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ON HARDY q-INEQUALITIES

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Abstract. Some q-analysis variants of Hardy type inequalities of the form

$$\int_0^b \left(x^{\alpha-1}\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\right)^p \mathrm{d}_q x \leqslant C\int_0^b f^p(t)\,\mathrm{d}_q t$$

with sharp constant C are proved and discussed. A similar result with the Riemann-Liouville operator involved is also proved. Finally, it is pointed out that by using these techniques we can also obtain some new discrete Hardy and Copson type inequalities in the classical case.

Keywords: inequality; Hardy type inequality; Hardy operator; Riemann-Liouville operator; q-analysis; sharp constant; discrete Hardy type inequality

MSC 2010: 26D10, 26D15, 39A13

1. Introduction and preliminaries

In recent years quantum calculus (q-calculus) has been actively developed. Many continuous scientific problems have their discrete versions by using the so-called q-calculus. This q-calculus has numerous applications in combinatorics, special functions, fractals, dynamical systems, number theory, computational methods, quantum mechanics, information technology, etc. (see [2], [8], [9], [10], [18]).

At present q-analogues of many inequalities from the classical analysis have been established but not q-inequalities of Hardy type (see, e.g., [15], [24] and [19], [26], [27]).

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The Hardy inequality and its various generalizations play an important role in classical analysis. Therefore during the last fifty years a huge amount of papers has been devoted to Hardy and Hardy type inequalities in various spaces. The main results and their applications in classical analysis are given in the books [22] and [20].

The main aim of this paper is to establish the q-analogue of the classical Hardy type inequalities

$$(1)\qquad \int_0^\infty \Big(x^{\alpha-1}\int_0^x t^{-\alpha}f(t)\,\mathrm{d}t\Big)^p\,\mathrm{d}x < \Big(\frac{p}{p-\alpha p-1}\Big)^p\int_0^\infty f^p(t)\,\mathrm{d}t,\quad f\geqslant 0,$$

where $\alpha < 1-1/p$ with either $p\geqslant 1$ (unless $f\equiv 0$) or $p<0$ and $f>0$ and (with the Riemann-Liouville operator involved)

$$(2)\quad \int_0^\infty \Big(\frac{1}{x^\alpha\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,\mathrm{d}t\Big)^p\,\mathrm{d}x < \Big[\frac{\Gamma(1-1/p)}{\Gamma(\alpha+1-1/p)}\Big]^p\int_0^\infty f^p(t)\,\mathrm{d}t,\qquad f\geqslant 0,$$

where $p>1,\alpha>0$, unless $f\equiv 0$ and with the best constant. For $\alpha=0$ inequality (1) becomes the classical Hardy inequality

$$(3)\qquad \int_0^\infty\Big(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\Big)^p\,\mathrm{d}x < \Big(\frac{p}{p-1}\Big)^p\int_0^\infty f^p(t)\,\mathrm{d}t,\quad f\geqslant 0,\ f\not\equiv 0,$$

and its corresponding discrete version reads

$$(4)\qquad \sum_{n=1}^\infty\Big(\frac{1}{n}\sum_{k=1}^n a_k\Big)^p < \Big(\frac{p}{p-1}\Big)^p\sum_{n=1}^\infty a_n^p,\quad p>1,\ a_n\geqslant 0,\ a_n\not\equiv 0.$$

All these estimates have numerous applications in analysis. We will prove the q-estimate

$$(5)\qquad \int_0^b\Big(x^{\alpha-1}\int_0^x t^{-\alpha}f(t)\,\mathrm{d}_qt\Big)^p\,\mathrm{d}_qx\leqslant C\int_0^b f^p(t)\,\mathrm{d}_qt$$

of Hardy type (1) for $b=\infty$ and $b=1$ with the sharp constants, and also the q-analogue of the estimate (2) with the sharp constant.

The paper is organized in the following way: after definitions and notation below, in Section 2 we prove the q-analogue of the inequality (1), that is, inequality (5) for $b=\infty$ and $b=1$ with the sharp constants.

In Section 3 we define a fractional q-analogue of the Riemann-Liouville operator I_q^α and prove a q-analogue of inequality (2) with the sharp constant.

Finally, in Section 4 we are pointing out that using the techniques of q-calculus we can also obtain some new discrete Hardy, Copson and matrix type inequalities in the classical case.

We now present some notation and definitions from the q-calculus, which are necessary for understanding this paper. They are taken mainly from the book [18].

Let $0 < q < 1$ be fixed. The *definite q-integral* or the *q-Jackson integral* (see [17] and [18]) of a function $f\colon [0,b) \to \mathbb{R}$, $0 < b \leqslant \infty$, is defined by the formula

$$\int_0^x f(t)\,\mathrm{d}_q t = (1-q)x\sum_{k=0}^{\infty} q^k f(q^k x) \quad \text{for } x \in (0,b), \tag{6}$$

and the *improper q-integral* of a function $f\colon [0,\infty) \to \mathbb{R}$ by the relation

$$\int_0^{\infty} f(t)\,\mathrm{d}_q t = (1-q)\sum_{k=-\infty}^{\infty} q^k f(q^k), \tag{7}$$

provided that the series on the right hand sides of (6) and (7) converge absolutely.

For $0 < a < b \leqslant \infty$ we define the q-integral

$$\int_a^b f(t)\,\mathrm{d}_q t = \int_0^b f(t)\,\mathrm{d}_q t - \int_0^a f(t)\,\mathrm{d}_q t.$$

In particular, for $x \in (0,\infty)$, this yields that

$$\int_x^{\infty} f(t)\,\mathrm{d}_q t = \int_0^{\infty} f(t)\,\mathrm{d}_q t - \int_0^x f(t)\,\mathrm{d}_q t. \tag{8}$$

In the theory of q-analysis the q-analogue $[\alpha]_q$ of a number $\alpha \in \mathbb{R}$ is defined by

$$[\alpha]_q = \frac{1-q^{\alpha}}{1-q}. \tag{9}$$

2. The Hardy inequality in q-analysis

We consider the q-integral analogue of the Hardy inequality of the form (1). Our first main result in this section reads:

Theorem 2.1. *Let $\alpha < (p-1)/p$. If either $1 \leqslant p < \infty$ and $f \geqslant 0$ or $p < 0$ and $f > 0$, then the inequality*

$$\int_0^\infty x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p \,\mathrm{d}_q x \leqslant C\int_0^\infty f^p(t)\,\mathrm{d}_q t \tag{10}$$

holds with the constant

$$C = \frac{1}{[(p-1)/p-\alpha]_q^p}. \tag{11}$$

In the case when $0 < p < 1$ the inequality (10) *for $f \geqslant 0$ holds in the reverse direction with the constant* (11)*. Moreover, in all the three cases the constant* (11) *is the best possible.*

Proof. Let $1 < p < \infty$. Consider the estimate (10) based on the definitions (6) and (7). We have

$$\begin{aligned}
L(f) &:= \int_0^\infty x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p \,\mathrm{d}_q x \\
&= \int_0^\infty x^{p(\alpha-1)}\Big((1-q)\sum_{i=0}^\infty x^{1-\alpha} q^{(1-\alpha)i} f(xq^i)\Big)^p \,\mathrm{d}_q x \\
&= (1-q)^{p+1}\sum_{j=-\infty}^\infty q^{jp(\alpha-1)}\Big(\sum_{i=0}^\infty q^{(i+j)(1-\alpha)} f(q^{i+j})\Big)^p q^j \\
&= (1-q)^{p+1}\sum_{j=-\infty}^\infty q^{j[p(\alpha-1)+1]}\Big(\sum_{i=j}^\infty q^{i(1-\alpha)} f(q^i)\Big)^p \equiv (1-q)^{p+1} I^p.
\end{aligned}$$

Let $g = \{g_k\}_{k=-\infty}^\infty \in l_{p'}(\mathbb{Z})$, $g \geqslant 0$, $\|g\|_{l_{p'}} = 1$, where $1/p + 1/p' = 1$. Moreover, let $\theta(z)$ be Heaviside's unit step function, that is, $\theta(z) = 1$ for $z \geqslant 0$ and $\theta(z) = 0$ for $z < 0$. Then, based on the duality principle in $l_p(\mathbb{Z})$, $p > 1$, and the Hölder-Rogers inequality (cf. [23] for the explanation why not only Hölder name should be here), we find that

$$\begin{aligned}
I &= \sup_{\|g\|_{l_{p'}}=1} \sum_i\sum_j g_j q^{j(\alpha-1/p')}\theta(i-j) q^{i(1/p'-\alpha)} q^{i/p} f(q^i) \\
&\leqslant \sup_{\|g\|_{l_{p'}}=1} \Big(\sum_i\sum_j g_j^{p'} q^{j(\alpha-1/p')}\theta(i-j) q^{i(1/p'-\alpha)}\Big)^{1/p'} \\
&\quad\times \Big(\sum_i\sum_j f^p(q^i) q^i q^{(1/p'-\alpha)i}\theta(i-j) q^{j(\alpha-1/p')}\Big)^{1/p}
\end{aligned}$$

$$\leqslant \sup_{\|g\|_{l_{p'}}=1} \biggl(\sum_j g_j^{p'} q^{j(\alpha-1/p')} \sum_{i=j}^{\infty} q^{i(1/p'-\alpha)}\biggr)^{1/p'}$$
$$\times \biggl(\sum_i f^p(q^i) q^i q^{i(1/p'-\alpha)} \sum_{j=-\infty}^{i} q^{j(\alpha-1/p')}\biggr)^{1/p} = \sup_{\|g\|_{l_{p'}}=1} I_1(g) I_2(f).$$

