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EQUIDISTRIBUTION IN THE DUAL GROUP OF THE S-ADIC INTEGERS

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Abstract. Let X be the quotient group of the S-adele ring of an algebraic number field by the discrete group of S-integers. Given a probability measure μ on X^d and an endomorphism T of X^d, we consider the relation between uniform distribution of the sequence $T^n\mathbf{x}$ for μ-almost all $\mathbf{x} \in X^d$ and the behavior of μ relative to the translations by some rational subgroups of X^d. The main result of this note is an extension of the corresponding result for the d-dimensional torus $\mathbb{T}^d$ due to B. Host.

Keywords: uniform distribution modulo 1; equidistribution in probability; algebraic number fields; S-adele ring; S-integer dynamical system; algebraic dynamics; topological dynamics; a-adic solenoid

MSC 2010: 11J71, 11K06, 54H20

1. Introduction and the main result

Given a probability measure μ on the d-dimensional torus $\mathbb{T}^d$ and an endomorphism T of $\mathbb{T}^d$, B. Host considered the relation between uniform distribution of the sequence $T^n t$ for μ-almost all $t \in \mathbb{T}^d$ and the behavior of μ relative to the translations by some rational subgroups of $\mathbb{T}^d$. In this paper we considerably extend Host's theorems ([8], Theorem 1 and Theorem 2) to the d-fold Cartesian product of the quotient group of the S-adele ring of an algebraic number field by the discrete group of S-integers.

Let k be an algebraic number field[1], i.e., a finite extension of the rational fieled $\mathbb{Q}$. It is known, that $k = \mathbb{Q}(\theta)$, where θ is an algebraic integer. The set of places, finite places and infinite places of k is denoted by $\mathcal{P} = \mathcal{P}(k)$, $\mathcal{P}_f = \mathcal{P}_f(k)$ and $\mathcal{P}_\infty = \mathcal{P}_\infty(k)$,

[1] For more details on the number theoretical notions appearing in this Introduction see §2.

respectively. Denote by k_v the completion of k under the metric $d_v(x,y) = |x-y|_v$ on k.

For a subset S of $\mathcal{P}_f(k)$, consider a discrete countable group R_S of S-integers,

$$R_S = \{x \in k\colon\ |x|_v \leqslant 1 \text{ for all } v \notin S \cup \mathcal{P}_\infty(k)\},$$

and, $k_{\mathbb{A}}(S)$ the S-adele ring of k (with a topology defined in §3)

$$k_{\mathbb{A}}(S) = \left\{ x = (x_v) \in \prod_{v \in S \cup \mathcal{P}_\infty(k)} k_v\colon\ |x_v|_v \leqslant 1 \text{ for all but finitely many } v \right\}.$$

For a given abelian group R_S of S-adic integers we consider its dual group $\widehat{R}_S$ (the set of all characters on R_S, i.e., the set of all continuous homomorphisms $R_S \to \mathbb{T}$) which is a compact abelian group (see [7]) and we denote it by

$$X = X^{(k,S)} := \widehat{R}_S. \tag{1.1}$$

Dynamical systems with the state space X were considered by Chothi, Everest and Ward in [3] (see also §3 for more details). Information on uniform distribution of sequences in the adelic setting can be found in the book by M.-J. Bertin et al. [2] (see also the references therein). In this paper we will be interested in higher dimensional spaces X^d and sequences of the form $T^n\mathbf{x}$, where T is a continuous endomorphism of X^d.

In what follows we assume that S is a finite set, and denote

$$m = m_S + m_\infty := \operatorname{card}(S) + \operatorname{card}(\mathcal{P}_\infty(k)).$$

Then

$$k_{\mathbb{A}}(S) = \prod_{v \in S \cup \mathcal{P}_\infty(k)} k_v$$

and, by Theorem 3.1,

$$X = k_{\mathbb{A}}(S)/R'_S,$$

where

$$R'_S = \{(\underbrace{x, \ldots, x}_{m \text{ times}})\colon\ x \in R_S\}.$$

By (1.1) it follows that for any positive integer d the Cartesian product X^d is the quotient group

$$X^d = \prod_{v \in S \cup \mathcal{P}_\infty(k)} k_v^d / R'_{S,d},$$

where

$$R'_{S,d} = \{(\underbrace{x, \dots, x}_{m \text{ times}})\colon\ x \in R_S^d\}.$$

Let, for an algebraic integer θ of degree t,

$$\mathbb{Z}[\theta] = \{x_0 + x_1\theta + \ldots + x_{t-1}\theta^{t-1}\colon\ x_j \in \mathbb{Z}\}$$

be the ring obtained from $\mathbb{Z}$ by adjoining θ. We introduce the following notation:

$$\mathbb{Z}[\theta]_{\leqslant n} = \{x_0 + x_1\theta + \ldots + x_{t-1}\theta^{t-1}\colon\ x_j \in \mathbb{Z} \text{ and } 0 \leqslant x_j \leqslant n\}.$$

For a rational integer $q > 1$, define the following subgroup[2] of X^d,

$$\mathcal{D}_q = \{(\underbrace{y/q^n, \dots, y/q^n}_{m \text{ times}}) + R'_{S,d}\colon\ y \in \mathbb{Z}[\theta]^d_{\leqslant q^n},\ n \geqslant 1\}. \tag{1.2}$$

We have,

$$\mathcal{D}_q = \bigcup_{n \geqslant 1} \mathcal{D}_{q,n},$$

where

$$\mathcal{D}_{q,n} = \{(\underbrace{y/q^n, \dots, y/q^n}_{m \text{ times}}) + R'_{S,d}\colon\ y \in \mathbb{Z}[\theta]^d_{\leqslant q^n}\} \tag{1.3}$$

are subgroups of X^d. Define the following sequence of measures on X^d,

$$\omega_n = \sum_{\mathbf{x} \in \mathcal{D}_{q,n}} \delta_{\mathbf{x}} * \mu.$$

Let

$$\varphi_k(\mathbf{x}) = \frac{\mathrm{d}\mu(\mathbf{x})}{\mathrm{d}\omega_k(\mathbf{x})} \tag{1.4}$$

be the Radon-Nikodym derivative (if it exists).

Definition 1.1. We say that the probability measure μ on X^d is $\mathcal{D}_q$-*conservative* if for every Borel set E with $\mu(E) > 0$, there exists $\mathbf{y} \in \mathcal{D}_q$, $\mathbf{y} \neq 0$, with $\mu(E \cap (\mathbf{y} + E)) > 0$.

[2] That $\mathcal{D}_q$ forms a subgroup follows from the fact that $\theta \in \mathcal{O}_k$, the ring of algebraic integers, and $\mathcal{O}_k = k \cap \bigcap_{w \in \mathcal{P}_f(k)} \{x \in k_w\colon\ |x|_w \leqslant 1\}$ (see [19], Theorem V.1).

