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# THE $\mathcal{L}_n^m$-PROPOSITIONAL CALCULUS

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*Abstract.* T. Almada and J. Vaz de Carvalho (2001) stated the problem to investigate if these Łukasiewicz algebras are algebras of some logic system. In this article an affirmative answer is given and the $\mathcal{L}_n^m$-propositional calculus, denoted by $\ell_n^m$, is introduced in terms of the binary connectives $\to$ (implication), $\twoheadrightarrow$ (standard implication), $\wedge$ (conjunction), $\vee$ (disjunction) and the unary ones $f$ (negation) and $D_i$, $1 \leqslant i \leqslant n-1$ (generalized Moisil operators). It is proved that $\ell_n^m$ belongs to the class of standard systems of implicative extensional propositional calculi. Besides, it is shown that the definitions of $L_n^m$-algebra and $\ell_n^m$-algebra are equivalent. Finally, the completeness theorem for $\ell_n^m$ is obtained.

*Keywords*: Łukasiewicz algebra of order $n$; $m$-generalized Łukasiewicz algebra of order $n$; equationally definable principal congruences; implicative extensional propositional calculus; completeness theorem

*MSC 2010*: 03G10, 06D99, 03B60

## 1. Introduction and preliminaries

In 1977, generalizing De Morgan algebras by omitting the polarity condition (i.e. the law of double negation), J. Berman [2] began the study of what he called distributive lattices with an additional unary operation. Two years later, A. Urquhart in [11] named them Ockham lattices. These algebras are the algebraic counterpart of logics provided with a negation operator which satisfies De Morgan laws. Then, recall that an Ockham algebra is an algebra $\langle L, \wedge, \vee, f, 0, 1\rangle$, where the reduct $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and $f$ is a unary operation satisfying the following conditions:

(O1) $f0 = 1$,
(O2) $f1 = 0$,
(O3) $f(x \wedge y) = fx \vee fy$,
(O4) $f(x \vee y) = fx \wedge fy$.

Ockham algebras, which are more closely related to De Morgan algebras, are the ones that satisfy the identity $f^{2m}x = x$ for some $m \geqslant 1$. The variety of these algebras will be denoted by $\mathcal{K}_{m,0}$. More details on these algebras can be consulted in [3]. Furthermore, for the notions of universal algebra including De Morgan algebras and $n$-valued Łukasiewicz-Moisil algebras outlined in this paper we refer the reader to [4], [5].

On the other hand, in 2001, T. Almada and J. Vaz de Carvalho [1] generalized Łukasiewicz-Moisil algebras of order $n$ by considering algebras of the same type which have a reduct in $\mathcal{K}_{m,0}$ instead of a reduct which is a De Morgan algebra. Hence, they introduced the variety $\mathcal{L}_n^m$ of $m$-generalized Łukasiewicz algebras of order $n$ which were defined as follows:

An $m$-generalized Łukasiewicz algebra of order $n$ (or $L_n^m$-algebra) is an algebra $\langle A, \vee, \wedge, f, D_1, \ldots, D_{n-1}, 0, 1\rangle$ of type $(2, 2, 1, \ldots, 1, 0, 0)$ such that

(GL$_1$) $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice for which $f$ is a dual endomorphism satisfying the identity $f^{2m}x = x$,

(GL$_2$) $D_i\Big(x \wedge \bigvee\limits_{p=0}^{m-1} f^{2p}y\Big) = D_ix \wedge D_i\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}y\Big)$, $1 \leqslant i \leqslant n-1$,

(GL$_3$) $D_ix \wedge D_jx = D_jx$, $1 \leqslant i \leqslant j \leqslant n-1$,

(GL$_4$) $D_ix \vee f\, D_ix = 1$, $1 \leqslant i \leqslant n-1$,

(GL$_5$) $D_if\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big) = fD_{n-i}\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big)$, $1 \leqslant i \leqslant n-1$,

(GL$_6$) $D_iD_jx = D_jx$, $1 \leqslant i, j \leqslant n-1$,

(GL$_7$) $x \vee D_1x = D_1x$,

(GL$_8$) $D_ix = D_i\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big)$, $1 \leqslant i \leqslant n-1$,

(GL$_9$) $(x \wedge fx) \vee y \vee fy = y \vee fy$,

(GL$_{10}$) $\bigvee\limits_{p=0}^{m-1} f^{2p}x \leqslant \bigvee\limits_{p=0}^{m-1} f^{2p}y \vee fD_i\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}y\Big) \vee D_{i+1}\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big)$, $1 \leqslant i \leqslant n-2$.

From the definition it follows that the identities listed below are also verified.

**Proposition 1.1** ([1]). *Let $A \in \mathcal{L}_n^m$. Then*

(GL$_{11}$) $D_i(x \vee y) = D_ix \vee D_iy$, $1 \leqslant i \leqslant n-1$,

(GL$_{12}$) $f^2D_ix = D_ix$, $1 \leqslant i \leqslant n-1$,

(GL$_{13}$) $D_ix \wedge fD_ix = 0$, $1 \leqslant i \leqslant n-1$,

(GL$_{14}$) $fx \vee D_1x = 1$, $f\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big) \wedge D_{n-1}\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big) = 0$,

(GL$_{15}$) $\bigvee\limits_{p=0}^{m-1} f^{2p}x \wedge D_{n-1}\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big) = D_{n-1}\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big)$,

(GL$_{16}$) $D_i0 = 0$, $D_i1 = 1$, $1 \leqslant i \leqslant n-1$.

Let $A \in \mathcal{L}_n^m$. The set $S_A = \{x \in A\colon\ f^2x = x\} = \left\{x \in A\colon \bigvee\limits_{p=0}^{m-1} f^{2p}x = x\right\}$ plays an important role in the study of these algebras. In particular, as a direct consequence of $(\mathrm{GL}_8)$ it follows that in $L_n^m$-algebras the operations $D_i$, $1 \leqslant i \leqslant n-1$ are determined by its restrictions to $S_A$. Besides, $S_A$ is a subalgebra of $A$ and it is the greatest subalgebra of $A$ that belongs to the variety of $L_n$-algebras ([1], Proposition 2.2).