Since

$$I_1^{p'}(g) = \sum_j g_j^{p'} q^{j(\alpha-1/p')} q^{j(1/p'-\alpha)} \sum_{i=0}^{\infty} q^{i(1/p'-\alpha)} = \frac{1}{1-q^{1/p'-\alpha}} \sum_j g_j^{p'}$$
$$= \frac{1}{1-q^{1/p'-\alpha}} \|g\|_{l_{p'}}^{p'} = \frac{1}{(1-q)[(p-1)/p-\alpha]_q} \|g\|_{l_{p'}}^{p'}$$

and

$$I_2^p(f) = \sum_i f^p(q^i) q^i q^{i(1/p'-\alpha)} q^{i(\alpha-1/p')} \sum_{j=0}^{\infty} q^{j(1/p'-\alpha)}$$
$$= \frac{1}{1-q^{1/p'-\alpha}} \sum_i q^i f^p(q^i) = \frac{1}{(1-q)^2[(p-1)/p-\alpha]_q} \int_0^{\infty} f^p(t)\, \mathrm{d}_q t,$$

it follows that

$$I^p \leqslant \frac{1}{(1-q)^{p+1}[(p-1)/p-\alpha]_q^p} \int_0^{\infty} f^p(t)\, \mathrm{d}_q t.$$

Putting the above calculations together we deduce that

$$\int_0^{\infty} x^{p(\alpha-1)} \biggl(\int_0^x t^{-\alpha} f(t)\, \mathrm{d}_q t\biggr)^p \mathrm{d}_q x = (1-q)^{p+1} I^p \leqslant \frac{1}{[(p-1)/p-\alpha]_q^p} \int_0^{\infty} f^p(t)\, \mathrm{d}_q t,$$

which means that inequality (10) holds with constant (11).

Now, we will show that the constant (11) is the best possible. For $\beta > -1/p$ let $f_\beta(t) = t^\beta \chi_{(0,1]}(t)$, $t > 0$. Then

$$\int_0^{\infty} f_\beta^p(t)\, \mathrm{d}_q t = (1-q) \sum_{i=-\infty}^{\infty} q^i f_\beta^p(q^i) = (1-q) \sum_{i=0}^{\infty} q^i q^{p\beta i}$$
$$= \frac{1-q}{1-q^{1+p\beta}} = \frac{1}{[1+p\beta]_q},$$

and

$$
\begin{aligned}
L(f_\beta) &= \int_0^\infty x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f_\beta(t)\,\mathrm{d}_q t\Big)^p \mathrm{d}_q x \\
&= (1-q)\sum_{j=-\infty}^{\infty} q^{j[p(\alpha-1)+1]}\bigg((1-q)\sum_{i=0}^{\infty} q^{(i+j)(1-\alpha)} f_\beta(q^{i+j})\bigg)^p \\
&\geqslant (1-q)^{p+1}\sum_{j=0}^{\infty} q^{j[p(\alpha-1)+1]}\bigg(\sum_{i=j}^{\infty} q^{i(1-\alpha)} q^{i\beta}\bigg)^p \\
&= (1-q)^{p+1}\sum_{j=0}^{\infty} q^{j(1+p\beta)}\bigg(\sum_{i=0}^{\infty} q^{i(1-\alpha+\beta)}\bigg)^p = \Big(\frac{1-q}{1-q^{1-\alpha+\beta}}\Big)^p \frac{1}{[1+p\beta]_q}.
\end{aligned}
$$

Since

$$
\sup_{\beta>-1/p} \frac{1}{1-q^{1-\alpha+\beta}} = \frac{1}{1-q^{1-1/p-\alpha}} = \frac{1}{1-q^{(p-1)/p-\alpha}},
$$

it follows that for the best constant C in (10) the following estimate is valid:

$$
C \geqslant \sup_{\beta>-1/p} \frac{L(f_\beta)}{\int_0^\infty f_\beta^p(t)\,\mathrm{d}_q t} \geqslant \sup_{\beta>-1/p} \Big(\frac{1-q}{1-q^{1-\alpha+\beta}}\Big)^p = \frac{1}{[(p-1)/p-\alpha]_q^p},
$$

which shows that the constant (11) is the sharp constant in (10).

If $p=1$, then

$$
\begin{aligned}
L(f) &= (1-q)^2 \sum_{j=-\infty}^{\infty} q^{j\alpha} \sum_{i=j}^{\infty} q^{i(1-\alpha)} f(q^i) = (1-q)^2 \sum_{i=-\infty}^{\infty} q^{i(1-\alpha)} f(q^i) \sum_{j=-\infty}^{i} q^{j\alpha} \\
&= (1-q)^2 \sum_{i=-\infty}^{\infty} q^i f(q^i) \sum_{j=0}^{\infty} q^{-j\alpha} = \frac{1}{[-\alpha]_q} \int_0^\infty f(t)\,\mathrm{d}_q t.
\end{aligned}
$$

Let $p<0$ and $f>0$. If we denote $\mu = (1/p' - \alpha)/p'$, then

$$
\begin{aligned}
L(f) &= \int_0^\infty x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p \mathrm{d}_q x \\
&= (1-q)^{p+1} \sum_{j=-\infty}^{\infty} q^{j[p(\alpha-1)+1]}\bigg(\sum_{i=j}^{\infty} q^{i(1-\alpha)} f(q^i)\bigg)^p \\
&= (1-q)^{p+1} \sum_{j=-\infty}^{\infty} q^{j[p(\alpha-1)+1]}\bigg(\sum_{i=j}^{\infty} q^{i\mu} q^{i(1-\alpha-\mu)} f(q^i)\bigg)^p.
\end{aligned}
$$

Taking into account the assumption $p<0$ and the fact that then the Hölder-Rogers inequality holds in the reverse direction, in this case we obtain

$$\begin{aligned}
L(f) &\leqslant (1-q)^{p+1}\sum_{j=-\infty}^{\infty} q^{j[p(\alpha-1)+1]}\Big(\sum_{i=j}^{\infty} q^{ip'\mu}\Big)^{p-1}\sum_{i=j}^{\infty} q^{ip(1-\alpha-\mu)} f^p(q^i)\\
&= (1-q)^{p+1}\Big(\sum_{i=0}^{\infty} q^{ip'\mu}\Big)^{p-1}\sum_{j=-\infty}^{\infty} q^{j[p(\alpha-1)+1+p\mu]}\sum_{i=j}^{\infty} q^{ip(1-\alpha-\mu)} f^p(q^i)\\
&= \frac{(1-q)^{p+1}}{(1-q^{(p-1)/p-\alpha})^{p-1}}\sum_{i=-\infty}^{\infty} f^p(q^i) q^{i(1+1/p'-\alpha)}\sum_{j=-\infty}^{i} q^{j(\alpha-1/p')}\\
&= \frac{(1-q)^{p+1}}{(1-q^{(p-1)/p-\alpha})^{p-1}}\sum_{j=0}^{\infty} q^{j(1/p'-\alpha)}\sum_{i=-\infty}^{\infty} q^i f^p(q^i)\\
&= \Big(\frac{1-q}{1-q^{(p-1)/p-\alpha}}\Big)^p\int_0^\infty f^p(t)\,\mathrm{d}_q t = \Big[\frac{p-1}{p}-\alpha\Big]_q^{-p}\int_0^\infty f^p(t)\,\mathrm{d}_q t.
\end{aligned}$$