Definition 1.2. We say that the probability measure μ on X^d is $\mathcal{D}_q$*-conservative with exponential decay* if

$$\liminf_{k\to\infty} -\frac{1}{k}\log\varphi_k(\mathbf{x}) > 0, \quad \mu\text{-a.e.}$$

Let R be a given ring and $d \in \mathbb{N}$. By $\mathrm{M}(d,R)$ we denote the set of all $d\times d$-matrices with element from R.

Definition 1.3. Let $T \in \mathrm{M}(d,R_S)$, $d \geqslant 1$. We say that the sequence $T^n\mathbf{x}$, $\mathbf{x} \in X^d$ is *equidistributed* if the sequence of probability measures $\mu_N = N^{-1}\sum_{n=0}^{N-1}\delta_{T^n\mathbf{x}}$ converges to the Haar measure in the weak-$*$ topology, i.e., for every $f \in C(X^d)$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(T^n\mathbf{x}) = \int_X f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Definition 1.4. Let $T \in \mathrm{M}(d,R_S)$, $d \geqslant 1$. According to [8], [9], we say that the sequence $T^n\mathbf{x}$, $\mathbf{x} \in X^d$ is *equidistributed in probability for the measure* μ if, for every weak-$*$ neighborhood U of the Haar measure on X^d,

$$\lim_{N\to\infty}\mu\Big\{\mathbf{x} \in X^d\colon\ \frac{1}{N}\sum_{n=0}^{N-1}\delta_{T^n\mathbf{x}} \notin U\Big\} = 0.$$

An excellent introduction into the topic of equidistribution theory can be found in the book of Kuipers and Niederreiter [11] or in the book of Drmota and Tichy [4].

The main result of this note is the following.

Theorem 1.1. *Let $k = \mathbb{Q}(\theta)$, where θ is an algebraic integer, S be the finite subset of $\mathcal{P}_f(k)$, and $T \in \mathrm{M}(d,R_S)$. Set $r = 2 + d(d-1)/2$. Let $\mathcal{D}_q$ be the subgroup of X^d defined in* (1.2).

Assume that

(i) *for every integer $k > 1$ the characteristic polynomial of T^k is irreducible over $\mathbb{Q}(\theta)$,*

(ii) *for every $v \in S$, $|q|_v = 1$,*

(iii) *the determinant $\det T$, considered as an element of the ring*

$$R := R_S/q^rR_S$$

is a unit in R.

Then

(1) *if the probability measure* μ *on* X^d *is* $\mathcal{D}_q$*-conservative then the sequence* $T^n\mathbf{x}$ *is equidistributed in probability for* μ*;*
(2) *if the probability measure* μ *on* X^d *is* $\mathcal{D}_q$*-conservative with exponential decay then for* μ*-a.e.* $\mathbf{x} \in X^d$ *the sequence* $T^n\mathbf{x}$ *is equidistributed.*

The outline of the rest of the paper is as follows. In §2 we recall some basic notions from algebraic number theory, in particular, the notion of places of algebraic number field and a definition of $\mathfrak{p}$-adic fields. We also consider additive characters and duality of local fields as well as logarithms and exponentials of a matrix with entries from $\mathfrak{p}$-adic fields.

In §3 we define an S-adele ring of an algebraic number field k and, following [3], the S-adic dynamical systems.

The next §4 contains some lemmas which are used in the proof of Theorem 1.1—which is given in §5.

Finally, in §6 we give some examples.

2. Preliminaries

2.1. p-adic fields. The basic references for this subsection are [5], [10], [12], [16]. Let $p \in \mathbb{P}$, the set of rational primes. The *p-adic norm* $|\cdot|_p$ on the field $\mathbb{Q}$ is defined by $|0|_p = 0$ and $|p^k n/m|_p = p^{-k}$ for $k, n, m \in \mathbb{Z}$ and $p \nmid nm$. The *p-adic field of rational numbers* $\mathbb{Q}_p$ is defined as the completion of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$. The p-adic field $\mathbb{Q}_p$ is a locally compact field and every $x \in \mathbb{Q}_p$ can be uniquely expressed as a convergent sum, in $|\cdot|_p$-norm (Hensel representation),

$$x = \sum_{k=t}^{\infty} x_k p^k, \tag{2.1}$$

for some $t \in \mathbb{Z}$ and $x_k \in \{0, 1, \ldots, p-1\}$. The *fractional part* of $x \in \mathbb{Q}_p$, denoted by $\{x\}_p$ or $\{x\}$, is 0 if the number t in the Hensel representation (2.1) is greater than or equal to 0, and equal to $\sum_{k<0} x_k p^k$, if $t < 0$.

The *integral part* $[x]_p$ (or simply $[x]$) of an element $x \in \mathbb{Q}_p$ is $\sum_{k\geqslant 0} x_k p^k$.

The closure of $\mathbb{Z}$ in $\mathbb{Q}_p$ is the compact ring $\mathbb{Z}_p$ of *p-adic integers.* An element $x \in \mathbb{Q}_p$ is a p-adic integer if it has a Hensel representation (2.1) with $t \geqslant 0$, that is, its fractional part $\{x\} = 0$.

2.2. Characters and duality of local fields. A good reference for this subsection is [15]. For a positive integer a, denote by $\mathbb{Z}[1/a]$ the ring obtained from $\mathbb{Z}$ by adjoining $1/a$. Thus, any $x \in \mathbb{Q}_p$ can be uniquely written as $x = [x] + \{x\}$, where $[x] \in \mathbb{Z}_p$ and the fractional part $\{x\} \in \mathbb{Z}[1/p] \cap [0, 1)$.

Define

$$e_p\colon\ \mathbb{Q}_p \to \mathbb{C}\colon\ x \mapsto \exp(2\pi\mathrm{i}\{x\}_p).$$

It is easy to see that the map e_p is a homomorphism and the additive group $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic with the group μ_{p^∞} of p-th power roots of unity in the complex field $\mathbb{C}$ (see [16]).

Recall that the dual group $\widehat{\mathbb{R}}$ is topologically isomorphic with $\mathbb{R}$. Moreover, $\widehat{\mathbb{Q}}_p$ is topologically isomorphic with $\mathbb{Q}_p$ and the action of the character $\chi_x \in \widehat{\mathbb{Q}}_p$ corresponding to $x \in \mathbb{Q}_p$ is $\chi_x(y) = e_p(xy) = \exp(-2\pi\mathrm{i}\{xy\}_p)$. This is very similar to the case of the action of the character from $\widehat{\mathbb{R}}$ on $\mathbb{R}$. For the field of complex numbers $\mathbb{C}$, the function

$$\chi(z) = \mathrm{e}^{-2\pi\mathrm{i}(z+\overline{z})} = \mathrm{e}^{-4\pi\,\mathrm{Re}(z)}$$

defines a non-trivial character on $\mathbb{C}$.

