submodule. Let $f\colon A \to B$ be an n-pure monomorphism. Then there is a homomorphism $h\colon B \to B$ such that $hf = 1_A$ where 1_A is the identity homomorphism of A.*

Proof. Since f is a monomorphism, we get a homomorphism $f^{-1}\colon \operatorname{Im}(f) \to A$. By hypothesis, there exists a homomorphism $h\colon B \to B$ such that the restriction $h|_{\operatorname{Im}(f)} = f^{-1}$. So $hf = 1_A$. □

Let $\mathscr{F}$ be a class of R-modules. We will denote by $\mathscr{F}^{\perp} = \{C\colon \operatorname{Ext}^1(F, C) = 0$ for all $F \in \mathscr{F}\}$ the right orthogonal class of $\mathscr{F}$.

A question posed by Pinzon [11], Remark 2.8, is whether R must necessarily be right coherent in order that every right R-module have a $(1,0)$-injective cover. The following theorem gives a partial answer to this question.

Theorem 3.13. *Let $n \geqslant 1$. Let R be a ring such that every $(n,0)$-injective right R-module is n-pure extending. If every right R-module has an $(n,0)$-injective cover, then R is right n-coherent.*

Proof. For convenience, we let $\mathscr{F}$ denote the class of $(n,0)$-injective right R-modules. Let $0 \to A \to B \to C \to 0$ be any exact sequence of right R-modules where A and $B \in \mathscr{F}$. We want to show that $C \in \mathscr{F}$. By hypothesis, C has an $\mathscr{F}$-cover $F \to C$. Then it is easy to see that $F \to C$ is an epimorphism. Using a pullback construction for

$$\begin{tikzcd} & F \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

with exact rows and columns. Since both $A \in \mathscr{F}$ and $F \in \mathscr{F}$, we have $M \in \mathscr{F}$ by Proposition 2.4. Since $F \to C$ is an $\mathscr{F}$-cover, $K = \ker(F \to C) \in \mathscr{F}^{\perp}$ by [8], Corollary 7.2.3, page 156. Note that $B \in \mathscr{F}$. So $\mathrm{Ext}^1(B, K) = 0$. Thus, the middle column of the diagram above is split exact, and hence $K \in \mathscr{F}$. We claim: $K = 0$. Let

$$0 \longrightarrow K \xrightarrow{\iota} E \xrightarrow{\pi} L \longrightarrow 0$$

be exact with E injective. By hypothesis, L has an $\mathscr{F}$-cover $\gamma\colon D \to L$. Then γ is an epimorphism. We construct the pullback diagram of (L, γ, π) and get the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r, "i"] \arrow[d, equal] & N \arrow[r, "f_2"] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d] & E \arrow[r, "\pi"] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. Similarly to the proof above, we have $N \in \mathscr{F}$ and the middle column of the diagram above is split exact. Hence $N \cong K_1 \oplus E$. That is, $E \cong G$ for an injective submodule $G \subset N$. Since $\mathrm{Im}(i) \cong K \cong \mathrm{Im}(\iota) \subset E \cong G \subset N$, we have a monomorphism $h_1\colon \mathrm{Im}(i) \to G$ which induces an exact sequence $0 \longrightarrow \mathrm{Im}(i) \xrightarrow{h_1} N \xrightarrow{h_2} P \longrightarrow 0$, where $P = N/\mathrm{Im}(h_1)$. Note that $\mathrm{Im}(i)$ is $(n, 0)$-injective. So both $\mathrm{Im}(i)$ and $\mathrm{Im}(h_1)$ are n-pure in N by Proposition 2.2. Thus, by Lemma 3.12, there is a homomorphism $\theta\colon N \to N$ such that $\theta h_1 = f_1$ where f_1 is the identity homomorphism of $\mathrm{Im}(i)$. On the other hand, the sequence $0 \longrightarrow \mathrm{Im}(i) \xrightarrow{f_1} N \xrightarrow{f_2} D \longrightarrow 0$ is exact. Thus, we obtain the rows exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Im}(i) \arrow[r, "h_1"] \arrow[d, equal] & N \arrow[r, "h_2"] \arrow[d, "\theta"] & P \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \mathrm{Im}(i) \arrow[r, "f_1"] & K_1 \arrow[r, "f_2"] & D \arrow[r] & 0
\end{tikzcd}$$