This implies that inequality (10) holds with the constant C in (11). Now, we will give a lower estimate for the best constant C in inequality (10). For $\alpha-1<\beta_1<-1/p<\beta_2$ let $f_{\beta_1,\beta_2}(t)=t^{\beta_1}\chi_{(0,1]}(t)+t^{\beta_2}\chi_{(1,\infty)}(t), t>0$. Then

$$\begin{aligned}
\int_0^\infty f^p_{\beta_1,\beta_2}(t)\,\mathrm{d}_q t &= (1-q)\Big(\sum_{i=-\infty}^{-1} q^{i(1+p\beta_2)}+\sum_{i=0}^{\infty} q^{i(1+p\beta_1)}\Big)\\
&= (1-q)\Big(\frac{q^{|1+p\beta_2|}}{1-q^{|1+p\beta_2|}}+\frac{1}{1-q^{1+p\beta_1}}\Big) := F^-(\beta_1,\beta_2)
\end{aligned}$$

and

$$\begin{aligned}
L(f_{\beta_1,\beta_2}) &= \int_0^\infty x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f_{\beta_1,\beta_2}(t)\,\mathrm{d}_q t\Big)^p \mathrm{d}_q x\\
&= (1-q)^{p+1}\sum_{i=-\infty}^{\infty} q^{i(1+p(\alpha-1))}\Big(\sum_{j=i}^{\infty} q^{j(1-\alpha)} f_{\beta_1,\beta_2}(q^j)\Big)^p\\
&= (1-q)^{p+1}\Big[\sum_{i=-\infty}^{-1} q^{i(1+p(\alpha-1))}\Big(\sum_{j=i}^{-1} q^{j(1-\alpha+\beta_2)}+\sum_{j=0}^{\infty} q^{j(1-\alpha+\beta_1)}\Big)^p\\
&\quad+\sum_{i=0}^{\infty} q^{i(1+p(\alpha-1))}\Big(\sum_{j=i}^{\infty} q^{j(1-\alpha+\beta_1)}\Big)^p\Big]\\
&> (1-q)^{p+1}\sum_{i=0}^{\infty} q^{i(1+p(\alpha-1))}\Big(\sum_{j=i}^{\infty} q^{j(1-\alpha+\beta_1)}\Big)^p
\end{aligned}$$

$$
\begin{aligned}
&= (1-q)^{p+1} \sum_{i=0}^{\infty} q^{i(1+p\beta_1)} \Big(\sum_{j=0}^{\infty} q^{j(1-\alpha+\beta_1)} \Big)^p \\
&= \frac{1-q}{1-q^{1+p\beta_1}} \Big(\frac{1-q}{1-q^{1-\alpha+\beta_1}} \Big)^p := F^+(\beta_1, \beta_2).
\end{aligned}
$$

If C is the best constant in (11), then

$$
\begin{aligned}
C &\geqslant \sup_{\alpha-1<\beta_1<-1/p} \lim_{\beta_2 \to \infty} \frac{F^+(\beta_1,\beta_2)}{F^-(\beta_1,\beta_2)} = \sup_{\alpha-1<\beta_1<-1/p} \Big(\frac{1-q}{1-q^{1-\alpha+\beta_1}} \Big)^p \\
&= \Big(\frac{1-q}{1-q^{1-\alpha-1/p}} \Big)^p = \frac{1}{[(p-1)/p-\alpha]_q^p}.
\end{aligned}
$$

The last estimate together with the earlier one shows that constant (11) is sharp in all cases.

Finally, we consider the case when $0 < p < 1$. Let us denote $\gamma = (p-1)/p - \alpha$. For any function $f \geqslant 0$ for which the right hand side of (10) is finite, we find that

$$
\begin{aligned}
[\gamma]_q^{-1} \int_0^{\infty} f^p(t)\, \mathrm{d}_q t &= \frac{(1-q)^2}{1-q^{\gamma}} \sum_{j=-\infty}^{\infty} q^j f^p(q^j) \\
&= (1-q)^2 \sum_{j=-\infty}^{\infty} q^j f^p(q^j) \sum_{i=0}^{\infty} q^{i\gamma} \\
&= (1-q)^2 \sum_{j=-\infty}^{\infty} q^j f^p(q^j) \sum_{i=-\infty}^{0} q^{-i\gamma} \\
&= (1-q)^2 \sum_{j=-\infty}^{\infty} q^{j(1+\gamma)} f^p(q^j) \sum_{i=-\infty}^{j} q^{-i\gamma} \\
&= (1-q)^2 \sum_{i=-\infty}^{\infty} q^{-i\gamma} \sum_{j=i}^{\infty} q^{j(1-p)\gamma} q^{jp(1-\alpha)} f^p(q^j) = J.
\end{aligned}
$$

Using the Hölder-Rogers inequality with powers $1/p$ and $1/(1-p)$ we obtain

$$
\begin{aligned}
J &\leqslant (1-q)^2 \sum_{i=-\infty}^{\infty} q^{-i\gamma} \Big(\sum_{k=i}^{\infty} q^{k\gamma} \Big)^{1-p} \Big(\sum_{j=i}^{\infty} q^{j(1-\alpha)} f(q^j) \Big)^p \\
&= [\gamma]_q^{p-1} (1-q)^{p+1} \sum_{i=-\infty}^{\infty} q^{-ip\gamma} \Big(\sum_{j=i}^{\infty} q^{j(1-\alpha)} f(q^j) \Big)^p \\
&= [\gamma]_q^{p-1} (1-q) \sum_{i=-\infty}^{\infty} q^i q^{ip(\alpha-1)} \Big((1-q) q^i \sum_{j=0}^{\infty} q^j q^{-(i+j)\alpha} f(q^{i+j}) \Big)^p \\
&= [\gamma]_q^{p-1} \int_0^{\infty} x^{p(\alpha-1)} \Big(\int_0^x t^{-\alpha} f(t)\, \mathrm{d}_q t \Big)^p \mathrm{d}_q x,
\end{aligned}
$$

which means that the following inequality holds:

$$\int_0^\infty f^p(t)\,\mathrm{d}_q t \leqslant [\gamma]_q^p \int_0^\infty x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p \,\mathrm{d}_q x \tag{12}$$

for all functions $f \geqslant 0$ for which the left hand side of (12) is finite.

Next, we show that the constant $[\gamma]_q^p = [(p-1)/p - \alpha]_q^p$ in (12) is sharp. For $\alpha - 1 < \beta < -1/p$ let $f_\beta(t) = t^\beta \chi_{[1,\infty)}(t)$, $t > 0$. Then

$$\begin{aligned}\int_0^\infty f_\beta^p(t)\,\mathrm{d}_q t &= (1-q)\sum_{i=-\infty}^{\infty} q^i f_\beta^p(q^i) = (1-q)\Big[\sum_{i=-\infty}^{0} q^i f_\beta^p(q^i) + \sum_{i=1}^{\infty} q^i f_\beta^p(q^i)\Big] \\ &= (1-q)\sum_{i=-\infty}^{0} q^{i(1+p\beta)} = (1-q)\sum_{i=0}^{\infty} q^{i|1+p\beta|} = \frac{1-q}{1-q^{|1+p\beta|}}\end{aligned}$$

and

$$\begin{aligned}L(f_\beta) &= \int_0^\infty x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f_\beta(t)\,\mathrm{d}_q t\Big)^p \,\mathrm{d}_q x \\ &= (1-q)\sum_{j=-\infty}^{\infty} q^{j[p(\alpha-1)+1]}\Big((1-q)q^j \sum_{i=0}^{\infty} q^i q^{-(i+j)\alpha} f_\beta(q^{i+j})\Big)^p \\ &= (1-q)^{p+1}\Big[\sum_{j=-\infty}^{0} q^{j[p(\alpha-1)+1]}\Big(\sum_{i=j}^{\infty} q^{i(1-\alpha)} f_\beta(q^i)\Big)^p \\ &\qquad + \sum_{j=1}^{\infty} q^{j[p(\alpha-1)+1]}\Big(\sum_{i=j}^{\infty} q^{i(1-\alpha)} f_\beta(q^i)\Big)^p\Big] \\ &= (1-q)^{p+1}\sum_{j=-\infty}^{0} q^{j[p(\alpha-1)+1]}\Big(\sum_{i=j}^{0} q^{i(1-\alpha+\beta)}\Big)^p \\ &= (1-q)^{p+1}\sum_{j=-\infty}^{0} q^{j[p(\alpha-1)+1]} q^{jp(1-\alpha+\beta)}\Big(\sum_{i=0}^{-j} q^{i(1-\alpha+\beta)}\Big)^p \\ &\leqslant \frac{(1-q)^{p+1}}{(1-q^{1-\alpha+\beta})^p}\sum_{j=-\infty}^{0} q^{j[1+p\beta]} = \frac{1-q}{1-q^{|1+p\beta|}}\Big(\frac{1-q}{1-q^{1-\alpha+\beta}}\Big)^p.\end{aligned}$$

If inequality (12) holds with the best constant $C > 0$, then

$$\begin{aligned}C &\geqslant \sup_{\beta\in(\alpha-1,-1/p)} \frac{\int_0^\infty f_\beta^p(t)\,\mathrm{d}_q t}{L(f_\beta)} \geqslant \sup_{\beta\in(\alpha-1,-1/p)}\Big(\frac{1-q^{1-\alpha+\beta}}{1-q}\Big)^p \\ &= \Big(\frac{1-q^{(p-1)/p-\alpha}}{1-q}\Big)^p = \Big[\frac{p-1}{p} - \alpha\Big]_q^p = [\gamma]_q^p,\end{aligned}$$

and this shows that the constant $[\gamma]_q^p$ in (12) is sharp. The proof of Theorem 2.1 is complete. □

Remark 2.2. The constant in the q-analogue of inequality (1) is smaller than the one in (1). In fact, if $\alpha < 1 - 1/p$ with $p \geqslant 1$ or $p < 0$, then

$$\frac{1}{[(p-1)/p-\alpha]_q} < \frac{p}{p-\alpha p-1} \quad \text{for } \alpha > -1/p. \tag{13}$$

Inequality (13) is reversed for $\alpha < -1/p$. For $\alpha = -1/p$ both sides in (13) are equal to 1.