Generally, let F be a local field (i.e., $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$) and let χ be any non-trivial additive character of F. For any $\alpha \in F$, we write χ_α for the character $x \mapsto \chi(\alpha x)$. Every character of F is of this form for some α, and the mapping $\alpha \mapsto \chi_\alpha$ is an isomorphism of topological groups. Thus the additive group of local field is self-dual.

Let F be a finite extension of $\mathbb{Q}_p$. We construct a non-trivial character χ as follows. It is a composition of four continuous homomorphisms,

$$\chi = e \circ \lambda \circ \mathrm{pr} \circ \mathrm{Tr}, \tag{2.2}$$

where $\mathrm{Tr}\colon\ F \to \mathbb{Q}_p$ is the trace map, the map pr is the natural projection $\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p$. Each coset of $\mathbb{Q}_p/\mathbb{Z}_p$ is represented by a unique p-adic number of the form $a_m p^{-m} + \ldots + a_1 p^{-1}$, hence $\mathrm{pr}(x) = \{x\}_p + \mathbb{Z}_p$. Since the fractional part $\{x\}_p \in [0, 1)$, the group homomorphism $\lambda\colon\ \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z}$, which sends a coset to its representative is well defined, and finally $e(x) = \mathrm{e}^{2\pi\mathrm{i}x}$.

2.3. Places. We follow the presentation contained in [17], page 60. Let k be an algebraic number field, i.e., a finite extension of the rational fieled $\mathbb{Q}$. An *absolute value* of k is a homomorphism $\phi\colon\ k \to \mathbb{R}^+ \cup \{0\}$ such that $\phi(x) = 0$ if and only if $x = 0$, and and there is a real number $c \geqslant 1$ such that for all $x, y \in k$, $\phi(xy) = \phi(x)\phi(y)$ and $\phi(x + y) \leqslant c \max\{\phi(x), \phi(y)\}$. The absolute value ϕ is *non-trivial* if $\phi(k) \supsetneq \{0, 1\}$. The absolute value ϕ is *non-Archimedean* if ϕ is non-trivial and we can set $c = 1$, and is said to be *Archimedean* otherwise. We say that two absolute values ϕ, ψ of k are *equivalent* if there is an $s > 0$ such that $\phi(x) = \psi(x)^s$ for every

$x \in k$. An equivalence class v of a non-trivial absolute value of k is called a *place* of k. A place v is *finite* if v contains a non-Archimedean absolute value, and *infinite* otherwise.

The set of places, finite places and infinite places of k is denoted by $\mathcal{P} = \mathcal{P}(k)$, $\mathcal{P}_f = \mathcal{P}_f(k)$ and $\mathcal{P}_\infty = \mathcal{P}_\infty(k)$, respectively.

By Ostrovski's theorem every non-trivial absolute value of $\mathbb{Q}$ is either equivalent to the usual absolute value $|\cdot|_\infty$, or to the p-adic absolute value $|\cdot|_p$ for some rational prime $p > 1$.

A place $w \in \mathcal{P}$ is said to lie above a place v of $\mathbb{Q}$, denoted $w \mid v$, if[3] $|\cdot|_w$ restricted to $\mathbb{Q}$ is equivalent with $|\cdot|_v$. Above every place v of $\mathbb{Q}$ there are at least one and at most finitely many places of k. Denote by k_w the completion of k under the metric $d_w(x,y) = |x-y|_w$ on k.

The infinite places of the algebraic number field k of degree n come from the n embeddings σ_i, $i = 1, \ldots, n$, of k into $\mathbb{C}$ and all of them lie above the unique infinite place $|\cdot|_\infty$ of $\mathbb{Q}$. If the place v comes from the embedding σ_i, $\sigma_i(k) \subset \mathbb{R}$ then v is called real, otherwise v is called complex.

2.4. $\mathfrak{p}$-adic fields. Let R_k be the ring of integers of an algebraic number field k. Let $\mathfrak{p}$ a prime ideal of R_k, v the (discrete) absolute value associated with $\mathfrak{p}$ ([13], Theorem 3.3). By $k_\mathfrak{p}$ or k_v we denote the completion of k under v, and we call $k_\mathfrak{p}$ the $\mathfrak{p}$-adic field. By κ we denote the quotient field $R_k/\mathfrak{p}$, the residue class field. The cardinality of this residue field we denote by $q = q_\mathfrak{p} = q_v$. The extension of v to $k_\mathfrak{p}$ will be also denoted by v. The ring of integers of $k_\mathfrak{p}$, $R_\mathfrak{p} = \{x \in k_\mathfrak{p} \colon v(x) \leqslant 1\}$ is the closure of the ring $R = \{x \in k \colon v(x) \leqslant 1\}$, and $\mathfrak{P} = \{x \in k_\mathfrak{p} \colon v(x) < 1\} = \mathfrak{p}R_\mathfrak{p}$ is a prime ideal of $R_\mathfrak{p}$, which is the closure of the prime ideal $\{x \in k \colon v(x) < 1\}$ of R. The invertible elements of $R_\mathfrak{p}$ form a group $U(R_\mathfrak{p}) = R_\mathfrak{p} \setminus \mathfrak{P}$ of units of $k_\mathfrak{p}$. The quotient fields $R_k/\mathfrak{p}$ and $R_\mathfrak{p}/\mathfrak{P}$ are isomorphic ([13], Proposition 5.1).

We define a uniformizer for v, or a local parameter, to be an element π, also denoted by π_v or $\pi_\mathfrak{p}$ of $k_\mathfrak{p}$ of maximal $v(\pi)$ less than 1. If we fix a uniformizer π, every element of $k_\mathfrak{p}^*$ can be written uniquely as $x = u\pi^m$ for some u with $v(u) = 1$ and $m \in \mathbb{Z}$. Moreover, each element $x \in k_\mathfrak{p}^*$ can be expressed in one and only one way as a convergent series

$$x = \sum_{i=m}^{\infty} r_i \pi^i,$$

where the coefficients r_i are taken from a (finite) set $\mathcal{R} \subset R_\mathfrak{p}$ of representatives of the residue classes in the field $\kappa_\mathfrak{p} := R_\mathfrak{p}/\mathfrak{P}$ (i.e., the canonical map $R_\mathfrak{p} \to \kappa_\mathfrak{p}$ induces a bijection of $\mathcal{R}$ onto $\kappa_\mathfrak{p}$).

[3] We slightly abuse notation and denote v by $|\cdot|_v$ if it is convenient.