Next we construct an exact sequence $0 \longrightarrow N \xrightarrow{\alpha} P \oplus N \xrightarrow{\beta} D \longrightarrow 0$. Define

$\alpha\colon N \to P \oplus N$ such that $\alpha(x) = (h_2(x), \theta(x))$ for any $x \in N$;

$\beta\colon P \oplus N \to D$ such that $\beta(p, x) = f_2(x) - \psi(p)$ for any $(p, x) \in P \oplus N$.

If $\alpha(x) = 0 = (h_2(x), \theta(x))$, then $h_2(x) = 0$. So there is $y \in \mathrm{Im}(i)$ such that $h_1(y) = x$. Hence $0 = \theta(x) = \theta h_1(y) = f_1(y)$. Noting that f_1 is monomorphic, we have $y = 0$. Thus, $x = h_1(y) = 0$, and we see that α is monomorphic. Clearly, β is epimorphic and $\beta\alpha = 0$.

Let $(p, x) \in \ker(\beta)$. Then $\beta(p, x) = f_2(x) - \psi(p) = 0$. Since h_2 is epimorphic, there is $x' \in N$ such that $h_2(x') = p$. So $f_2\theta(x') = \psi h_2(x') = \psi(p) = f_2(x)$, and $f_2(\theta(x') - x) = 0$. Thus, there is $y \in \mathrm{Im}(i)$ such that $\theta h_1(y) = f_1(y) = \theta(x') - x$. This means that $x = \theta(x') - \theta h_1(y) = \theta(x' - h_1(y))$. In addition, $h_2(x' - h_1(y)) = h_2(x') - h_2 h_1(y) = h_2(x') = p$. So $(p, x) = (h_2(x' - h_1(y)), \theta(x' - h_1(y))) \in \mathrm{Im}(\alpha)$. Now we get an exact sequence

$$0 \longrightarrow N \xrightarrow{\alpha} P \oplus N \xrightarrow{\beta} D \longrightarrow 0$$

with N, $D \in \mathscr{F}$. Hence $P \oplus N \in \mathscr{F}$ and so $P \in \mathscr{F}$. Since G is injective, we get a commutative diagram

$$\begin{tikzcd}
 & G & \\
0 \arrow[r] & K \arrow[u, "h_1 i"] \arrow[r, "\iota"'] & E \arrow[ul, "\phi"']
\end{tikzcd}$$

and hence the diagram

$$\begin{tikzcd}
 & N & \\
0 \arrow[r] & K \arrow[u, "h_1 i"] \arrow[r, "\iota"'] & E \arrow[ul, "\phi"']
\end{tikzcd}$$

is also commutative. So the rows exact diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d, equal] & E \arrow[r, "\pi"] \arrow[d, "\phi"] & L \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r, "h_1 i"] & N \arrow[r, "h_2"] & P \arrow[r] & 0
\end{tikzcd}$$

can be completed to a commutative diagram. Similarly to the proof above, we obtain an exact sequence

$$0 \longrightarrow E \longrightarrow L \oplus N \longrightarrow P \longrightarrow 0.$$

Note that E, $P \in \mathscr{F}$. Hence $L \oplus N \in \mathscr{F}$ and so $L \in \mathscr{F}$. But then $\mathrm{Ext}^1(L, K) = 0$ again by [8], Corollary 7.2.3, page 156. Thus, the sequence $0 \to K \to E \to L \to 0$ is split exact, and hence K is injective. Since K is the kernel of the $\mathscr{F}$-cover $F \to C$, by [14], Corollary 1.2.8, page 13, K is zero. Hence $C \cong F$, and so C is $(n, 0)$-injective. It follows that R is right n-coherent by Proposition 2.1. The proof is complete. □

When $n = 1$ in Theorem 3.13, we have

Corollary 3.14. *Let R be a ring such that every absolutely pure right R-module is pure extending. If every right R-module has an absolutely pure cover, then R is right coherent.*

Proposition 3.15. *The following assertions are equivalent for a ring R and $n \geqslant 1$:*
(1) *R is right n-coherent.*
(2) *Every n-pure submodule of an $(n,1)$-injective right R-module is $(n,1)$-injective.*

Proof. (1) $\Rightarrow$ (2) holds by Lemma 3.1.