In addition to the properties $(\mathrm{GL}_{11})$ through $(\mathrm{GL}_{16})$, we show other ones that will be useful throughout this paper.

**Proposition 1.2** ([7]). *Let $A \in \mathcal{L}_n^m$. Then the following properties are verified:*

$(\mathrm{g}_1)$ $D_j f D_i x = f D_i x$, $1 \leqslant i,\ j \leqslant n-1$,
$(\mathrm{g}_2)$ $f D_i x$ *is the Boolean complement of* $D_i x$, $1 \leqslant i \leqslant n-1$,
$(\mathrm{g}_3)$ $D_i x \leqslant D_i y$ *if and only if* $f D_i x \vee D_i y = 1$, $1 \leqslant i \leqslant n-1$,
$(\mathrm{g}_4)$ $D_j(D_i x \wedge D_i y) = D_i x \wedge D_i y$, $1 \leqslant i, j \leqslant n-1$,
$(\mathrm{g}_5)$ $x \wedge f D_1 x = 0$,
$(\mathrm{g}_6)$ $(f D_i x \wedge f D_i y) \vee (D_i x \wedge D_i y) = (D_i x \vee f D_i y) \wedge (D_i y \vee f D_i x)$, $1 \leqslant i \leqslant n-1$,
$(\mathrm{g}_7)$ $z \in S_A$ *implies* $D_i(x \wedge z) = D_i x \wedge D_i z$, $1 \leqslant i \leqslant n-1$.

Bearing in mind some unpublished results established by M. Sequeira in the context of congruences on algebras of certain subvarieties of Ockham algebras some of which are $\mathcal{K}_{m,0}$, J. Vaz de Carvalho considered certain elements which we will describe in what follows.

Let $A \in \mathcal{L}_n^m$ and $T = \{0, 1, \ldots, m-1\}$. For each $z \in A$ and $s \in \{1, \ldots, m\}$ take

$$q_s z = \bigwedge_{\substack{J \subseteq T \\ |J| = s}} \bigvee_{j \in J} f^{2j} z.$$

The same author asserted that it is straightforward to see the following statements.

**Lemma 1.1** ([12]). *Let $A \in \mathcal{L}_n^m$. Then*

(i) $f^2 q_s z = q_s z$, $s \in \{1, \ldots, m\}$,
(ii) $q_s z \leqslant q_{s+1} z$, $s \in \{1, \ldots, m-1\}$,
(iii) $q_1 z = \bigwedge\limits_{p=0}^{m-1} f^{2p} z$ *and* $\bigvee\limits_{p=0}^{m-1} f^{2p} z = q_m z$,
(iv) $z \in S_A$ *implies* $q_s z = z$, $s \in \{1, \ldots, m\}$,
(v) $x \leqslant z$ *implies* $q_s x \leqslant q_s z$, $s \in \{1, \ldots, m\}$.

On the other hand, in [7], we introduced a new binary operation $\to$ on $L_n^m$-algebras, called *weak implication*, as follows:

$$x \to y = D_1 f x \vee y.$$

The deductive systems associated with this implication enable us to establish an isomorphism between the congruence lattice of an $m$-generalized Łukasiewicz algebra $A$ of order $n$ and the lattice of all the deductive systems of $A$. This result turns out to be quite useful for characterizing the principal congruences on these algebras. Furthermore, it is worth noting that from this operation the one considered by R. Cignoli [6] for $L_n$-algebras is deduced.

**Proposition 1.3.** *Let $A \in \mathcal{L}_n^m$. Then the following statements hold:*

(W$_1$) $x \to 1 = 1$,

(W$_2$) $x \to x = 1$,

(W$_3$) $1 \to x = x$,

(W$_4$) $x \to (y \to x) = 1$,

(W$_5$) $x \leqslant y$ *implies* $x \to y = 1$,

(W$_6$) $x \to (y \to z) = (x \to y) \to (x \to z)$,

(W$_7$) $x \to (x \wedge y) = x \to y$,

(W$_8$) $(x \to y) \to ((x \to z) \to (x \to (y \wedge z))) = 1$,

(W$_9$) $(x \wedge y) \to z = x \to (y \to z)$,

(W$_{10}$) $D_i x \to D_i y = f D_i x \vee D_i y$, $1 \leqslant i \leqslant n-1$,

(W$_{11}$) $D_i x \to D_i y = 1$ *if and only if* $D_i x \leqslant D_i y$, $1 \leqslant i \leqslant n-1$,

(W$_{12}$) $D_i q_s(x \vee y) \to D_i q_s(x \wedge y) = 1$ *if and only if* $x = y$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(W$_{13}$) $((x \wedge z) \to (y \wedge z)) \to (z \to (x \to y)) = 1$,

(W$_{14}$) $D_i q_s x \to D_1 x = 1$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(W$_{15}$) $D_i q_s(x \wedge fx) \to D_i q_s((x \wedge fx) \wedge (y \vee fy)) = 1$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(W$_{16}$) $D_i q_s\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big) \to D_i q_s\Big(\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big) \wedge \Big(\bigvee\limits_{p=0}^{m-1} f^{2p}y \vee f D_i\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}y\Big) \vee D_{i+1}\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}x\Big)\Big)\Big) = 1$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(W$_{17}$) $D_i q_s x \to D_i q_s(x \wedge f^{2m-1}(fx \wedge fy)) = 1$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(W$_{18}$) $D_i q_s(x \wedge f^{2m-1}(fy \wedge fz) \vee f^{2m-1}(f(z \wedge x) \wedge f(y \wedge x))) \to D_i q_s(x \wedge f^{2m-1}(fy \wedge fz) \wedge f^{2m-1}(f(z \wedge x) \wedge f(y \wedge x))) = 1$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(W$_{19}$) $x \wedge \bigwedge\limits_{s=1}^{m} \bigwedge\limits_{i=1}^{n-1} (D_i q_s(x \vee y) \to D_i q_s(x \wedge y)) = y \wedge \bigwedge\limits_{s=1}^{m} \bigwedge\limits_{i=1}^{n-1} (D_i q_s(x \vee y) \to D_i q_s(x \wedge y))$,