In what follows we consider the normalized valuation, i.e., if $v \mid p$, $p \in \mathbb{P}$, then

$$|x|_v = v(x) = f^{-m},$$

where m is the unique integer such that $x = u\pi^m$ for some unit u, and $f > 1$ is chosen so that

$$|p|_v = p^{-1}. \tag{2.3}$$

Let k be a field with a valuation v. Then k is a $\mathfrak{p}$-adic field with the $\mathfrak{p}$-adic valuation if and only if k is a finite extension of $\mathbb{Q}_p$ for a suitable p. (See [13], Theorem 5.10.)

2.5. Logarithms and exponentials of a matrix. We refer to [13], [14] for the general theory. Consider an algebraic number field k_v with the ring of integers R_v, where $v \in \mathcal{P}_f(k)$. Let $A = (a_{ij}) \in \mathrm{M}(d, k_v)$ and $x = (x_1, \ldots, x_d)^t \in k_v^d$ be a column vector. Here and in what follows all vectors are column vectors unless explicitly written as transposed. For a finite place $v \mid p$, $p \in \mathbb{P}$, let $|\cdot|_v$ denote the normalized as in (2.3) absolute value. We define the norms of A and x by

$$\|A\|_v = \max_{i,j} |a_{ij}|_v \quad \text{and} \quad \|x\|_v = \max_{j} |x_j|_v.$$

Let $A \in \mathrm{M}(d, R_v)$ and $\|I_d - A\|_v \leqslant f^{-1}$. Since $|1/n|_v \leqslant n$ for every $n \geqslant 1$ it follows that the following series

$$\log A := \sum_{n=1}^{\infty} -\frac{1}{n}(I_d - A)^n$$

converges in $\mathrm{M}(d, k_v)$ and $\log A \in \mathrm{M}(d, R_v)$ satisfies $\|\log A\|_v \leqslant \|I_d - A\|_v$. Moreover, if $A \in \mathrm{M}(d, R_v)$ and $\|A\|_v \leqslant f^{-2}$ then one can define $\exp A$ as the series

$$\exp A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$

In fact, since $|1/n!|_v \leqslant p^n$ for every $n \geqslant 0$, the above series converges in $\mathrm{M}(d, k_v)$, and one has $\exp A \in \mathrm{M}(d, R_v)$. We have

$$\|\exp(A) - I_d - A\|_v \leqslant p^2 \|A^2\|_v. \tag{2.4}$$

Now, following [3], Definition 2.1, we can define the dynamical system associated to a set S of finite places.

Let $S \subset \mathcal{P}(k) \setminus \mathcal{P}_\infty(k)$ and define the discrete countable group R_S of S-*integers* as

$$R_S = \{x \in k \colon |x|_w \leqslant 1 \text{ for all } w \notin S \cup \mathcal{P}_\infty(k)\},$$

and define its dual group (see [7] for definition),

$$X = \widehat{R}_S.$$

Hence, X is a compact abelian group.

For a given element $\xi \in k^*$, and any set $S \subset \mathcal{P}(k) \setminus \mathcal{P}_\infty(k)$ with the property that $|\xi|_w \leqslant 1$ for all $w \notin S \cup \mathcal{P}_\infty(k)$, we define a dynamical system as

$$(X, \alpha) = (X^{(k,S)}, \alpha^{(k,S,\xi)}),$$

where the continuous group endomorphism

$$\alpha\colon X \to X$$

is dual to the monomorphism

$$\widehat{\alpha}\colon R_S \to R_S$$

defined by

$$\widehat{\alpha}\colon x \mapsto \xi x.$$

Example 3.1. Let $k = \mathbb{Q}$, $S = \{2\}$, and $\xi = 1$. Then $R_S = R_{\{2\}} = \mathbb{Z}[1/2]$, and $X = \widehat{R}_S$ is the 2-adic solenoid (of finite type) in this case (see [1] or [7] for more information on a-adic solenoids). The automorphism α of X is dual to the automorphism $x \mapsto 2x$ of $R_{\{2\}}$.

3.1. S-adele ring. Let $S \subset \mathcal{P}(k) \setminus \mathcal{P}_\infty(k)$. The S-*adele ring* of k is the ring

$$k_{\mathbb{A}}(S) = \left\{x = (x_v) \in \prod_{v \in S \cup \mathcal{P}_\infty(k)} k_v \colon |x_v|_v \leqslant 1 \text{ for all but finitely many } v\right\}$$

furnished with the topology in which for every finite set $S' \subset S$, the subring

$$k_{\mathbb{A}}^{S'} := \prod_{v \in S' \cup \mathcal{P}_\infty(k)} k_v \times \prod_{v \in S \setminus S'} R_v$$

carries the product topology (so that is locally compact) and is open in $k_{\mathbb{A}}(S)$, and a fundamental system of open neighborhoods of 0 in the additive group of $k_{\mathbb{A}}(S)$ is given by a fundamental system of neighborhoods of 0 in any one of the subrings $k_{\mathbb{A}}^{S'}$.

Since for every $v \in \mathcal{P}(k)$, the ring R_v is compact it follows that the S-adele ring is locally compact.

Let

$$\imath\colon\ R_S \to k_{\mathbb{A}}(S)$$

be the diagonal embedding

$$\imath(x) = (x, x, x, \ldots).$$

The following theorem taken from [3] is an extension (to arbitrary set of places) of some results proved in [19], Chapter IV.2.

Theorem 3.1 ([3], Theorem 3.1)**.** *The map $\imath\colon\ R_S \to k_{\mathbb{A}}(S)$ embeds R_S as a discrete cocompact subring in the S-adele ring of k. There is an isomorphism between the S-adele ring $k_{\mathbb{A}}(S)$ and its dual, which induces an isomorphism between $X = \widehat{R}_S$ and $k_{\mathbb{A}}(S)/\imath(R_S)$.*

4. Lemmas

The following theorem is classical.

Theorem 4.1 ([6], Theorem 1)**.** *Let X be a compact metrizable abelian group and $T\colon\ X \to X$ a surjective continuous endomorphism. The Haar measure on X is ergodic for T if and only if the trivial character $\chi \equiv 1$ is the only $\chi \in \widehat{X}$ satisfying $\chi \circ T^n = \chi$ for some $n > 0$.*

As a corollary we get, as in [3], the following

Lemma 4.1. *Let $(X, \alpha) = (X^{(k,S)}, \alpha^{(k,S,\xi)})$ be an S-integer dynamical system. Then α is ergodic if and only if ξ is not a root of unity.*

Proof. The map α is non-ergodic if and only if there is an $r \in R_S \setminus \{0\}$ with $\xi^m r = r$ for some $m \neq 0$. This is possible in a field if and only if ξ is a root of unity. $\square$

The formula given in the following lemma can be view via an adelic covering lemma that makes this just a volume calculation in some finite product of p-adic fields.