Estimate (13) means that $(1-q)/(1-q^{(p-1)/p-\alpha}) < p/(p-\alpha p-1)$ for any $0 < q < 1$, which is true since the function $h(q) := p(1-q^{(p-1)/p-\alpha})/(p-\alpha p-1)+q-1$ has the derivative $h'(q) = -q^{-1/p-\alpha}+1 < 0$ for $\alpha > -1/p$, and so $h(q) > h(1) = 0$.

Next, we consider the Hardy inequality on a finite interval. Without loss of generality we consider only the interval $[0,1]$, since in the q-integral we are allowed to change variables in the form $z = xl$, $0 < l < \infty$ (see [18]). Therefore, a q-integral on the interval $[0,l]$ naturally can be reduced to a q-integral on the interval $[0,1]$.

Hence, we consider inequality (5) with $b = 1$ and formulate our next main theorem in this section.

Theorem 2.3. *Let $\alpha < 1 - 1/p$. If either $1 \leqslant p < \infty$ and $f \geqslant 0$ or $p < 0$ and $f > 0$, then the strict inequality*

$$\int_0^1 x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p \mathrm{d}_q x < \frac{1}{[(p-1)/p-\alpha]_q^p}\int_0^1 f^p(t)\,\mathrm{d}_q t \tag{14}$$

holds (unless $f \equiv 0$) and the constant $[(p-1)/p-\alpha]_q^{-p}$ is sharp.

P r o o f. Theorem 2.3 can be proved in a way similar to Theorem 2.1. Hence, we will only point out some differences of the corresponding relations. In the case when $p > 1$ we have

$$\begin{aligned}\int_0^1 x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p \mathrm{d}_q x &= (1-q)^{p+1}\sum_{j=0}^{\infty} q^{j[p(\alpha-1)+1]}\Big(\sum_{i=j}^{\infty} q^{i(1-\alpha)} f(q^i)\Big)^p \\ &= (1-q)^{p+1} I^p,\end{aligned}$$

and

$$\begin{aligned} I &< \sup_{\|g\|_{p'}=1, g\geqslant 0}\Big(\sum_{j=0}^{\infty} g_i^{p'} q^{i(\alpha-1/p')}\sum_{i=j}^{\infty} q^{i(1/p'-\alpha)}\Big)^{1/p'} \\ &\quad\times\Big(\sum_{i=0}^{\infty} f^p(q^i) q^i q^{i(1/p'-\alpha)}\sum_{j=-\infty}^{i} q^{i(\alpha-1/p')}\Big)^{1/p} \\ &= \sup_{\|g\|_{p'}=1, g\geqslant 0} I_1(g) I_2(f),\end{aligned}$$

respectively. If $p = 1$, then

$$\begin{aligned}
\int_0^1 x^{\alpha-1} \int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\,\mathrm{d}_q x &= (1-q)^2 \sum_{j=0}^{\infty} q^{j\alpha} \sum_{i=j}^{\infty} q^{i(1-\alpha)} f(q^i) \\
&= (1-q)^2 \sum_{i=0}^{\infty} q^{i(1-\alpha)} f(q^i) \sum_{j=0}^{i} q^{j\alpha} \\
&= (1-q)^2 \sum_{i=0}^{\infty} q^i f(q^i) \sum_{j=0}^{i} q^{-j\alpha} < \frac{1}{[-\alpha]_q} \int_0^1 f(t)\,\mathrm{d}_q t.
\end{aligned}$$

The last strict inequalities give the validity of strict inequality (14). The best constant in (14) can be found by using the test functions $f_\beta(t) = t^\beta$ if $0 < t < 1$, where $\beta > -1/p$. In the case when $p < 0$ the proof of estimate (14) can be done by use of the same method as in Theorem 2.1 for F. In fact, we have

$$\begin{aligned}
L(f) &< \frac{(1-q)^{p+1}}{(1-q^{(p-1)/p-\alpha})^{p-1}} \sum_{i=0}^{\infty} f^p(q^i) q^{i(1+1/p'-\alpha)} \sum_{j=-\infty}^{i} q^{j(\alpha-1/p')} \\
&= \frac{1}{[(p-1)/p-\alpha]_q^p} \int_0^1 f^p(t)\,\mathrm{d}_q t.
\end{aligned}$$

This implies the strict inequality in (14). In order to obtain a lower estimate we consider the test functions $f_\beta(t) = t^\beta \chi_{(0,1]}(t)$ for $\alpha - 1 < \beta < -1/p$. Then

$$\int_0^1 f_\beta^p(t)\,\mathrm{d}_q t = \frac{1-q}{1-q^{1+p\beta}} := F^-(\beta)$$

and

$$\begin{aligned}
L(f_\beta) &= \int_0^1 x^{p(\alpha-1)} \left(\int_0^x t^{-\alpha} f_\beta(t)\,\mathrm{d}_q t \right)^p \mathrm{d}_q x \\
&= (1-q)^{p+1} \sum_{i=0}^{\infty} q^{i(1+p(\alpha-1))} \left(\sum_{j=i}^{\infty} q^{j(1-\alpha+\beta)} \right)^p \\
&> (1-q)^{p+1} \frac{1}{1-q^{1+p\beta}} \left(\frac{1}{1-q^{1-\alpha+\beta}} \right)^p \\
&= \frac{1-q}{1-q^{1+p\beta}} \left(\frac{1-q}{1-q^{1-\alpha+\beta}} \right)^p := F^+(\beta).
\end{aligned}$$

Hence, if $C > 0$ is the best constant in inequality (14), then we obtain the estimate

$$C \geqslant \lim_{\beta \to -1/p} \frac{F^+(\beta)}{F^-(\beta)} = \frac{1}{[(p-1)/p-\alpha]_q^p},$$

and the proof of Theorem 2.3 is complete. □

Next, we present some corresponding sharp reverse inequalities with additional terms for the case $0 < p < 1$.

Theorem 2.4. *Let $0 < p < 1$ and $\alpha < (p-1)/p$. Then the following strict inequalities hold:*

$$\int_0^1 f^p(t)(1 - t^{(p-1)/p-\alpha})\,\mathrm{d}_q t < C \int_0^1 x^{p(\alpha-1)} \Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p \,\mathrm{d}_q x, \tag{15}$$

$$\int_0^1 f^p(t)\,\mathrm{d}_q t < C \int_0^1 \Big(1 + \frac{\chi_{(q,1]}(x)}{[(p-1)/p-\alpha]_q}\Big) x^{p(\alpha-1)} \Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p \,\mathrm{d}_q x \tag{16}$$

for all functions $f \geqslant 0$ with the finite left hand side of (16) *unless $f \equiv 0$, and with the best constant*

$$C = \Big[\frac{p-1}{p} - \alpha\Big]_q^p. \tag{17}$$

P r o o f. Let $f \geqslant 0$ and $\int_0^1 f^p(t)\,\mathrm{d}_q t < \infty$. Denoting $\gamma = (p-1)/p - \alpha$ and $c_q = (1-q)(1-q^\gamma)$ we obtain

$$\begin{aligned}
\int_0^1 f^p(t)\,\mathrm{d}_q t &= (1-q)\sum_{j=0}^\infty q^j f^p(q^j) = c_q \sum_{j=0}^\infty q^j f^p(q^j) \sum_{i=-\infty}^0 q^{-i\gamma} \\
&= c_q \sum_{j=0}^\infty q^{j(1+\gamma)} f^p(q^j) \sum_{i=-\infty}^j q^{-i\gamma} \\
&= c_q \sum_{j=0}^\infty q^{j(1+\gamma)} f^p(q^j) \sum_{i=0}^j q^{-i\gamma} + c_q \sum_{i=-\infty}^{-1} q^{-i\gamma} \sum_{j=0}^\infty q^{j(1+\gamma)} f^p(q^j) \\
&= c_q \sum_{j=0}^\infty q^{j(1+\gamma)} f^p(q^j) \sum_{i=0}^j q^{-i\gamma} + (1-q) q^\gamma \sum_{j=0}^\infty q^{j(1+\gamma)} f^p(q^j) \\
&< c_q \sum_{i=0}^\infty q^{-i\gamma} \sum_{j=i}^\infty q^{j(1+\gamma)} f^p(q^j) + (1-q) \sum_{j=0}^\infty q^{j(1+\gamma)} f^p(q^j) := I_1 + I_2.
\end{aligned}$$