(W$_{20}$) $D_j q_k x \wedge \bigwedge\limits_{s=1}^{m} \bigwedge\limits_{i=1}^{n-1} (D_i q_s(x \vee y) \to D_i q_s(x \wedge y)) = D_j q_k y \wedge \bigwedge\limits_{s=1}^{m} \bigwedge\limits_{i=1}^{n-1} (D_i q_s(x \vee y) \to D_i q_s(x \wedge y))$, $1 \leqslant j \leqslant n-1$, $1 \leqslant k \leqslant m$,

(W$_{21}$) $D_{n-1} q_1 x \wedge \bigwedge\limits_{s=1}^{m} \bigwedge\limits_{i=1}^{n-1} (D_i q_s(x \vee y) \to D_i q_s(x \wedge y)) = D_{n-1} q_1 y \wedge \bigwedge\limits_{s=1}^{m} \bigwedge\limits_{i=1}^{n-1} (D_i q_s(x \vee y) \to D_i q_s(x \wedge y))$.

P r o o f. We will only prove ($W_{12}$), ($W_{13}$), ($W_{14}$), ($W_{15}$), ($W_{18}$), ($W_{19}$) and ($W_{20}$) since the proof of the remaining properties is routine.

($W_{12}$): It is a direct consequence of [12], Proposition 4.2.

($W_{13}$): From ($W_9$) and ($W_6$) we have that $((x \wedge z) \to (y \wedge z)) \to (z \to (x \to y)) = ((x \wedge z) \to (y \wedge z)) \to ((x \wedge z) \to y)$. Hence, by ($W_5$) and ($W_1$) we conclude that $((x \wedge z) \to (y \wedge z)) \to (z \to (x \to y)) = (x \wedge z) \to 1 = 1$.

($W_{14}$): From (ii) in Lemma 1.1 and ($GL_{11}$) we infer that $D_i q_s x \leqslant D_i q_m x$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$. On the other hand, by (iii) in Lemma 1.1, ($GL_8$) and ($GL_3$) we have that $D_i q_m x = D_i x \leqslant D_1 x$, $1 \leqslant i \leqslant n-1$ and so, by ($W_{11}$) we conclude that $D_i q_s x \to D_1 x = 1$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$.

($W_{15}$): From ($GL_9$) we have that $x \wedge fx = (x \wedge fx) \wedge (y \vee fy)$ and so, $D_i q_s(x \wedge fx) = D_i q_s((x \wedge fx) \wedge (y \vee fy))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$. Hence, by ($W_2$) we conclude the proof.

($W_{18}$): It is a direct consequence of the fact that $x \wedge f^{2m-1}(fy \wedge fz) = f^{2m-1}(f(z \wedge x) \wedge f(y \wedge x))$ and ($W_2$).

($W_{19}$): By virtue of ($g_1$) and the definition of the weak implication we have that $D_i q_s(x \wedge y) \vee f D_i q_s(x \vee y) = D_i q_s(x \wedge y) \vee D_1 f D_i q_s(x \vee y) = D_i q_s(x \vee y) \to D_i q_s(x \wedge y)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$ and so, by [12], Proposition 3.5, we conclude that $x \wedge \bigwedge_{s=1}^{m} \bigwedge_{i=1}^{n-1} (D_i q_s(x \vee y) \to D_i q_s(x \wedge y)) = y \wedge \bigwedge_{s=1}^{m} \bigwedge_{i=1}^{n-1} (D_i q_s(x \vee y) \to D_i q_s(x \wedge y))$.

($W_{20}$): Following reasoning analogous to that in ($W_{19}$) we obtain the proof. $\square$

Next, in order to simplify reading we will summarize the fundamental concepts we use on the class of standard systems of implicative extensional propositional calculi ([9], VIII).

Let $\mathcal{L} = (A^0, F)$ be a formalized language of zero order ([9], VIII 1). A system $\mathcal{S} = (\mathcal{L}, C_{\mathcal{L}})$, where $C_{\mathcal{L}}$ is determined by a set $\mathcal{A}$ of logical axioms and by a set $\{r_1, \ldots, r_k\}$ of rules of inference, belongs to the class **S** of standard systems of implicative extensional propositional calculi provided that the following conditions are satisfied:

(s1) the set $\mathcal{A}$ of logical axioms is closed under substitutions,

(s2) the rules of inference $r_i$, $i = 1, \ldots, k$, are invariant under substitutions,

(s3) for every formula $\alpha \in F$, $\alpha \Rightarrow \alpha \in C_{\mathcal{L}}(\emptyset)$,

(s4) for all formulas $\alpha, \beta \in F$ and for every set $H \subseteq F$, if $\alpha, \alpha \Rightarrow \beta \in C_{\mathcal{L}}(H)$, then $\beta \in C_{\mathcal{L}}(H)$,

(s5) for all formulas $\alpha, \beta, \gamma \in F$ and for every set $H \subseteq F$, if $\alpha \Rightarrow \beta$, $\beta \Rightarrow \gamma \in C_{\mathcal{L}}(H)$, then $\alpha \Rightarrow \gamma \in C_{\mathcal{L}}(H)$,

(s6) for every formula $\alpha \in F$ and for every set $H \subseteq F$ the condition $\alpha \in C_{\mathcal{L}}(H)$ implies that for every formula $\beta \in F$, $\beta \Rightarrow \alpha \in C_{\mathcal{L}}(H)$, (s7) for all formulas $\alpha, \beta \in F$ and for every set $H \subseteq F$ the condition $\alpha \Rightarrow \beta$, $\beta \Rightarrow \alpha \in C_{\mathcal{L}}(H)$ implies that for each unary connective $\circ$ of $\mathcal{L}$, $\circ\alpha \Rightarrow \circ\beta \in C_{\mathcal{L}}(H)$,

(s8) for all formulas $\alpha, \beta, \gamma, \delta \in F$ and for every set $H \subseteq F$ the condition $\alpha \Rightarrow \beta$, $\beta \Rightarrow \alpha$, $\gamma \Rightarrow \delta$, $\delta \Rightarrow \gamma \in C_{\mathcal{L}}(H)$ implies that for each binary connective $*$ of $\mathcal{L}$, $(\alpha * \gamma) \Rightarrow (\beta * \delta) \in C_{\mathcal{L}}(H)$.