Lemma 4.2 ([3], Lemma 5.2). *Let $(X,\alpha) = (X^{(k,S)}, \alpha^{(k,S,\xi)})$ be an S-integer dynamical system. Then $F_n(\alpha)$, the number of points of period $n \geqslant 1$, is finite if α is ergodic, and*

$$|F_n(\alpha)| = \prod_{v \in S \cup \mathcal{P}_\infty(k)} |\xi^n - 1|_v,$$

where v is normalized so that the product formula holds.[4]

Lemma 4.3. *For every $l \in \mathbb{N}$, the group*

$$R_S^d / l R_S^d \simeq (R_S / l R_S)^d$$

is finite and its cardinality is bounded by $l^{m_\infty d}$, where $m_\infty = \operatorname{card}(\mathcal{P}_\infty(k))$.

Proof. The cardinality c of $R_S^d / l R_S^d$ is the number of points fixed by the endomorphism $x \mapsto (1-l)x$ on X^d. This endomorphism is ergodic by Lemma 4.1. Hence, it follows by Lemma 4.2 and the product formula (see footnote in Lemma 4.2) that

$$c = \Bigl(\prod_{v \in S \cup \mathcal{P}_\infty(k)} |l|_v \Bigr)^d \leqslant \Bigl(\prod_{v \in \mathcal{P}_\infty(k)} |l|_v \Bigr)^d = l^{m_\infty d}.$$

□

Lemma 4.4. *Let $T \in \mathrm{M}(d, R_S)$, and let $l \in \mathbb{N}$. Assume that $\det T$, considered as an element of the ring*

$$R := R_S / l R_S,$$

is invertible in R. Then there exists a number $\tau \in \mathbb{N}$ such that

$$T^\tau \equiv I_d \bmod l R_S^d,$$

where I_d stands for the identity $d \times d$-matrix.

Proof. For a given T, define the matrix

$$\widetilde{T} \in \mathrm{M}(d, R),$$

with entries

$$\tilde{t}_{ij} = t_{ij} \bmod l R_S = t_{ij} + l R_S.$$

[4] The product formula says that $\prod_{\mathcal{P}(k)} |x|_v = 1$, for all $x \in k \setminus \{0\}$ (see [13], [14], [15], [19]).

By Lemma 4.3 the matrix $\widetilde{T}$ acts naturally on the finite module

$$R^d = (R_S/lR_S)^d$$

over the finite ring R. Thus we have an action of the semigroup $\mathbb{N}$ on R^d, given by

$$k.\,x = T^k x, \quad k \in \mathbb{N},\ x \in R^d.$$

We have that $\det \widetilde{T}$ is invertible in R, hence $\widetilde{T} \in \mathrm{GL}(d,R)$. Thus $\{\widetilde{T}^k\colon\ k \in \mathbb{N}\}$ is a semigroup contained in the finite group $\mathrm{GL}(d,R)$; it follows that $\{\widetilde{T}^k\colon\ k \in \mathbb{N}\}$ is a group. Thus there exists a τ such that $\widetilde{T}^\tau = I_d$, and the lemma is proved. □

Let $T = (t_{ij}) \in \mathrm{M}(d, R_S)$. Set

$$r = 2 + d(d-1)/2. \tag{4.1}$$

For an integer q satisfying (ii) of Theorem 1.1 consider, as in the proof of Lemma 4.4, the matrix

$$\widetilde{T} \in \mathrm{M}(d, R_S/q^r R_S),$$

with entries

$$\tilde{t}_{ij} = t_{ij} \bmod q^r R_S = t_{ij} + q^r R_S.$$

Denote

$$I(N) = \{0, 1, \ldots, N-1\}^d.$$

Let us fix some $\varepsilon \in (0,1)$, and let α be an integer so large that the set

$$\Lambda = \{n \in \mathbb{N}^d\colon\ n_i \neq n_j \bmod p^\alpha \text{ for all } i \neq j \text{ and all prime divisors } p \text{ of } q\} \tag{4.2}$$

satisfies

$$\mathrm{card}(I(N) \cap \Lambda) \geqslant (1 - \varepsilon^d) N^d \quad \text{for all } N \text{ large enough.}$$

Let M be the transpose matrix of T^τ, where τ is as in Lemma 4.4 (with $l = q^r$), that is,

$$M = (T^\tau)^t. \tag{4.3}$$

Now we are able to generalize the fundamental bound (and its proof) from [8], §4, to our setting and get the following result.

Lemma 4.5. *Under the assumptions of Theorem* 1.1 *there exists an integer* $l > 0$ *such that for* $k \geqslant l$, $m, n \in \Lambda$, *and* $b \in R_S^d$ *if* $m = n \bmod q^l$ *and* $\sum_{i=1}^{d} M^{m_i} b = \sum_{i=1}^{d} M^{n_i} b \bmod q^{l+k} R_S$ *then* $m = n \bmod q^k$.

P r o o f. By Lemma 4.4 each entry of the matrix $I_d - T^\tau$ is equal to 0 modulo $q^r R_S$. Thus, also $(I_d - M)_{ij} = 0 \bmod q^r R_S$. Hence, the ij-th entry of the matrix $I_d - M$ belongs to $q^r R_S$, i.e.,

$$(I_d - M)_{ij} = q^r a_{ij}, \quad \text{where } a_{ij} \in R_S. \tag{4.4}$$

Let $P = \{p_1, \ldots, p_s\}$ be the set of different prime numbers such that for every $j = 1, \ldots, s$ there exists a place $v \in S$ such that $v \mid p_j$, i.e., P is the set of all places of $\mathbb{Q}$ that lie below the places from S. By the assumption (ii) of Theorem 1.1, $|q|_v = 1$ for every $v \in S$. Hence, $|q|_{p_j} = 1$ for every $p_j \in P$. Thus if p is a prime divisor of q then also $|p|_{p_j} = 1$ for every $p_j \in P$, and we conclude that $p \neq p_j$, $j = 1, \ldots, s$. Hence, it follows that if $v \in \mathcal{P}_f(k)$ and $v \mid p$, where p is a prime divisor of q, then $v \notin S$. So, using (4.4) we can write

$$\begin{aligned} \|I_d - M\|_v &= \max_{i,j} |(I_d - M)_{ij}|_v \\ &= \max_{i,j} |q^r a_{ij}|_v = |q^r|_v \max_{i,j} |a_{ij}|_v \\ &\leqslant |q^r|_v = p^{-r}. \end{aligned} \tag{4.5}$$

This together with the results of §2.5 implies that the following matrices are well defined

$$A = p^{-r} \log(M) \in \mathrm{M}(d, R_v) \tag{4.6}$$

and

$$M^x := \exp(x p^r A) \in \mathrm{M}(d, R_v), \quad \text{for } x \in R_v. \tag{4.7}$$