By using the Hölder-Rogers inequality with powers $1/(1-p)$ and $1/p$ we can estimate I_1 as

$$\begin{aligned}
I_1 &= c_q \sum_{i=0}^\infty q^{-i\gamma} \sum_{j=i}^\infty q^{j(1-p)\gamma} q^{jp(1-\alpha)} f^p(q^j) \\
&< c_q \sum_{i=0}^\infty q^{-i\gamma} \Big(\sum_{k=i}^\infty q^{k\gamma}\Big)^{1-p} \Big(\sum_{j=i}^\infty q^{j(1-\alpha)} f(q^j)\Big)^p
\end{aligned}$$

$$
\begin{aligned}
&= c_q\Big(\frac{1}{1-q^{\gamma}}\Big)^{1-p}\sum_{i=0}^{\infty} q^{i(1+p(\alpha-1))}\Big(\sum_{j=i}^{\infty} q^{j(1-\alpha)} f(q^j)\Big)^p \\
&= \Big(\Big[\frac{p-1}{p}-\alpha\Big]_q\Big)^p\int_0^1 x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p\,\mathrm{d}_q x.
\end{aligned}
$$

Since $I_2 = (1-q)\sum\limits_{j=0}^{\infty} q^{j(1+\gamma)} f^p(q^j) = \int_0^1 t^{(p-1)/p-\alpha} f^p(t)\,\mathrm{d}_q t$ it follows from the above calculations that estimate (15) holds with the constant $C \leqslant [(p-1)/p-\alpha]_q^p$.

Now, we will show also the validity of inequality (16). For this purpose we estimate I_2, using the Hölder-Rogers inequality with powers $1/(1-p)$ and $1/p$, and obtain

$$
\begin{aligned}
I_2 &= (1-q)\sum_{j=0}^{\infty} q^{j(1-p)\gamma} q^{jp(1-\alpha)} f^p(q^j) < (1-q)\Big(\sum_{k=0}^{\infty} q^{k\gamma}\Big)^{1-p}\Big(\sum_{j=0}^{\infty} q^{j(1-\alpha)} f(q^j)\Big)^p \\
&= (1-q)(1-q^{\gamma})^{p-1}\Big(\sum_{j=0}^{\infty} q^{j(1-\alpha)} f(q^j)\Big)^p = \Big[\frac{p-1}{p}-\alpha\Big]_q^{p-1}\Big(\int_0^1 t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p.
\end{aligned}
$$

Hence, again from the above calculations we obtain

$$
\begin{aligned}
\int_0^1 f^p(t)\,\mathrm{d}_q t < \Big[\frac{p-1}{p}-\alpha\Big]_q^p\Big[&\int_0^1 x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}_q t\Big)^p\,\mathrm{d}_q x \\
&+ \frac{1}{[(p-1)/p-\alpha]_q}\Big(\int_0^1 x^{-\alpha} f(x)\,\mathrm{d}_q x\Big)^p\Big].
\end{aligned}
$$

This means that (16) holds with the constant (17). Next, we show that the constant $[\gamma]_q^p = [(p-1)/p-\alpha]_q^p$ in both the inequalities (15) and (16) is the best possible. To see this we consider the function $f_\beta(t) = t^\beta$ for $0 < t \leqslant 1$, where $\beta > -1/p$. Then

$$
\begin{aligned}
&\int_0^1 t^{(p-1)/p-\alpha} f_\beta^p(t)\,\mathrm{d}_q t = \frac{1-q}{1-q^{1+p\beta+\gamma}},\quad \int_0^1 f_\beta^p(t)\,\mathrm{d}_q t = \frac{1-q}{1-q^{1+p\beta}}, \\
&\int_0^1 x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f_\beta(t)\,\mathrm{d}_q t\Big)^p\,\mathrm{d}_q x = \Big(\frac{1-q}{1-q^{1-\alpha+\beta}}\Big)^p\frac{1-q}{1-q^{1+p\beta}},
\end{aligned}
$$

and

$$
\frac{1}{[(p-1)/p-\alpha]_q}\Big(\int_0^1 t^{-\alpha} f_\beta(t)\,\mathrm{d}_q t\Big)^p = \frac{1}{[(p-1)/p-\alpha]_q}\Big(\frac{1-q}{1-q^{1-\alpha+\beta}}\Big)^p.
$$

If $C > 0$ is the sharp constant in inequality (15), then

$$
C \geqslant \Big(\frac{1-q^{1-\alpha+\beta}}{1-q}\Big)^p\frac{1-q-(1-q^{1+p\beta})(1-q)/(1-q^{1+p\beta+\gamma})}{1-q},
$$

and by letting $\beta \to -1/p$ we find that $C \geqslant [(p-1)/p-\alpha]_q^p$. Moreover, if $C > 0$ is the sharp constant in inequality (16), then

$$C \geqslant \Big(\frac{1-q^{1-\alpha+\beta}}{1-q}\Big)^p \frac{1-q}{1-q+(1-q^{1+p\beta})/[(p-1)/p-\alpha]_q}.$$

Again, by letting $\beta \to -1/p$, we obtain $C \geqslant [(p-1)/p-\alpha]_q^p$. The proof is complete. □

Remark 2.5. From Theorem 2.1 with $\alpha = 0$ we obtain the q-analogue of the classical Hardy inequality

$$\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}_q t\Big)^p \mathrm{d}_q x \leqslant \frac{1}{[(p-1)/p]_q^p}\int_0^\infty f^p(t)\,\mathrm{d}_q t, \quad f \geqslant 0,$$

if $p > 1$ or $p < 0$ and $f > 0$. Moreover, the constant $1/[(p-1)/p]_q^p$ is the best possible and $1/[(p-1)/p]_q^p < (p/(p-1))^p$.

Remark 2.6. If $f \geqslant 0$ is a continuous function on $[0,1]$, then by passing to the limit as $q \to 1^-$ in (15) and (16) we get

$$\text{(18)}\quad \int_0^1 f^p(t)(1-t^{(p-1)/p-\alpha})\,\mathrm{d}t \leqslant \Big(\frac{p-1}{p}-\alpha\Big)^p \int_0^1 x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}t\Big)^p \mathrm{d}x,$$

and

$$\int_0^1 f^p(t)\,\mathrm{d}t \leqslant \Big(\frac{p-1}{p}-\alpha\Big)^p \int_0^1 x^{p(\alpha-1)}\Big(\int_0^x t^{-\alpha} f(t)\,\mathrm{d}t\Big)^p \mathrm{d}\mu(x)$$

where $\mathrm{d}\mu(x) = (1+p/(p-p\alpha-1)\delta(1-x))\,\mathrm{d}x$ and $\delta(\cdot)$ is the Dirac delta function.

The inequality (18) is one of the cases recently proved in [25], Theorem 2.4 (b).

3. A new sharp inequality for the Riemann-Liouville operator in q-analysis

We need definitions and formulas from the q-calculus to be able to define a q-analogue of fractional integration Riemann-Liouville operator of order $\alpha > 0$. These facts are taken mainly from the book [18] (see also [1] and [26]).

If $x \geqslant t > 0$, then the q-analogue of the polynomial $(x-t)^k$ of order $k \in \mathbb{N}$ and the generalized polynomial $(x-t)^\alpha$ of order $\alpha \in \mathbb{R}$ are defined by the relations

$$\text{(19)}\qquad (x-t)_q^k = x^k\Big(\frac{t}{x};q\Big)_k \quad\text{and}\quad (x-t)_q^\alpha = x^\alpha\Big(\frac{t}{x};q\Big)_\alpha,$$

respectively, where the q-analogue of the Pochhammer symbol (q-shifted factorial) is defined by

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i) \quad \text{for } k \in \mathbb{N}\cup\{\infty\} \text{ and } (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}.$$

In q-analysis the gamma function Γ_q has the form

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad \text{for } x \in \mathbb{R}\setminus\{0,-1,-2,\dots\},$$

and the beta function $B_q(\cdot,\cdot)$ is defined in the following way:

$$B_q(a,b) = \int_0^1 t^{a-1}(qt;q)_{b-1}\,\mathrm{d}_q t = (1-q)\sum_{i=0}^{\infty} q^{ia}(q^{i+1};q)_{b-1}.$$

Moreover, the following relations are valid:

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x) \quad \text{and} \quad B_q(a,b) = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}.$$

Finally, the q-analogue of the fractional integration Riemann-Liouville operator of order $\alpha > 0$ has the form

$$I_q^\alpha f(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)_q^{\alpha-1} f(t)\,\mathrm{d}_q t. \tag{20}$$

Our main result in this section is the following q-analogue of inequality (2).