If $\mathcal{S}$ is a system in $\mathbf{S}$ and there exists a formula $\alpha$ of $\mathcal{L}$ such that $\alpha \notin C_{\mathcal{L}}(\emptyset)$ we will say that $\mathcal{S}$ is consistent.

On the other hand, any system $\mathcal{S} \in \mathbf{S}$ determines a class of algebras called $\mathcal{S}$-algebras in the following way: an algebra $\mathcal{U} = \langle A, \Rightarrow, *_1, \ldots, *_k, o_1, \ldots, o_t, e_1, \ldots, e_m, \vee \rangle$ associated with the formalized language $\mathcal{L}$ ([9], VIII 1) is an $\mathcal{S}$-algebra provided that

(a1) if a formula $\alpha$ of $\mathcal{L}$ belongs to the set $\mathcal{A}$ of logical axioms of $\mathcal{S}$, then $v(\alpha) = \vee$ for every valuation $v$ of $\mathcal{L}$ in $\mathcal{U}$,

(a2) if a rule of inference $r$ in $\mathcal{S}$ assigns to the premises $\alpha_1, \ldots, \alpha_n$ the conclusion $\beta$, then for every valuation $v$ of $\mathcal{L}$ in $\mathcal{U}$ the condition $v(\alpha_1) = \ldots = v(\alpha_n) = \vee$ implies $v(\beta) = \vee$,

(a3) for all $a, b, c \in A$, if $a \Rightarrow b = \vee$ and $b \Rightarrow c = \vee$, then $a \Rightarrow c = \vee$,

(a4) for all $a, b \in A$, if $a \Rightarrow b = \vee$ and $b \Rightarrow a = \vee$, then $a = b$.

Let $\mathcal{S} = (\mathcal{L}, C_{\mathcal{L}})$ be a consistent system in $\mathbf{S}$. A formula $\alpha \in \mathcal{L}$ is valid in an algebra $\mathcal{U}$ associated with $\mathcal{L}$ provided that $v(\alpha) = \vee$ for every valuation $v$ of $\mathcal{L}$ in $\mathcal{U}$. Furthermore, $\alpha$ is $\mathcal{S}$-valid if it is valid in every $\mathcal{S}$-algebra. Taking into account that if $\alpha$ is derivable in $\mathcal{S}$ ([9], VIII 5), then $v(\alpha) = \vee$ for every valuation $v$ of $\mathcal{L}$ in every $\mathcal{S}$-algebra $\mathcal{U}$ ([9], VIII 6.1), every formula derivable in $\mathcal{S}$ is $\mathcal{S}$-valid. The converse statement is also true and this equivalence is known as the completeness theorem for propositional calculi in the class $\mathbf{S}$ ([9], VIII 7.2).

## 2. The standard implication

In order to establish an implicative extensional propositional calculus (see [9]) which has $\mathcal{L}_n^m$-algebras as the algebraic counterpart, we introduce another implication operation $\twoheadrightarrow$ on these algebras by means of the formula

$$x \twoheadrightarrow y = D_{n-1}q_1 y \vee \bigwedge_{s=1}^{m} \bigwedge_{i=1}^{n-1} (D_i q_s(x \vee y) \to D_i q_s(x \wedge y))$$

and we call it *standard implication*. Furthermore, this implication allows us to obtain a new description of the congruence lattice $\mathrm{Con}(A)$ of an $\mathcal{L}_n^m$-algebra $A$ which plays an important role in what follows.

**Proposition 2.1.** *Let $A \in \mathcal{L}_n^m$. Then the following statements hold:*

(S1) $x \twoheadrightarrow 1 = 1$,

(S2) $x \twoheadrightarrow x = 1$,

(S3) $1 \twoheadrightarrow x = D_{n-1}q_1x$,

(S4) $D_{n-1}q_1x \wedge (x \twoheadrightarrow y) = D_{n-1}q_1y \wedge (y \twoheadrightarrow x)$,

(S5) $x \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) = y \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x)$,

(S6) $(x \twoheadrightarrow y) \rightarrow ((y \twoheadrightarrow z) \rightarrow (x \twoheadrightarrow z)) = 1$,

(S7) $(D_{n-1}q_1x \wedge (x \twoheadrightarrow y)) \rightarrow y = 1$,

(S8) $f^2(x \twoheadrightarrow y) = x \twoheadrightarrow y$,

(S9) $D_i(x \twoheadrightarrow y) = x \twoheadrightarrow y$, $1 \leqslant i \leqslant n-1$.

P r o o f. We will only prove (S4), (S5), (S6) and (S9), since the proof of the others is straightforward.

(S4): Taking into account the definition of the standard implication and $(\mathrm{W}_{21})$ we have that $D_{n-1}q_1x \wedge (x \twoheadrightarrow y) = (D_{n-1}q_1x \wedge D_{n-1}q_1y) \vee \Big(D_{n-1}q_1x \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big) = (D_{n-1}q_1x \wedge D_{n-1}q_1y) \vee \Big(D_{n-1}q_1y \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big) = D_{n-1}q_1y \wedge (y \twoheadrightarrow x)$.