By (2.4),

$$\|M^x - I_d - x \log M\|_v \leqslant p^2 \|x\|_v^2 \|I_d - M\|_v^2. \tag{4.8}$$

For a given non-zero $b \in R_S$ and $v \mid p$, where p is a prime divisor of q, we define the function F_v on R_v^d by formula

$$F_v(x) = \sum_{i=1}^{d} M^{x_i} b, \quad x \in R_v^d. \tag{4.9}$$

Since $|b|_v \leqslant 1$,

$$F_v\colon\ R_v^d \to R_v^d.$$

Let $D \in k_v$ is the determinant of the vectors $b, Ab, \ldots, A^{d-1}b$ in k_v^d,

$$D = \det(b, Ab, \ldots, A^{d-1}b), \quad 0 \neq b \in R_S. \tag{4.10}$$

The following lemma is a straightforward generalization of [8], Lemma 1. We include its proof for the sake of completeness. We also note that the proof of Lemma 4.6 is the only place where condition (i) of Theorem 1.1 is used.

Lemma 4.6. *Under the assumptions of Theorem* 1.1 *we have*

$$\det A \neq 0 \quad \textit{and} \quad D \neq 0,$$

where A and D are as in (4.6) *and* (4.10), *respectively.*

Proof. We follow the proof of [8], Lemma 1. Let $v \in \mathcal{P}_f(k)$ and $v \mid p$, where p is a prime divisor of q. By (4.6), $A \in \mathrm{M}(d, R_v)$. Suppose that $\det A = 0$. Then there is a non-zero $x \in k_v^d$ such that $Ax = 0$. Since $\exp(p^r A) = M$, where M is defined in (4.3), it follows that $(I_d - M)x = 0$. Consequently $\det(I_d - T^\tau) = \det(I_d - M^t) = \det(I_d - M)$, and we get that 1 is an eigenvalue of T^τ. This gives us a contradiction with the condition (i) of Theorem 1.1.

Now, suppose that $D = 0$. Therefore, the vectors $b, Ab, \ldots, A^{d-1}b$ in k_v^d are linearly dependent. Thus there is a non-trivial linear map $\xi\colon\ k_v^d \to k_v$ such that $\xi(A^n b) = 0$ for $0 \leqslant n \leqslant d-1$. The Cayley-Hamilton theorem allows us to express A^n for $n > d-1$ as a linear combination of the lower matrix powers of A, hence $\xi(A^n b) = 0$ for all $n \geqslant 0$. Hence, it follows from (4.7) that $\xi(M^n b) = 0$ for $n \geqslant 0$, and so $b, Mb, \ldots, M^{d-1}b$ are linearly dependent over k_v. Hence, $\det(b, Mb, \ldots, M^{d-1}b) = 0$. Since $M \in \mathrm{M}(d, R_S)$ and $b \in R_S$, the coordinates of the vectors $b, Mb, \ldots, M^{d-1}b$ are also from R_S. Thus the vectors $b, Mb, \ldots, M^{d-1}b$ are not linearly independent over $k = \mathbb{Q}(\theta)$, and this gives us a contradiction with the condition (i) of Theorem 1.1. $\square$

We will also need the following.

Lemma 4.7. *Let $v \in \mathcal{P}_f(k)$ and $v \mid p$, where p is a prime divisor of q. Let $x \in R_v^d$ and $x_i \neq x_j$ for $i \neq j$. Then for all $y \in R_v^d$ such that $\|y\|_v < p^{r-2}\delta_v |V(x)|_v$ we have*

$$\|F_v(x+y) - F_v(x)\|_v \geqslant p^{-r}\delta_v |V(x)|_v \|y\|_v,$$

where

$$V(x)=\prod_{1\leqslant i<j\leqslant d}(x_j-x_i)\quad \textit{and}\quad \delta_v=\left|D\det(A)\prod_{i=0}^{d-1}\frac{p^{ri}}{i!}\right|_v,$$

and F_v *is defined in* (4.9).

Proof. It goes along the lines of the proof of [8], Lemma 3, where the case of the function F on $\mathbb{Z}_p^d$ was considered.

Let for $x\in R_v^d$,

$$K=[K_{ij}]=[(AM^{x_j}b)_i]_{1\leqslant i,j\leqslant d}\in \mathrm{M}(d,R_v).$$

Then

$$Ky=\sum_{j=1}^d y_jAM^{x_j}b,\quad y\in R_v^d.$$

By (4.5), (4.6) and (4.8) we get

$$\begin{aligned}\text{(4.11)}\qquad &\|F_v(x+y)-F_v(x)-p^rKy\|_v\\ &=\left\|\sum_{j=1}^d((M^{y_j}-I_d)M^{x_j}b-p^ry_jAM^{x_j}b)\right\|_v\\ &\leqslant p^2\|y\|_v^2\|I_d-M\|_v^2\leqslant p^{2-2r}\|y\|_v^2.\end{aligned}$$

The same argument as in the proof of [8], Lemma 2, shows that

$$|\det K|_v=\delta_v|V(x)|_v\neq 0.$$

Hence, $\|K^{-1}\|_v\leqslant 1/|\det K|_v$, and consequently

$$\|Ky\|_v\geqslant |\det K|_v\|y\|_v=\delta_v|V(x)|_v\|y\|_v.$$

Using our assumption, $\|y\|_v<p^{r-2}\delta_v|V(x)|_v$, we get

$$\|p^rKy\|_v>\|y\|_v^2p^{2-2r}.$$

This together with (4.11) finishes the proof. □

Now we proceed as in [8], Section 4.3. For $n\in\mathbb{N}^d$ we have $F_v(n)\in R_S^d$. By (4.2), for all $n\in\Lambda$ and every $v\mid p$, where p is a prime divisor of q, we have,

$$|V(n)|_v\geqslant p^{-d(d-1)\alpha/2}.$$

We take an integer $\beta > 0$ such that $\beta \geqslant 2 - r - \log \delta_v / \log p + d(d-1)/2\alpha$ for all $v \mid p$, where p is a prime divisor of q. It follows from Lemma 4.7 that for all prime divisors p of q, and all $v \mid p$,

$$\begin{aligned} m, n \in \Lambda \quad \text{and} \quad \|m - n\|_v \leqslant p^{-\beta} \\ \Rightarrow \|F_v(m) - F_v(n)\|_v \geqslant p^{-\beta-2r+2}\|m-n\|_v. \end{aligned} \tag{4.12}$$