(2) $\Rightarrow$ (1). Let $0 \to A \to B \to C \to 0$ be any exact sequence of right R-modules where A and B are $(n,0)$-injective. Then we get a long exact sequence

$$0 = \mathrm{Ext}^1(P,B) \to \mathrm{Ext}^1(P,C) \to \mathrm{Ext}^2(P,A)$$

for any n-presented right R-module P. Let $0 \to A \to E$ be exact with E injective. Then A is $(n,1)$-injective by (2). So $\mathrm{Ext}^2(P,A) = 0$ and hence $\mathrm{Ext}^1(P,C) = 0$. Thus, (1) follows. $\square$

4. Applications to (n,d)-rings

In this section, we will say that R is a *right (n,d)-FC ring* provided it is right n-coherent and R_R is (n,d)-injective. We note that right $(1,0)$-FC rings are also called right FC rings [5], and $(0,0)$-FC rings coincide with QF rings.

Recall that an (n,d)-injective envelope $\phi\colon M \to E$ of M has the *unique mapping property* [6] if for any homomorphism $f\colon M \to A$ with A (n,d)-injective, there is a unique homomorphism $g\colon E \to A$ such that $g\phi = f$. The concept of an (n,d)-injective cover with the unique mapping property can be defined similarly.

Theorem 4.1. *Let R be a ring. Then the following assertions are equivalent:*
(1) *R is a right (n,d)-ring;*
(2) *R is a right (n,d)-FC ring, and the kernel of any (n,d)-injective cover of a right R-module is (n,d)-injective;*
(3) *R is a right (n,d)-FC ring, and the cokernel of any (n,d)-injective preenvelope of a right R-module is (n,d)-injective;*
(4) *R is a right (n,d)-FC ring, and every factor module of a right (n,d)-injective R-module is (n,d)-injective;*
(5) *R_R is (n,d)-injective, and every right R-module has a monomorphic (n,d)-injective cover;*
(6) *every right R-module has an epimorphic (n,d)-injective cover with the unique mapping property;*

(7) *every right R-module has an (n,d)-injective envelope with the unique mapping property;*

(8) *R is a right (n,d)-FC ring, and R is a right $(n,d+m)$-ring for some $m \geqslant 0$;*

(9) *R is a right (n,d)-FC ring, and every right R-module has an $(n,d+m)$-injective cover with the unique mapping property for some $m \geqslant 0$;*

(10) *R is a right (n,d)-FC ring, and every right R-module has an $(n,d+m)$-injective envelope with the unique mapping property for some $m \geqslant 0$.*

Proof. $(1) \Rightarrow (3)$, $(1) \Rightarrow (4)$ and $(1) \Rightarrow (8)$ are clear.

$(3) \Rightarrow (2)$. Let M be any right R-module. Then, by Corollary 3.10, M has an epimorphic (n,d)-injective cover $E \to M$ with E (n,d)-injective. On the other hand, by [10], Theorem 3.9, $K = \ker(E \to M)$ has a monomorphic (n,d)-injective preenvelope $g\colon K \to E^1$. So we get the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E^1 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. Since $K \to E^1$ is an (n,d)-injective preenvelope, the homomorphism $K \to E$ can be extended to a homomorphism $E^1 \to E$ and so the diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & E \\
 & E^1 \arrow[ur] &
\end{tikzcd}$$

is commutative. It induces a homomorphism $L \to N$ so that the diagram

$$\begin{tikzcd}
 & & N \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[ur] & L \arrow[r] & 0
\end{tikzcd}$$

is also commutative. This implies that the sequence $0 \to E \to N \to L \to 0$ is split exact. Now applying $\mathrm{Hom}(A, -)$ to the first commutative diagram with A (n,d)-