Theorem 3.1. *If $p > 1$ and $\alpha > 0$, then the inequality*

$$\int_0^\infty \left[\frac{1}{x^\alpha\Gamma_q(\alpha)}\int_0^x (x-qt)_q^{\alpha-1} f(t)\,\mathrm{d}_q t\right]^p \mathrm{d}_q x \leqslant C\int_0^\infty f^p(t)\,\mathrm{d}_q t, \quad f \geqslant 0, \tag{21}$$

holds with the best constant

$$C = \left[\frac{\Gamma_q(1-1/p)}{\Gamma_q(\alpha+1-1/p)}\right]^p. \tag{22}$$

P r o o f. Let $f \geqslant 0$. Based on (6), (19) and (20) we have

$$I_q^\alpha f(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)_q^{\alpha-1} f(t)\,\mathrm{d}_q t = \frac{x^\alpha}{\Gamma_q(\alpha)}(1-q)\sum_{i=0}^{\infty}(q^{i+1};q)_{\alpha-1} f(xq^i)q^i. \tag{23}$$

Then, in view of (7) and (23), we find that

$$\int_0^\infty \left(\frac{I_q^\alpha f(x)}{x^\alpha}\right)^p \mathrm{d}_q x = (1-q)\left(\frac{1-q}{\Gamma_q(\alpha)}\right)^p \sum_{j=-\infty}^{\infty}\left(\sum_{i=0}^{\infty}(q^{i+1};q)_{\alpha-1} f(q^{i+j}) q^i\right)^p q^j$$

$$= (1-q)\left(\frac{1-q}{\Gamma_q(\alpha)}\right)^p \sum_{j=-\infty}^{\infty} q^{j(1-p)}\left(\sum_{i=j}^{\infty}(q^{i-j+1};q)_{\alpha-1} f(q^i) q^i\right)^p$$

$$= (1-q)\left(\frac{1-q}{\Gamma_q(\alpha)}\right)^p (J_\alpha)^p.$$

By applying the duality principle in $l_p(\mathbb{Z})$ and by using the Hölder-Rogers inequality we obtain

$$\begin{aligned} J_\alpha &= \sup_{\|g\|_{l_{p'}}=1,\, g\geqslant 0} \sum_{j=-\infty}^{\infty} g_j q^{-j/p'} \sum_{i=j}^{\infty} (q^{i-j+1};q)_{\alpha-1} f(q^i) q^i \\ &= \sup_{\|g\|_{l_{p'}}=1,\, g\geqslant 0} \sum_j \sum_i g_j q^{(i-j)/p'} \theta(i-j)(q^{i-j+1};q)_{\alpha-1} f(q^i) q^{i/p} \\ &\leqslant \sup_{\|g\|_{l_{p'}}=1,\, g\geqslant 0} \left(\sum_j \sum_i g_j^{p'} q^{(i-j)/p'} \theta(i-j)(q^{i-j+1};q)_{\alpha-1}\right)^{1/p'} \\ &\quad\times \left(\sum_i \sum_j f^p(q^i) q^i q^{(i-j)/p'} \theta(i-j)(q^{i-j+1};q)_{\alpha-1}\right)^{1/p} \\ &= \sup_{\|g\|_{l_{p'}}=1,\, g\geqslant 0} J_{\alpha,p'}(g) J_{\alpha,p}(f), \end{aligned}$$

where

$$J_{\alpha,p'}(g)^{p'} = \sum_j \sum_i g_j^{p'} q^{(i-j)/p'} \theta(i-j)(q^{i-j+1};q)_{\alpha-1}$$

and

$$J_{\alpha,p}(f)^p = \sum_i \sum_j f^p(q^i) q^i q^{(i-j)/p'} \theta(i-j)(q^{i-j+1};q)_{\alpha-1}.$$

By formulas for beta and gamma functions, we get

$$\begin{aligned} \sup_{\|g\|_{l_{p'}}=1,\, g\geqslant 0} J_{\alpha,p'}(g)^{p'} &= \sup_{\|g\|_{l_{p'}}=1,\, g\geqslant 0} \sum_j \sum_i g_j^{p'} q^{(i-j)/p'} \theta(i-j)(q^{i-j+1};q)_{\alpha-1} \\ &= \sup_{\|g\|_{l_{p'}}=1,\, g\geqslant 0} \sum_j g_j^{p'} \sum_{i=j}^{\infty} q^{(i-j)/p'} (q^{i-j+1};q)_{\alpha-1} \\ &= \sup_{\|g\|_{l_{p'}}=1,\, g\geqslant 0} \sum_j g_j^{p'} \sum_{i=0}^{\infty} q^{i/p'} (q^{i+1};q)_{\alpha-1} \\ &= \frac{B_q(1/p';\alpha)}{1-q} = \frac{\Gamma_q(1-1/p)\Gamma_q(\alpha)}{\Gamma_q(\alpha+1-1/p)}\,\frac{1}{1-q}, \end{aligned}$$

and

$$\begin{aligned}
J_{\alpha,p}(f)^p &= \sum_i \sum_j f^p(q^i) q^i q^{(i-j)/p'} \theta(i-j)(q^{i-j+1};q)_{\alpha-1} \\
&= \sum_i f^p(q^i) q^i \sum_{j=-\infty}^{i} q^{(i-j)/p'} (q^{i-j+1};q)_{\alpha-1} \\
&= \sum_i f^p(q^i) q^i \sum_{j=0}^{\infty} q^{j/p'} (q^{j+1};q)_{\alpha-1} \\
&= \frac{1}{(1-q)^2} \frac{\Gamma_q(1-1/p)\Gamma_q(\alpha)}{\Gamma_q(\alpha+1-1/p)} \int_0^\infty f^p(t)\,\mathrm{d}_q t.
\end{aligned}$$

By combining the above calculations we find that for $f \geqslant 0$ we have

$$\begin{aligned}
\int_0^\infty \left(\frac{I_q^\alpha f(x)}{x^\alpha}\right)^p \mathrm{d}_q x &= \int_0^\infty \left(\frac{1}{x^\alpha \Gamma_q(\alpha)} \int_0^x (x-qt)_q^{\alpha-1} f(t)\,\mathrm{d}_q t\right)^p \mathrm{d}_q x \\
&= (1-q)\left(\frac{1-q}{\Gamma_q(\alpha)}\right)^p (J_\alpha)^p \\
&\leqslant (1-q)\left(\frac{1-q}{\Gamma_q(\alpha)}\right)^p \sup_{\|g\|_{l_{p'}}=1,\, g\geqslant 0} J_{\alpha,p'}(g)^p J_{\alpha,p}(f)^p \\
&\leqslant (1-q)\left(\frac{1-q}{\Gamma_q(\alpha)}\right)^p \left[\frac{\Gamma_q(1-1/p)\Gamma_q(\alpha)}{\Gamma_q(\alpha+1-1/p)} \frac{1}{1-q}\right]^{p-1} \\
&\quad\times \frac{1}{(1-q)^2} \frac{\Gamma_q(1-1/p)\Gamma_q(\alpha)}{\Gamma_q(\alpha+1-1/p)} \int_0^\infty f^p(t)\,\mathrm{d}_q t \\
&= \left[\frac{\Gamma_q(1-1/p)}{\Gamma_q(\alpha+1-1/p)}\right]^p \int_0^\infty f^p(t)\,\mathrm{d}_q t,
\end{aligned}$$

which means that inequality (21) holds with the estimate $C \leqslant [\Gamma_q(1-1/p)/\Gamma_q(\alpha+1-1/p)]^p$ for the best constant C.