Notice that $m = n \bmod q^k$ means that $m - n = q^k a$ with $a \in \mathbb{Z}^d$, and this is equivalent to $\|m-n\|_p \leqslant p^{-k}$, and consequently to

$$\|m - n\|_v = \|q^k a\|_v \leqslant p^{-k}.$$

Similarly, using (ii) of Theorem 1.1 we see that the condition $\sum_{i=1}^{d} M^{m_i} b = \sum_{i=1}^{d} M^{n_i} b$ (mod $q^{l+k} R_S$) means that

$$\|F_v(m) - F_v(n)\|_v \leqslant p^{-(l+k)}. \tag{4.12}$$

Thus, by (4.12) and (4.13),

$$\|m - n\|_v \leqslant \mathsf{q}_v^{-l-k+\beta+2r-2}.$$

Now it is enough to choose $l \in \mathbb{N}$ so that

$$-l + \beta + 2r - 2 \leqslant 0 \quad \text{and} \quad l \geqslant \beta,$$

and Lemma 4.5 follows. □

5. Proof of Theorem 1.1

Let S be a finite subset of $\mathcal{P}_f(k)$, where $k = \mathbb{Q}(\theta)$. Then the product

$$k_{\mathbb{A}}^d(S) = \prod_{v \in \mathcal{P}_\infty \cup S} k_v^d$$

may be thought of as the "covering space" of X^d. Let $P \subset \mathbb{P}$ be the set corresponding to S, i.e., the set of different rational primes $\{p_1, \dots, p_s\}$ such that for every $p \in P$ there is a $v \in S$ such that $v \mid p$. Since $\widehat{X}^d = R_S^d$, the characters of X^d are indexed by vectors $b \in R_S^d$, and are of the form

$$\chi_b(\mathbf{x} + R'_{S,d}) = \prod_{v \in \mathcal{P}_\infty \cup S} \chi_{b,v}(x_v), \quad \mathbf{x} = (x_v)_{v \in \mathcal{P}_\infty \cup \mathcal{P}_f} \in k_{\mathbb{A}}^d(S),$$

where $\chi_{b,v}$ is given by

$$\chi_{b,v}(x_v) = \begin{cases} \exp(2\pi \mathrm{i}\lambda \circ \mathrm{pr} \circ \mathrm{Tr}^{k_v}_{\mathbb{Q}_p}(x_v)) & \text{if } v \notin \mathcal{P}_\infty, \\ \exp(-2\pi \mathrm{i} x_v) & \text{if } v \text{ is real}, \\ \exp(-4\pi \mathrm{i}\,\mathrm{Re}(x_v)) & \text{if } v \text{ is complex}, \end{cases}$$

with functions λ and pr defined as in (2.2). Hence,

$$\begin{aligned} &\chi_b(\mathbf{x} + R'_{S,d}) \\ &\quad = \prod_{v \text{ real}} \mathrm{e}^{-2\pi\mathrm{i}\langle b, x_v\rangle} \prod_{v \text{ complex}} \mathrm{e}^{-4\pi\mathrm{i}\,\mathrm{Re}\langle b, x_v\rangle} \prod_{v\in S} \chi_{b,v}(x_v) \prod_{j=1}^{s} \prod_{v|p} \mathrm{e}^{2\pi\mathrm{i}\{\mathrm{Tr}^{k_v}_{\mathbb{Q}_p}(\langle b, x_v\rangle)\}_{p_j}}, \\ &\qquad\qquad \mathbf{x} = (x_v)_{v\in\mathcal{P}_\infty\cup\mathcal{P}_f} \in k^d_{\mathbb{A}}(S), \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the standard real inner product.

For a given non-zero $b \in R^d_S$, let

$$S_N(\mathbf{x}) = \frac{1}{N}\sum_{n=0}^{N-1} \chi_b(T^n\mathbf{x})$$

and

$$S^\tau_N(\mathbf{x}) = \frac{1}{N}\sum_{n=0}^{N-1} \chi_b(T^{n\tau}\mathbf{x}),$$

where τ is as in Lemma 4.4. Since for every matrix A, $\langle x, Ay\rangle = \langle A^t x, y\rangle$,

$$S^\tau_N(\mathbf{x}) = \frac{1}{N}\sum_{n=0}^{N-1} \chi_{M^n b}(\mathbf{x}),$$

where M is the transpose matrix of T^τ.

We have

$$(S^\tau_N(\mathbf{x}))^d = \frac{1}{N^d}\sum_{n\in I(N)} \chi_{\sum_{j=1}^d M^{n_j} b}(\mathbf{x}),$$

where $I(N) = \{0, 1, \ldots, N-1\}^d$. Let

$$\widetilde{S}^\tau_N(\mathbf{x}) = \frac{1}{N^d}\sum_{n\in I(N)\cap\Lambda} \chi_{\sum_{j=1}^d M^{n_j} b}(\mathbf{x}),$$

where Λ is defined in (4.2). Then, for N large enough,

$$|(S^\tau_N(\mathbf{x}))^d - \widetilde{S}^\tau_N(\mathbf{x})| \leqslant \varepsilon^d. \tag{5.1}$$

Lemma 5.1. *There exists a constant $C > 0$ such that for all $k \geqslant 2l$, where l is from Lemma 4.5, and for all $N \geqslant q^{kt}$, where $t = \deg_{\mathbb{Q}} \theta$,*

$$\int_{\Omega_a^d} \frac{|\widetilde{S}_N^\tau(\mathbf{x})|^2}{\varphi_k(\mathbf{x})}\, \mathrm{d}\mu(\mathbf{x}) \leqslant C,$$

where φ_k is defined in (1.4).

Proof. We note that $\mathrm{card}(\mathcal{D}_{q,k}) = q^{tdk}$. Using the orthogonality of characters, i.e., the fact that for every non-zero element $b \in R_S^d$,

$$\sum_{\mathbf{x} \in \mathcal{D}_{q,k}} \chi_b(\mathbf{x}) = 0,$$

we get, in the same way as in [8], §2.3, the following estimate

$$\begin{aligned} (5.2)\quad & \int_{\Omega_a^d} \frac{|\widetilde{S}_N^\tau(\mathbf{x})|^2}{\varphi_k(\mathbf{x})}\, \mathrm{d}\mu(\mathbf{x}) \\ & \leqslant q^{tdk} \sum_{\mathbf{j} \in (R_S/q^k R_S)^d} \left(\frac{1}{N^d} \mathrm{card}\{n \in I(N) \cap \Lambda : \sum_{i=1}^d M^{n_j} b = \mathbf{j} \bmod q^k R_S\} \right)^2. \end{aligned}$$

By Lemma 4.3,

$$\mathrm{card}((R_S/q^k R_S)^d) \leqslant q^{tkd}.$$

Lemma 4.5 provides a bound for the cardinality of the set in (5.2) of the form[5] $(Nq^{2l-k} + q^l)^{2d}$. Hence, since $N \geqslant q^{kt}$ and $k \geqslant 2l$, we get the required bound

$$q^{2tdk} N^{-2d} (Nq^{2l-k} + q^l)^{2d} = q^{2tdk} (q^{2l-k} N^{-1} + q^l N^{-1})^{2d} \leqslant (q^{2l-k} + q^l)^{2d}$$

with $C = (1 + q^l)^{2d}$. □

Proof of Theorem 1.1 (1). By the classical results on uniformly distributed sequences in compact groups [11] we have to show that for every non-zero $b \in R_S^d$,

$$\lim_{N \to \infty} S_N(\mathbf{x}) = 0 \quad \text{in } \mu\text{-probability}.$$

Clearly, it is enough to prove that

$$\lim_{N \to \infty} S_N^\tau(\mathbf{x}) = 0 \quad \text{in } \mu\text{-probability}.$$

We will need the following

[5] We divide the set $I(N)$ into q^l equivalence classes modulo q^l, and count the points $n \in I(N) \cap \Lambda$ in each equivalence class which have the same value of $F_v(n) \bmod q^k$, getting at most $Nq^{l-k} + 1$ elements in each equivalence class.