(S5): Taking into account (i) and (iii) in Lemma 1.1 we infer that $D_{n-1}q_1x \leqslant q_1x \leqslant x$ and so we have that $x \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) = x \wedge \Big(D_{n-1}q_1y \vee \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big) \wedge \Big(D_{n-1}q_1x \vee \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big) = \Big(D_{n-1}q_1y \vee \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big) \wedge \Big((x \wedge D_{n-1}q_1x) \vee \Big(x \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big)\Big) = \Big(D_{n-1}q_1y \vee \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big) \wedge \Big(D_{n-1}q_1x \vee \Big(x \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big)\Big) = (D_{n-1}q_1x \wedge D_{n-1}q_1y) \vee \Big(D_{n-1}q_1y \wedge x \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big) \vee \Big(D_{n-1}q_1x \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big) \vee \Big(x \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))\Big)$.

Analogously, we have that $y \wedge ((x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x)) = (D_{n-1}q_1y \wedge D_{n-1}q_1x) \vee \Big(D_{n-1}q_1x \wedge y \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y)) \vee \Big(D_{n-1}q_1y \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y)) \vee \Big(y \wedge \bigwedge\limits_{s=1}^{m}\bigwedge\limits_{i=1}^{n-1}(D_iq_s(x \vee y) \rightarrow D_iq_s(x \wedge y))$. Hence, taking into account $(\mathrm{W}_{19})$ and $(\mathrm{W}_{21})$ we infer that $x \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) = y \wedge ((x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x))$.

(S6): Let $A$ be a subdirectly irreducible $L_n^m$-algebra, then by ([1], Proposition 4.1) we have that the set of Boolean elements of $S_A$ is $\{0,1\}$. Hence, by (i) in Lemma 1.1 and ($g_2$) we have that $D_i q_s(a \vee b) \to D_i q_s(a \wedge b) \in \{0,1\}$ for all $a, b \in A$. Suppose now that there are $x, y \in A$ such that $D_i q_s(x \vee y) \to D_i q_s(x \wedge y) = 1$. Hence, by ($W_{12}$) it follows that $x = y$ and so, by ($W_2$) we have that $(x \twoheadrightarrow y) \to ((y \twoheadrightarrow z) \to (x \twoheadrightarrow z)) = (x \twoheadrightarrow x) \to ((x \twoheadrightarrow z) \to (x \twoheadrightarrow z)) = 1$. On the other hand, if we suppose that there are $x, y, z \in A$ such that $D_i q_s(y \vee z) \to D_i q_s(y \wedge z) = 1$ or $D_i q_s(x \vee z) \to D_i q_s(x \wedge z) = 1$, following an analogously reasoning we prove (S6). Finally, if $D_i q_s(x \vee y) \to D_i q_s(x \wedge y) = D_i q_s(y \vee z) \to D_i q_s(y \wedge z) = D_i q_s(x \vee z) \to D_i q_s(x \wedge z) = 0$, then $y \twoheadrightarrow z = D_{n-1} q_1 z = x \twoheadrightarrow z$ and so, by ($W_2$) and ($W_1$) we conclude the proof.

(S9): It follows as a consequence of ($g_1$), ($GL_{11}$), ($GL_{12}$), ($g_7$) and ($GL_6$). $\square$

For any $A \in \mathcal{L}_n^m$ we will denote by $\mathcal{D}(A)$ the set of all deductive systems of $A$ associated with $\to$, which are defined as usual ([7]).

**Lemma 2.1.** *Let $A \in \mathcal{L}_n^m$ and $F \in \mathcal{D}(A)$. Then the following conditions are equivalent for all $x, y \in A$:*

(i) *there is* $u \in F$ *such that* $D_{n-1} u \to f x = D_{n-1} u \to f y$,

(ii) *there is* $w \in F$ *such that* $x \wedge D_{n-1} w = y \wedge D_{n-1} w$,

(iii) $x \twoheadrightarrow y, y \twoheadrightarrow x \in F$.

Proof. Taking into account [7], Remark 2.11, we will only prove the equivalence between (ii) and (iii).

(ii) ⇒ (iii): From the hypothesis and [7], Theorem 2.14, we have that $(x, y) \in R_F = \{(a, b) \in A^2 \colon$ there is $w \in F$ such that $a \wedge D_{n-1} w = b \wedge D_{n-1} w\}$ and so, $(D_i q_s(x \vee y), D_i q_s(x \wedge y)) \in R_F$. Hence, $(D_i q_s(x \vee y) \to D_i q_s(x \wedge y), 1) \in R_F$ which implies that $\left(D_{n-1} q_1 y \vee \bigwedge\limits_{i=1}^{n-1} \bigwedge\limits_{s=1}^{m} (D_i q_s(x \vee y) \to q_s(x \wedge y)), 1\right) \in R_F$. Therefore, $x \twoheadrightarrow y \in F$. Similarly, we get that $y \twoheadrightarrow x \in F$.

(iii) ⇒ (ii): From the hypothesis and (S8) we have that $w = (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) \in F \cap S_A$ and taking into account that $S_A$ is an $L_n$-algebra we have that $D_{n-1} w \leqslant w$. Hence, by (S5) we conclude that $x \wedge D_{n-1} w = y \wedge D_{n-1} w$. $\square$

From now on, for any $A \in \mathcal{L}_n^m$ we will denote by $A/R$ the quotient algebra of $A$ by $R$ for any $R \in \mathrm{Con}(A)$. Besides, for $x \in A$ the equivalence class of $x$ modulo $R$ will be denoted by $[x]_R$.