injective, we obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}(A,K) \arrow[r] \arrow[d] & \mathrm{Hom}(A,E) \arrow[r] \arrow[d] & \mathrm{Hom}(A,M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(A,E^1) \arrow[r] \arrow[d] & \mathrm{Hom}(A,N) \arrow[r] \arrow[d] & \mathrm{Hom}(A,M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Hom}(A,L) \arrow[r] \arrow[d] & \mathrm{Hom}(A,L) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Clearly, the bottom row and the right two columns are exact. Since $E \to M$ is an (n,d)-injective cover, the top row is also exact. So the middle row is exact. It follows that the left column is also exact by [12], Lemma 6.31, page 354. Note that L is (n,d)-injective by (3). So by setting $E = L$, we see that $0 \to K \to E^1 \to L \to 0$ is split exact, and hence K is (n,d)-injective, as desired.

(2) $\Rightarrow$ (1). Let M be any right R-module. Then, by Corollary 3.10, M has an epimorphic (n,d)-injective cover $E \to M$ with E (n,d)-injective. So we get an exact sequence $0 \to K \to E \to M \to 0$. By (2), K is (n,d)-injective. It follows that M is also (n,d)-injective by Proposition 2.5, as desired.

(5) $\Rightarrow$ (1). Let M be any right R-module. By hypothesis, M has a monomorphic (n,d)-injective cover $F \to M$. Since R_R is (n,d)-injective, it is easy to see that $F \to M$ is an epimorphism. So M is (n,d)-injective and (1) follows.

(4) $\Rightarrow$ (1). Let M be any right R-module. Then by Corollary 3.10, M has an epimorphic (n,d)-injective cover $g\colon E \to M$ with E (n,d)-injective. So $M \cong \mathrm{coker}(g)$, and (4) implies that M is (n,d)-injective, as desired.

(8) $\Rightarrow$ (1). If $m = 0$ then we are done. So assume $m \geqslant 1$. Let M be any right R-module. Since R is right n-coherent and R_R is (n,d)-injective, by Corollary 3.10, M has an epimorphic (n,d)-injective cover $f\colon E \to M$ with E (n,d)-injective, which yields the exactness of the sequence $0 \to K \to E \to M \to 0$. So we get a long exact sequence

$$\mathrm{Ext}^{d+m}(P,E) \longrightarrow \mathrm{Ext}^{d+m}(P,M) \longrightarrow \mathrm{Ext}^{d+m+1}(P,K)$$

for any n-presented right R-module P. By (8), K is $(n,d+m)$-injective, and so $\mathrm{Ext}^{d+m+1}(P,K) = 0$. But $\mathrm{Ext}^{d+m}(P,E) = 0$ since R is right n-coherent and E is (n,d)-injective. Hence $\mathrm{Ext}^{d+m}(P,M) = 0$, and so M is $(n,d+m-1)$-injective. Thus R is a right $(n,d+m-1)$-ring. Repeat this procedure to obtain R is a right (n,d)-ring.

(6) $\Rightarrow$ (1). For any right R-module M, let $g\colon E \to M$ be an epimorphic (n,d)-injective cover of M with the unique mapping property, where E is (n,d)-injective. By (6), $K = \ker(g)$ has an epimorphic (n,d)-injective cover $f\colon E' \to K$. So, we obtain the following row exact commutative diagram:

$$\begin{tikzcd}
 & & E' \arrow[dl, "f"'] \arrow[d, "if"] \arrow[dr, "0"] & & \\
0 \arrow[r] & K \arrow[r, "i"] & E \arrow[r, "g"] & M \arrow[r] & 0.
\end{tikzcd}$$

Since $g(if) = 0$, we have $if = 0$ by uniqueness. Note that f is an epimorphism. Hence $K = \mathrm{Im}(f) \subseteq \ker(i) = 0$. Hence, M is (n,d)-injective. So (1) follows.

(1) $\Rightarrow$ (5), (1) $\Rightarrow$ (6) and (1) $\Rightarrow$ (7). Let M be any right R-module. Then M is (n,d)-injective by (1). Now it is easy to verify that the identity homomorphism on M is an (n,d)-injective cover with the unique mapping property. It is also an (n,d)-injective envelope of M which has the unique mapping property. Thus (5), (6) and (7) hold.