Now, we give also a lower estimate for the best constant C in (21). Let $f_\beta(t) = t^\beta \chi_{(0,1]}(t)$ with $\beta > -1/p$. Then $\int_0^\infty f_\beta^p(t)\,\mathrm{d}_q t = (1-q)/(1-q^{1+p\beta})$ (cf. proof of Theorem 2.1 in the case $1 < p < \infty$) and

$$\begin{aligned}
\int_0^\infty \left(\frac{I_q^\alpha f_\beta(x)}{x^\alpha}\right)^p \mathrm{d}_q x &= \frac{(1-q)^{p+1}}{\Gamma_q^p(\alpha)} \sum_{j=-\infty}^{\infty} q^{j(1-p)} \left(\sum_{i=j}^{\infty} (q^{i-j+1};q)_{\alpha-1} f_\beta(q^i) q^i\right)^p \\
&\geqslant \frac{(1-q)^{p+1}}{\Gamma_q^p(\alpha)} \sum_{j=0}^{\infty} q^{j(1-p)} \left(\sum_{i=j}^{\infty} (q^{i-j+1};q)_{\alpha-1} f_\beta(q^i) q^i\right)^p
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-q)^{p+1}}{\Gamma_q^p(\alpha)} \sum_{j=0}^{\infty} q^{j(1-p)} \left(\sum_{i=j}^{\infty} (q^{i-j+1};q)_{\alpha-1} q^{i(1+\beta)} \right)^p \\
&= \frac{(1-q)^{p+1}}{\Gamma_q^p(\alpha)} \sum_{j=0}^{\infty} q^{j(1+p\beta)} \left(\sum_{i=0}^{\infty} (q^{i+1};q)_{\alpha-1} q^{i(1+\beta)} \right)^p \\
&= \frac{1-q}{\Gamma_q^p(\alpha)(1-q^{1+p\beta})} B_q^p(\beta+1,\alpha).
\end{aligned}
$$

If inequality (21) holds with the best constant $C > 0$, then

$$
C \geqslant \sup_{\beta > -1/p} \left(\frac{B_q(\beta+1,\alpha)}{\Gamma_q(\alpha)} \right)^p = \left(\frac{B_q(1-1/p,\alpha)}{\Gamma_q(\alpha)} \right)^p = \left(\frac{\Gamma_q(1-1/p)}{\Gamma_q(\alpha+1-1/p)} \right)^p,
$$

which shows that constant (22) is sharp. The proof is complete. □

From Theorem 3.1 we obtain immediately the validity of the following statement:

Corollary 3.2. *Let $p > 1$ and $\alpha > 0$. Then the following inequality is valid:*

$$
\int_0^1 \left(\frac{I_q^\alpha f(x)}{x^\alpha} \right)^p \mathrm{d}_q x < \left(\frac{\Gamma_q(1-1/p)}{\Gamma_q(\alpha+1-1/p)} \right)^p \int_0^1 f^p(t)\, \mathrm{d}_q t.
$$

Moreover, the constant $(\Gamma_q(1-1/p)/\Gamma_q(\alpha+1-1/p))^p$ is the best possible.

The strict inequality we are getting as before in the estimate of $J_{\alpha,p}(f)$. In fact, for the finite interval of integration the sum inside of the expression J_α is going from 0 to ∞,

$$
\begin{aligned}
J_{\alpha,p}(f) &= \sum_{i=0}^{\infty} f^p(q^i) q^i \sum_{j=0}^{i} q^{(i-j)/p'} (q^{i-j+1};q)_{\alpha-1} \\
&< \sum_{i=0}^{\infty} f^p(q^i) q^i \sum_{j=0}^{\infty} q^{j/p'} (q^{j+1};q)_{\alpha-1} \\
&= \frac{1}{(1-q)^2} \frac{\Gamma_q(1-1/p)\Gamma_q(\alpha)}{\Gamma_q(\alpha+1-1/p)} \int_0^1 f^p(t)\, \mathrm{d}_q t.
\end{aligned}
$$

4. Remarks on classical discrete Hardy inequalities

The Hardy discrete inequality (4) follows from the Hardy integral inequality (3) by putting in (3) a simple nonincreasing function (cf. [16], page 248, and [20], pages 155–156, [21], page 726).

Up to now there is no sharp discrete analogue of the Hardy integral inequality (1) except for $\alpha = 0$ and this fact was the motivation for many authors to establish the discrete inequalities

$$(24)\qquad \sum_{n=1}^{\infty}\Big(\frac{1}{n^{1-\alpha}}\sum_{k=1}^{n}[k^{1-\alpha}-(k-1)^{1-\alpha}]a_k\Big)^p \leqslant \Big(\frac{(1-\alpha)p}{p-\alpha p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p,\quad a_n\geqslant 0,$$

and

$$(25)\qquad \sum_{n=1}^{\infty}\Big(\frac{1}{\sum_{k=1}^{n}k^{-\alpha}}\sum_{k=1}^{n}k^{-\alpha}a_k\Big)^p \leqslant \Big(\frac{(1-\alpha)p}{p-\alpha p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p,\quad a_n\geqslant 0.$$

For fixed $p > 1$, thanks to a result of Cass and Kratz [7], Theorem 2, we know that inequalities (24) and (25) can only hold for $\alpha < 1-1/p$ and if they hold for some $\alpha < 1-1/p$, then the constant $[(1-\alpha)p/(p-\alpha p-1)]^p = [p/(p-1/(1-\alpha))]^p$ is the best possible since for $\alpha < 1$ we have $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}[k^{1-\alpha}-(k-1)^{1-\alpha}]/n[n^{1-\alpha}-(n-1)^{1-\alpha}] = 1/(1-\alpha)$ and $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}k^{-\alpha}/n^{1-\alpha} = 1/(1-\alpha)$ (see also [14], pages 374–375).

Both the inequalities were stated by Bennett in [3], pages 40–41, whenever $p > 1$, $\alpha < 1-1/p$ and $\alpha \leqslant 0$. No proofs were given in [3]. The proof of (24) for $p > 1$, $\alpha < 1-1/p$ and $\alpha < 0$ (for $\alpha = 0$ this is just the classical discrete Hardy inequality (4)) was given by Bennett [4], pages 401–402, 407, and the proof for $p > 1$, $\alpha < 1-1/p$ by Bennett [5], Theorem 1, pages 31–32, [6], Theorem 1, page 803, and Theorem 18, page 829, and Gao [11], Corollary 3.1.

Inequality (25) was proved independently by Gao [11], Corollary 3.2, and Bennett [6], Theorem 7, for $p > 1$, $\alpha < 1-1/p$ and if either $\alpha \leqslant -1$ or $0 \leqslant \alpha < 1$. Moreover, Gao [12], Theorem 1.1, has shown that the inequality holds for $p \geqslant 2$ and $-1/p \leqslant \alpha \leqslant 0$ or $1 < p \leqslant 4/3$ and $-1 \leqslant \alpha \leqslant -1/p$. In [13], Theorem 6.1, he extended the proof to $p \geqslant 2$ and $-1 \leqslant \alpha \leqslant 0$. This means that they are still some regions with no proof of (25).

Now, let us comment which discrete Hardy inequalities we are getting from the Hardy q-inequalities. Directly from the proof of Theorems 2.1 and 2.3 we obtain the following discrete inequalities of independent interest: for $0 < q < 1$ and $\alpha < 1-1/p$

we have

$$\sum_{j=-\infty}^{\infty}\left(q^{j(\alpha+1/p-1)}\sum_{i=j}^{\infty}q^{i(1-1/p-\alpha)}a_i\right)^p\leqslant\frac{1}{(1-q^{1-1/p-\alpha})^p}\sum_{i=-\infty}^{\infty}a_i^p,\quad a_i\geqslant 0,$$
$$\sum_{j=0}^{\infty}\left(q^{j(\alpha+1/p-1)}\sum_{i=j}^{\infty}q^{i(1-1/p-\alpha)}a_i\right)^p\leqslant\frac{1}{(1-q^{1-1/p-\alpha})^p}\sum_{i=0}^{\infty}a_i^p,\quad a_i\geqslant 0,$$

if either $p>1$ or $p<0$ and $a_i>0$ ($i\in\mathbb{Z}$ or $i\in\mathbb{N}\cup\{0\}$, respectively) with the best constant $(1-q^{1-1/p-\alpha})^{-p}$.

The above two inequalities we can rewrite by putting $\lambda=1-1/p-\alpha>0$ to the following new sharp discrete inequalities: if $0<q<1$ and either $p>1$ or $p<0$ and $a_n>0$ ($n\in\mathbb{Z}$ or $n\in\mathbb{N}\cup\{0\}$, respectively), then with the best constant

$$\sum_{n=-\infty}^{\infty}\left(\frac{1}{q^{\lambda n}}\sum_{k=n}^{\infty}q^{\lambda k}a_k\right)^p\leqslant\frac{1}{(1-q^{\lambda})^p}\sum_{n=-\infty}^{\infty}a_n^p,\quad a_n\geqslant 0,\tag{26}$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{q^{\lambda n}}\sum_{k=n}^{\infty}q^{\lambda k}a_k\right)^p\leqslant\frac{1}{(1-q^{\lambda})^p}\sum_{n=0}^{\infty}a_n^p,\quad a_n\geqslant 0.\tag{27}$$

For $0<p<1$ inequality (26) holds in the reverse direction. If $p>1$, then in view of (26) and (27) by passing to the dual inequalities with substitution of p by p' we obtain

$$\sum_{n=-\infty}^{\infty}\left(q^{\lambda n}\sum_{k=-\infty}^{n}q^{-\lambda k}a_k\right)^p\leqslant\frac{1}{(1-q^{\lambda})^p}\sum_{n=-\infty}^{\infty}a_n^p,\quad a_n\geqslant 0,\tag{28}$$

$$\sum_{n=0}^{\infty}\left(q^{\lambda n}\sum_{k=0}^{n}q^{-\lambda k}a_k\right)^p\leqslant\frac{1}{(1-q^{\lambda})^p}\sum_{n=0}^{\infty}a_n^p,\quad a_n\geqslant 0.\tag{29}$$