**Theorem 2.1.** *Let $A \in \mathcal{L}_n^m$. Then the following statements hold:*

(i) $\mathrm{Con}(A) = \{R(F)\colon F \in \mathcal{D}(A)\}$ *where* $R(F) = \{(x,y) \in A^2\colon x \twoheadrightarrow y,\ y \twoheadrightarrow x \in F\}$,

(ii) *the lattices* $\mathrm{Con}(A)$ *and* $\mathcal{D}(A)$ *are isomorphic considering the applications* $\theta \longmapsto [1]_\theta$ *and* $F \longmapsto R(F)$ *which are inverse to each other.*

Proof. It is a direct consequence of Lemma 2.1 and [7], Theorem 2.14. $\square$

Let $A \in \mathcal{L}_n^m$ and $z \in A$. We will denote by $[z)$ the principal filter of $A$ generated by $z$ (i.e., $[z) = \{x \in A\colon z \leqslant x\}$).

**Lemma 2.2.** *Let $A \in \mathcal{L}_n^m$ and $a, b \in A$. If $w = (a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a)$, then $[w)$ is a deductive system of $A$.*

Proof. Taking into account [7], Proposition 2.6, it only remains to prove that $fD_{n-1}fx \in [w)$ for all $x \in [w)$. By (S8), (GL$_5$), (g$_7$) and (S9) we have that $fD_{n-1}fw = fD_{n-1}f((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a)) = f^2D_1((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a)) = f^2(D_1(a \twoheadrightarrow b) \wedge D_1(b \twoheadrightarrow a)) = f^2((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a)) = w$. From this assertion the proof is straightforward. $\square$

Taking into account the above results we obtain a characterization of the principal congruences on $L_n^m$-algebras. For any $A \in \mathcal{L}_n^m$ and $a, b \in A$ we will denote by $\theta(a,b)$ the principal congruence of $A$ generated by $(a,b)$.

**Theorem 2.2.** *Let $A \in \mathcal{L}_n^m$ and $a, b \in A$. Then $\theta(a,b) = \{(x,y) \in A^2\colon x \wedge ((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a)) = y \wedge ((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a))\}$.*

Proof. Let $T = \{(x,y) \in A^2\colon x \wedge ((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a)) = y \wedge ((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a))\}$. By (S5) we have that $(a,b) \in T$. Besides, by (S9) and (S8) it follows that $T = \{(x,y) \in A^2\colon x \wedge D_{n-1}((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a)) = y \wedge D_{n-1}((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a))\}$ and so, by Lemma 2.2, Lemma 2.1 and [7], Theorem 2.14, we conclude that $T \in \mathrm{Con}(A)$.

On the other hand, let $R \in \mathrm{Con}(A)$ be such that $(a,b) \in R$ and suppose that $(x,y) \in T$. Hence, we have that $((a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a), 1) \in R$ and so, $(x \wedge (a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a), x) \in R$ and $(y \wedge (a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a), y) \in R$. From these last assertions and the fact that $(x,y) \in T$ we conclude that $(x,y) \in R$. Therefore, $T = \theta(a,b)$. $\square$

Example 2.1. Let us consider the $L_3^2$-algebra $A$ shown in Figure 1, where the operations $f$, $D_i$, $1 \leqslant i \leqslant 2$ and $q_i$, $1 \leqslant i \leqslant 2$ are defined as follows:

If $w = (a \twoheadrightarrow b) \wedge (b \twoheadrightarrow a) = h$, by Lemma 2.2 we have that $F = [h) = \{h, i, j, k, m, n, 1\}$ is a deductive system of $A$. Hence, by Theorem 2.1 we have that $A/R(F) = \{[0]_{R(F)}, [1]_{R(F)}\}$ where $[1]_{R(F)} = F$ and $[0]_{R(F)} = \{0, a, b, c, d, e, g\}$.

| $x$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $g$ |
|---|---|---|---|---|---|---|---|
| $fx$ | 1 | $n$ | $m$ | $k$ | $i$ | $j$ | $h$ |
| $D_1x$ | 0 | $g$ | $g$ | $g$ | $g$ | $g$ | $g$ |
| $D_2x$ | 0 | 0 | 0 | 0 | $g$ | $g$ | $g$ |
| $q_1x$ | 0 | 0 | 0 | $c$ | $c$ | $c$ | $g$ |
| $q_2x$ | 0 | $c$ | $c$ | $c$ | $g$ | $g$ | $g$ |

| $x$ | $h$ | $i$ | $j$ | $k$ | $m$ | $n$ | 1 |
|---|---|---|---|---|---|---|---|
| $fx$ | $g$ | $e$ | $d$ | $c$ | $a$ | $b$ | 0 |
| $D_1x$ | $h$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $D_2x$ | $h$ | $h$ | $h$ | $h$ | 1 | 1 | 1 |
| $q_1x$ | $h$ | $h$ | $h$ | $k$ | $k$ | $k$ | 1 |
| $q_2x$ | $h$ | $k$ | $k$ | $k$ | 1 | 1 | 1 |

On the other hand, by (S1) and (S3) we have that $g \twoheadrightarrow 1 = 1$ and $1 \twoheadrightarrow g = g$. Then, taking into account Theorem 2.2 we obtain that $\theta(g, 1) = \{(x, y) \in A^2\colon\ x \wedge g = y \wedge g\} = \mathrm{Id}_A \cup \{(g, 1), (1, g), (d, m), (m, d), (n, e), (e, n), (c, k), (k, c), (a, i), (i, a), (b, j), (j, b), (0, h), (h, 0)\}$.

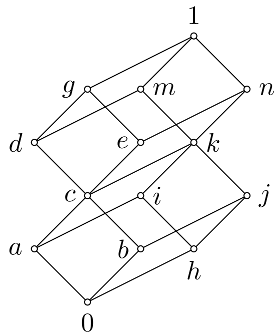


Figure 1.

From Theorem 2.2 it is easy to verify Proposition 2.2, which will be quite useful in the development of the $\mathcal{L}_n^m$-propositional calculus.