(7) $\Rightarrow$ (1). For any right R-module M, let $f\colon M \to E$ be an (n,d)-injective envelope of M with the unique mapping property, where E is (n,d)-injective. By (7), $L = \mathrm{coker}(f)$ has an (n,d)-injective envelope $g\colon L \to E'$. Therefore we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[dr, "0"] & E \arrow[r, "\pi"] \arrow[d, "g\pi"] & L \arrow[r] \arrow[dl, "g"'] & 0 \\
 & & E' & &
\end{tikzcd}$$

with exact row. Since $(g\pi)f = 0$, we have $g\pi = 0$ by uniqueness. Note that g is a monomorphism. Hence, $L = \mathrm{Im}(\pi) \subseteq \ker(g) = 0$. So M is (n,d)-injective, and (1) follows.

(8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10). The proofs are analogous to those of (1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7). $\square$

It is well-known that a ring R is a right $(0,0)$-ring (or $(0,1)$-ring, $(1,0)$-ring, $(1,1)$-ring) if and only if R is semisimple (or right hereditary, von Neumann regular, right semihereditary, respectively) (see [4], Theorem 1.3; or [16], Corollary 2.7). Specializing Theorem 4.1, we have

Corollary 4.2. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is von Neumann regular;*

(2) *R is a right FC ring, and the kernel of any FP-injective cover of a right R-module is FP-injective;*

(3) *R is a right FC ring, and the cokernel of any FP-injective preenvelope of a right R-module is FP-injective;*

(4) *R_R is right FP-injective, and every factor module of a right FP-injective R-module is FP-injective;*

(5) *R_R is FP-injective, and R is right semihereditary;*

(6) *R_R is FP-injective, and every right R-module has a monomorphic FP-injective cover;*

(7) *every right R-module has an epimorphic FP-injective cover with the unique mapping property;*

(8) *every right R-module has an FP-injective envelope with the unique mapping property;*

(9) *R is a right FC ring, and R is a right $(1, m)$-ring for some $m \geqslant 0$;*

(10) *R is a right FC ring, and every right R-module has a $(1, m)$-injective cover with the unique mapping property for some $m \geqslant 0$;*

(11) *R is a right FC ring, and every right R-module has a $(1, m)$-injective envelope with the unique mapping property for some $m \geqslant 0$;*

(12) *R is a right FC ring and $wD(R) < \infty$.*

P r o o f. Due to Theorem 4.1, we need only to show that (4) $\Rightarrow$ (5) and (9) $\Leftrightarrow$ (12).

(4) $\Rightarrow$ (5). Let M be any right R-module. There is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. So for any finitely presented right R-module P we have $\mathrm{Ext}^2(P, M) \cong \mathrm{Ext}^1(P, L) = 0$ since L is FP-injective by(4). Hence M is $(1, 1)$-injective, and (5) follows.

(9) $\Leftrightarrow$ (12). This follows from [16], Proposition 2.6; and [16], Corollary 2.7. $\square$

Corollary 4.3. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is semisimple;*

(2) *R is a QF ring, and the kernel of any injective cover of a right R-module is injective;*

(3) *R is a QF ring, and the cokernel of any injective envelope of a right R-module is injective;*

(4) *R is a QF ring, and R is right hereditary;*

(5) *R_R is injective, and every right R-module has a monomorphic injective cover;*

(6) *every right R-module has an epimorphic injective cover with the unique mapping property;*

(7) *every right R-module has an injective envelope with the unique mapping property;*

(8) *R is a QF ring, and $rD(R) < \infty$;*

(9) *R is a QF ring, and R is a $(0,m)$-ring for some $m \geqslant 0$;*

(10) *R is a QF ring, and every right R-module has a $(0,m)$-injective cover with the unique mapping property for some $m \geqslant 0$;*

(11) *R is a QF ring, and every right R-module has a $(0,m)$-injective envelope with the unique mapping property for some $m \geqslant 0$.*

By Corollary 3.8, we see that R is right Noetherian if and only if every right R-module has a $(0,d)$-injective cover, for any non-negative integer d. We end the paper with

Remark 4.4. Let $n \geqslant 1$. The question whether R must necessarily be right n-coherent in order that every right R-module have an (n,d)-injective cover for any non-negative integer d, is open.

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