In recent years the following weighted Hardy and weighted Copson inequalities have been frequently investigated (see, e.g. [6], [11], [13] and the references given there):

$$\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n}\lambda_k a_k}{\sum_{k=0}^{n}\lambda_k}\right)^p\leqslant A\sum_{n=0}^{\infty}a_n^p,\quad \sum_{n=0}^{\infty}\left(\frac{\sum_{k=n}^{\infty}\lambda_k a_k}{\sum_{k=n}^{\infty}\lambda_k}\right)^p\leqslant B\sum_{n=0}^{\infty}a_n^p,\tag{30}$$

$$\sum_{n=-\infty}^{\infty}\left(\frac{\sum_{k=-\infty}^{n}\lambda_k a_k}{\sum_{k=-\infty}^{n}\lambda_k}\right)^p\leqslant C\sum_{n=-\infty}^{\infty}a_n^p,\quad \sum_{n=-\infty}^{\infty}\left(\frac{\sum_{k=n}^{\infty}\lambda_k a_k}{\sum_{k=n}^{\infty}\lambda_k}\right)^p\leqslant D\sum_{n=-\infty}^{\infty}a_n^p,\tag{31}$$

where $\lambda_n>0$, $a_n\geqslant 0$, $n\in\mathbb{N}\cup\{0\}$ or $n\in\mathbb{Z}$, respectively. However, in general, the best constants in the above inequalities have not been found yet.

If $\lambda_k=k^{-\alpha}$ for $k\in\mathbb{N}$ and $\lambda_0=a_0=0$, then the first inequality in (30) becomes (25) and it holds with the best constant $A=((1-\alpha)p/(p-\alpha p-1))^p$ for parameters which have been mentioned at the beginning of this part.

Since $\sum\limits_{k=n}^{\infty} q^{\lambda k} = q^{\lambda n}/(1-q^{\lambda})$, estimates (27) and (26) imply that the second inequalities in (30) and (31) (the Copson inequalities) with $\lambda_k = q^{\lambda k}$ $(0<q<1,\ \lambda>0,\ k\in\mathbb{N}\cup\{0\}$ or $k\in\mathbb{Z}$, respectively) for $p>1$ or $p<0$ hold with the best constants $B=1$ and $D=1$, respectively.

Also, since $\sum\limits_{k=-\infty}^{n} q^{-\lambda k} = q^{-\lambda n}/(1-q^{\lambda})$, estimate (28) implies the first inequality in (31) (the Hardy inequality) with $\lambda_k = q^{-\lambda k}$ $(0<q<1,\ \lambda>0,\ k\in\mathbb{Z})$ for $p>1$ with the best constant $C=1$. In the case $0<p<1$ the second inequality in (31) holds in the reverse direction.

Inequality (29) and the obvious estimate $\sum\limits_{k=0}^{n} q^{-\lambda k} \geqslant q^{-\lambda n}$ imply that the first inequality in (30) holds with $\lambda_k = q^{-\lambda k}$ $(0<q<1,\ \lambda>0,\ k=0,1,2,\ldots)$ for $p>1$ with the estimate $A\leqslant (1-q^{\lambda})^{-p}$ for the best constant.

From the proof of Theorem 2.4 we obtain that if $\lambda>0$, $0<q<1$, $a_n\geqslant 0$ $(n=0,1,2,\ldots)$ and $0<p<1$, then the following discrete inequalities hold with the best constants:

$$\sum_{n=0}^{\infty}\Big(q^{-\lambda n}\sum_{k=n}^{\infty} q^{\lambda k}a_k\Big)^p > \frac{1}{(1-q^{\lambda})^p}\sum_{n=0}^{\infty}(1-q^{\lambda n})a_n^p,$$

and

$$\sum_{n=0}^{\infty}\Big(q^{-\lambda n}\sum_{k=n}^{\infty} q^{\lambda k}a_k\Big)^p + \frac{1}{1-q^{\lambda}}\Big(\sum_{n=0}^{\infty} q^{\lambda n}a_n\Big)^p > \frac{1}{(1-q^{\lambda})^p}\sum_{n=0}^{\infty} a_n^p.$$

The proof of the q-inequality for the Riemann-Liouville operator gives estimates for matrix operators. In fact, from the proof of Theorem 3.1 we obtain the following inequalities: if $0<q<1$, $\alpha>0$ and $p>1$, then

$$\sum_{n=-\infty}^{\infty}\Big(q^{-n/p'}\sum_{k=n}^{\infty}(q^{k-n+1};q)_{\alpha-1}q^{k/p'}a_k\Big)^p \leqslant E\sum_{n=-\infty}^{\infty} a_n^p,\quad a_n\geqslant 0, \tag{32}$$

$$\sum_{n=0}^{\infty}\Big(q^{-n/p'}\sum_{k=n}^{\infty}(q^{k-n+1};q)_{\alpha-1}q^{k/p'}a_k\Big)^p \leqslant E\sum_{n=0}^{\infty} a_n^p,\quad a_n\geqslant 0, \tag{33}$$

with the best constant $E = \Big(\sum\limits_{n=0}^{\infty} q^{n/p'}(q^{n+1};q)_{\alpha-1}\Big)^p$.

Since $\sum\limits_{k=n}^{\infty} q^{k/p'}(q^{k-n+1};q)_{\alpha-1} = q^{n/p'}\sum\limits_{k=0}^{\infty} q^{k/p'}(q^{k+1};q)_{\alpha-1}$, then denoting

$$\overline{Q}_n = \sum_{k=n}^{\infty} q^{k/p'}(q^{k-n+1};q)_{\alpha-1}$$

we can rewrite inequalities (32) and (33) in the following forms:

$$\sum_{n=-\infty}^{\infty}\left(\frac{1}{\overline{Q}_n}\sum_{k=n}^{\infty}q^{k/p'}(q^{k-n+1};q)_{\alpha-1}a_k\right)^p \leqslant \sum_{n=-\infty}^{\infty}a_n^p, \quad a_n\geqslant 0 \tag{34}$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{\overline{Q}_n}\sum_{k=n}^{\infty}q^{k/p'}(q^{k-n+1};q)_{\alpha-1}a_k\right)^p \leqslant \sum_{n=0}^{\infty}a_n^p, \quad a_n\geqslant 0. \tag{35}$$

Moreover, by passing to the dual inequality in (32) and substituting p by p' we obtain

$$\begin{aligned}\sum_{n=-\infty}^{\infty}&\left(q^{n/p}\sum_{k=-\infty}^{n}(q^{n-k+1};q)_{\alpha-1}q^{-k/p}a_k\right)^p\\ &\leqslant\left(\sum_{n=0}^{\infty}q^{n/p}(q^{n+1};q)_{\alpha-1}\right)^p\sum_{n=0}^{\infty}a_n^p, \quad a_n\geqslant 0.\end{aligned} \tag{36}$$

Since $\sum\limits_{k=-\infty}^{n}(q^{n-k+1};q)_{\alpha-1}q^{-k/p} = q^{n/p}\sum\limits_{j=0}^{\infty}(q^{k+1};q)_{\alpha-1} =: Q_n$, the inequality (36) can be written in the form

$$\sum_{n=-\infty}^{\infty}\left(\frac{1}{Q_n}\sum_{k=-\infty}^{n}(q^{n-k+1};q)_{\alpha-1}q^{-k/p}a_k\right)^p \leqslant \sum_{n=0}^{\infty}a_n^p, \quad a_n\geqslant 0. \tag{37}$$

Inequalities (34), (35) and (37) are examples of sharp matrix inequalities of the forms

$$\sum_{n=-\infty}^{\infty}\left(\frac{\sum_{k=n}^{\infty}\lambda_{n,k}a_k}{\sum_{k=n}^{\infty}\lambda_{n,k}}\right)^p \leqslant \sum_{n=-\infty}^{\infty}a_n^p, \quad \sum_{n=0}^{\infty}\left(\frac{\sum_{k=n}^{\infty}\lambda_{n,k}a_k}{\sum_{k=n}^{\infty}\lambda_{n,k}}\right)^p \leqslant \sum_{n=0}^{\infty}a_n^p,$$
$$\text{and}\quad \sum_{n=-\infty}^{\infty}\left(\frac{\sum_{k=-\infty}^{n}\lambda_{n,k}a_k}{\sum_{k=-\infty}^{n}\lambda_{n,k}}\right)^p \leqslant \sum_{n=-\infty}^{\infty}a_n^p, \quad a_n\geqslant 0,$$

where $p>1$ and $\lambda_{n,k}>0$ are of special form. They are generalizations of three of the inequalities in (30) and (31).

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