**Proposition 2.2.** *Let $A \in \mathcal{L}_n^m$. Then the following statements hold:*

(S10) $D_i x \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) = D_i y \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x)$, $1 \leqslant i \leqslant n-1$,

(S11) $D_i q_s(fx \vee fy) \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) = D_i q_s(fx \wedge fy) \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(S12) $D_i q_s((x \wedge z) \vee (y \wedge z)) \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) = D_i q_s((x \wedge z) \wedge (y \wedge z)) \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(S13) $D_i q_s((x \to z) \vee (y \to z)) \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) = D_i q_s((x \to z) \wedge (y \to z)) \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(S14) $D_i q_s((x \twoheadrightarrow z) \vee (y \twoheadrightarrow z)) \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x) = D_i q_s((x \twoheadrightarrow z) \wedge (y \twoheadrightarrow z)) \wedge (x \twoheadrightarrow y) \wedge (y \twoheadrightarrow x)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$.

P r o o f. It is routine. □

## 3. A CHARACTERIZATION OF $k_{2m}$-LATTICES

The goal of this section is to find an equivalent formulation to $(\mathrm{GL}_1)$ with a simpler proof than the previous one. To this end, we take into account the results established in [8].

**Definition 3.1.** A $k_{2m}$-lattice, $m \in \mathbb{N}$, is an algebra $\langle A, \vee, \wedge, f\rangle$ such that $\langle A, \vee, \wedge\rangle$ is a distributive lattice and $f$ is a unary operation on $A$ verifying the following conditions:

(r1) $f^{2m}x = x$,
(r2) $f(x \vee y) = fx \wedge fy$.

Theorem 3.2 enables us to characterize $k_{2m}$-lattices by means of the operations of infimum $\wedge$ and the dual endomorphism $f$. This characterization results easier by the use of Sholander's characterization of distributive lattices as follows:

**Theorem 3.1** ([10]). *An algebra $\langle A, \wedge, \vee\rangle$ of type $(2, 2)$ is a distributive lattice if and only if it verifies the conditions*

(l1) $a = a \wedge (a \vee b)$,
(l2) $a \wedge (b \vee c) = (c \wedge a) \vee (b \wedge a)$.

**Theorem 3.2.** *Let $\langle A, \wedge, f\rangle$ be an algebra of type $(2, 1)$. Define $(s)$: $a \vee b = f^{2m-1}(fa \wedge fb)$ for all $a, b \in A$. Then $\langle A, \wedge, \vee, f\rangle$ is a $k_{2m}$-lattice, $m \in \mathbb{N}$, if and only if the following conditions are verified:*

(m1) $a = a \wedge f^{2m-1}(fa \wedge fb)$,
(m2) $a \wedge f^{2m-1}(fb \wedge fc) = f^{2m-1}(f(c \wedge a) \wedge f(b \wedge a))$.

P r o o f. From (l1), (l2) and taking into account the definition of $\vee$ we have that (m1) and (m2) immediately follow. In order to prove the converse we will first show that $A$ is a distributive lattice, which is a consequence of the fact that (l1) and (l2) hold. Indeed, from (m1), (m2) and (s) we have (l1): $a \wedge (a \vee b) = f^{2m-1}(fa \wedge fb) \wedge a = a$ and (l2): $(c \wedge a) \vee (b \wedge a) = f^{2m-1}(f(c \wedge a) \wedge f(b \wedge a)) = a \wedge f^{2m-1}(fb \wedge fc) = a \wedge (b \vee c)$. Hence, by (m1) and (m2) we obtain (r1): $a = a \wedge f^{2m-1}(fa \wedge fa) = f^{2m-1}(f(a \wedge a) \wedge f(a \wedge a)) = f^{2m-1}(fa \wedge fa) = f^{2m-1}fa$. Finally, from (r1) and (s) we get (r2): $f(a \vee b) = ff^{2m-1}(fa \wedge fb) = fa \wedge fb$. □

## 4. The $\mathcal{L}_n^m$-propositional calculus

In this section, which is the core of this paper, we describe a propositional calculus and show that it has $L_n^m$-algebras as the algebraic counterpart. We are interested in finding a calculus which belongs to the class of standard systems of implicative propositional calculi. The complexity of the standard implication together with the fact that $L_n^m$-algebras do not verify Moisil's determination principle and that the operators $D_i$ are not $\wedge$-homomorphisms have made that in this calculus the number of axioms and inference rules are greater than in $n$-valued Łukasiewicz propositional calculus ([4]). The terminology and symbols used here coincide with those used in [9].

Let $\mathcal{L} = (A^0, F)$ be a formalized language of zero order where in the alphabet $A^0 = (V, L_0, L_1, L_2, U)$ the set

(i) $V$ of propositional variables is countable;
(ii) $L_0$ is empty;
(iii) $L_1$ contains $n$ elements denoted by $f$, $D_i$ for $1 \leqslant i \leqslant n-1$, called negation sign and generalized Moisil operators signs, respectively;
(iv) $L_2$ contains four elements denoted by $\wedge$, $\vee$, $\to$ and $\twoheadrightarrow$ called conjunction sign, disjunction sign, weak implication sign and standard implication sign, respectively;
(v) $U$ contains two elements denoted by ( , ).

In what follows, for any $\alpha_1, \ldots, \alpha_k$ in the set $F$ of all formulas over $A^0$, $\bigvee\limits_{p=0}^{k} \alpha_p$, $\bigwedge\limits_{p=0}^{k} \alpha_p$ will mean $\alpha_0 \vee (\ldots \vee (\alpha_{k-1} \vee \alpha_k) \ldots)$ and $\alpha_0 \wedge (\ldots \wedge (\alpha_{k-1} \wedge \alpha_k) \ldots)$, respectively. Besides, for any $\alpha$ in $F$, $f^t\alpha$ is the result of applying $f$ $t$ times to $\alpha$ if $t > 0$, or $\alpha$ if $t = 0$. Furthermore, for any $\alpha$, $\beta$ in $F$, we will write for brevity $\alpha \leftrightarrow \beta$, $\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta$ and $q_s\alpha$ instead of $(\alpha \to \beta) \wedge (\beta \to \alpha)$, $(\alpha \twoheadrightarrow \beta) \wedge (\beta \twoheadrightarrow \alpha)$ and $\bigwedge\limits_{J \subseteq T, |J| = s} \bigvee\limits_{j \in J} f^{2j}\alpha$, where $T = \{0, 1, \ldots, m-1\}$ and $s \in \{1, \ldots, m\}$, respectively.