Lemma 5.2. *A probability measure μ on X^d is $\mathcal{D}_q$-conservative if and only if $\varphi_k(\mathbf{x}) \to 0$ μ-a.e. as k tends to ∞.*

P r o o f. Is the same as the proof of the corresponding result for the 1-dimensional torus [9], Lemma 2. $\square$

Now we proceed as in [8]. By Lemma 5.2 for every $\varepsilon > 0$, there exists a Borel subset $E \subset X^d$ with $\mu(E) > 1 - \varepsilon$ and $k > 0$ such that $\varphi_k(\mathbf{x}) < \varepsilon^{2d+1}$ for all $\mathbf{x} \in E$. By Lemma 5.1 we have, for N sufficiently large,

$$\int_E |\widetilde{S}_N^\tau(\mathbf{x})|^2 \,\mathrm{d}\mu(\mathbf{x}) \leqslant \varepsilon^{2d+1} \int_E \frac{|\widetilde{S}_N^\tau(\mathbf{x})|^2}{\varphi_k(\mathbf{x})} \,\mathrm{d}\mu(\mathbf{x}) \leqslant \varepsilon^{2d+1} C.$$

Hence, by (5.1),

$$\begin{aligned}\mu\{\mathbf{x}\colon\ |S_N^\tau(\mathbf{x})| \geqslant 2\varepsilon\} &\leqslant \mu\{\mathbf{x}\colon\ |\widetilde{S}_N^\tau(\mathbf{x})| \geqslant \varepsilon^d\} \\ &\leqslant \varepsilon + \varepsilon^{-2d} \int_E |\widetilde{S}_N^\tau(\mathbf{x})|^2 \,\mathrm{d}\mu(\mathbf{x}) \\ &\leqslant (1+C)\varepsilon,\end{aligned}$$

for N sufficiently large, and part (1) of Theorem 1.1 is proved. $\square$

P r o o f of Theorem 1.1 (2). We have to show that

$$\lim_{N\to\infty} S_N^\tau(\mathbf{x}) = 0 \quad \text{for } \mu\text{-a.e. } \mathbf{x}.$$

The proof given in [8] works in this case again. We include here the main steps for the convenience of the reader.

The measure μ is $\mathcal{D}_q$-conservative with exponential decay. Hence, for every $\varepsilon > 0$, we can find $\eta > 0$ and the set F with $\mu(F) > 1 - \varepsilon/2$, such that

$$\liminf_{k\to\infty} -\frac{1}{k} \log \varphi_k(\mathbf{x}) > \eta \quad \text{for } \mathbf{x} \in F.$$

Hence, there is a set E with $\mu(E) > 1 - \varepsilon$ and $K \in \mathbb{N}$, $K \geqslant 2l$, where l is from Lemma 4.5, such that

$$\varphi_k(\mathbf{x}) < \mathrm{e}^{-k\eta} \quad \text{for } \mathbf{x} \in E \text{ and } k \geqslant K.$$

Using Lemma 5.1, similarly as in the proof of part (1) above, we get

$$\int_E |\widetilde{S}_N(\mathbf{x})|^2 \leqslant C\mathrm{e}^{-k\eta} \quad \text{for } k \geqslant K \text{ and } N \geqslant q^k,$$

and consequently, taking $k = [\log N/\log q]$,

$$\int_E |\widetilde{S}_N(\mathbf{x})|^2 \leqslant Ce^{\eta} N^{-\eta/\log q} \quad \text{for } N \text{ sufficiently large.}$$

This shows that if $m\eta/\log q > 1$ then $\lim\limits_{N\to\infty} \widetilde{S}_{N^m} = 0$ a.e. on E. This implies, in a standard way, that

$$\limsup_{N\to\infty} |S_N(\mathbf{x})| \leqslant \varepsilon \quad \text{for } \mu\text{-a.e. } \mathbf{x} \in E,$$

and the result follows. $\square$

6. Examples

Example 6.1. If $k = \mathbb{Q}$ and $S = \{p_1, \ldots, p_s\} \subset \mathbb{P}$ is a subset of different rational primes then $R_S = \mathbb{Z}[1/a]$, where $a = p_1 \ldots p_s$, and

$$X^d = \mathbb{R}^d \times \mathbb{Q}_{p_1}^d \times \ldots \times \mathbb{Q}_{p_s}^d / B^d,$$

where

$$B^d = \{\underbrace{(b, b, \ldots, b)}_{s \text{ times}}\colon\ b \in \mathbb{Z}[1/a]^d\}.$$

Let $q = q_1^{\alpha_1} \ldots q_m^{\alpha_m} > 1$, where $q_i \in \mathbb{P}$, $\alpha_i \geqslant 1$. In this case X^d is the so called a-adic solenoid (see [1], [7]). The analogue of Theorem 1.1 in this case was proved in [18]. In this particular case condition (iii) of Theorem 1.1 reads $|\det T|_{q_j} = 1$ for $j = 1, \ldots, m$.

Example 6.2. Consider $k = \mathbb{Q}(\sqrt{2})$. Let $P = \{3, 5\} \subset \mathbb{P}$ be a subset of different prime numbers, and set $a = 3 \cdot 5$. Take $q = 7$ and $d = 2$. Consider the set of finite places

$$S = \{v \in \mathcal{P}_f(k)\colon\ \exists p \in P \text{ such that } v \mid p\}.$$

In this case

$$R_S = \mathbb{Z}[1/a] + \mathbb{Z}[1/a]\sqrt{2} = \mathbb{Z}[1/a, \sqrt{2}].$$

Let

$$T = \begin{bmatrix} \frac{\sqrt{2}}{3} & 5 \\ \frac{1-\sqrt{2}}{5} & 3\sqrt{2} \end{bmatrix}.$$

Then the conditions of Theorem 1.1 are satisfied.

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