We assume that the set $\mathcal{A}_l$ of logical axioms consists of all formulas of the following form, where $\alpha$, $\beta$, $\gamma$ are any formulas in $F$:

(A1) $\alpha \to (\beta \to \alpha)$,
(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$,
(A3) $\alpha \to (\alpha \vee \beta)$,
(A4) $\beta \to (\alpha \vee \beta)$,
(A5) $(\alpha \wedge \beta) \to \alpha$,
(A6) $(\alpha \wedge \beta) \to \beta$,
(A7) $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to (\beta \wedge \gamma)))$,
(A8) $\alpha \to D_1\alpha$,
(A9) $D_j D_i \alpha \leftrightarrow D_i \alpha$, $1 \leqslant i, j \leqslant n-1$,

(A10) $D_i \bigvee_{p=0}^{m-1} f^{2p}\alpha \leftrightarrow D_i\alpha$, $1 \leqslant i \leqslant n-1$,

(A11) $((\alpha \wedge \gamma) \to (\beta \wedge \gamma)) \to (\gamma \to (\alpha \to \beta))$,

(A12) $D_i\alpha \vee fD_i\alpha$, $1 \leqslant i \leqslant n-1$,

(A13) $D_iq_s(\alpha \vee \alpha) \to D_iq_s(\alpha \wedge \alpha)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A14) $D_iq_s\alpha \to D_iq_s(\alpha \wedge D_1\alpha)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A15) $D_iq_s(f^2D_i\alpha \vee D_i\alpha) \to D_iq_s(f^2D_i\alpha \wedge D_i\alpha)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A16) $D_iq_s\Big(D_i\Big(\alpha \wedge \bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee (D_i\alpha \wedge D_i\beta)\Big) \to$

$D_iq_s\Big(D_i\Big(\alpha \wedge \bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \wedge (D_i\alpha \wedge D_i\beta)\Big)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A17) $D_iq_s(D_j\alpha \vee (D_i\alpha \wedge D_j\alpha)) \to D_iq_s(D_j\alpha \wedge (D_i\alpha \wedge D_j\alpha))$, $1 \leqslant i \leqslant j \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A18) $D_iq_s\Big(D_if\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \vee fD_{n-i}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big) \to$

$D_iq_s\Big(D_if\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge fD_{n-i}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A19) $D_iq_s(\alpha \wedge f\alpha) \to D_iq_s((\alpha \wedge f\alpha) \wedge (\beta \vee f\beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A20) $D_iq_s\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \to$

$D_iq_s\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha \wedge \Big(\bigvee_{p=0}^{m-1} f^{2p}\beta \vee fD_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee D_{i+1}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)\Big)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A21) $D_iq_s\alpha \to D_iq_s(\alpha \wedge f^{2m-1}(f\alpha \wedge f\beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A22) $D_iq_s((\alpha \wedge f^{2m-1}(f\beta \wedge f\gamma)) \vee f^{2m-1}(f(\gamma \wedge \alpha) \wedge f(\beta \wedge \alpha))) \to D_iq_s(\alpha \wedge f^{2m-1}(f\beta \wedge f\gamma) \wedge f^{2m-1}(f(\gamma \wedge \alpha) \wedge f(\beta \wedge \alpha)))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A23) $\alpha \twoheadrightarrow \beta \leftrightarrow D_{n-1}q_1\beta \vee \bigwedge_{i=1}^{n-1} \bigwedge_{s=1}^{m} (D_iq_s(\alpha \vee \beta) \to D_iq_s(\alpha \wedge \beta))$,

(A24) $(D_iq_s(f\alpha \vee f\beta) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta)) \to (D_iq_s(f\alpha \wedge f\beta) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A25) $(D_i\alpha \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta)) \to (D_i\beta \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta))$, $1 \leqslant i \leqslant n-1$,

(A26) $(D_iq_s((\alpha \wedge \gamma) \vee (\beta \wedge \gamma)) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta)) \to (D_iq_s((\alpha \wedge \gamma) \wedge (\beta \wedge \gamma)) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A27) $(\alpha \wedge \gamma) \twoheadrightarrow (\gamma \wedge \alpha)$,

(A28) $(D_iq_s((\gamma \to \alpha) \vee (\gamma \to \beta)) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta)) \to (D_iq_s((\gamma \to \alpha) \wedge (\gamma \to \beta)) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A29) $(D_iq_s((\alpha \to \gamma) \vee (\beta \to \gamma)) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta)) \to (D_iq_s((\alpha \to \gamma) \wedge (\beta \to \gamma)) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A30) $(D_iq_s((\gamma \twoheadrightarrow \alpha) \vee (\gamma \twoheadrightarrow \beta)) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta)) \to (D_iq_s((\gamma \twoheadrightarrow \alpha) \wedge (\gamma \twoheadrightarrow \beta)) \wedge (\alpha \twoheadleftarrow\!\twoheadrightarrow \beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, (A31) $(D_iq_s((\alpha \twoheadrightarrow \gamma)\vee(\beta \twoheadrightarrow \gamma))\wedge(\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta)) \to (D_iq_s((\alpha \twoheadrightarrow \gamma)\wedge(\beta \twoheadrightarrow \gamma)\wedge(\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta)))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(A32) $(\alpha \twoheadrightarrow \beta) \to ((\beta \twoheadrightarrow \gamma) \to (\alpha \twoheadrightarrow \gamma))$,

(A33) $(f^{2m-1}(f\alpha \wedge f\beta) \twoheadrightarrow (\alpha \vee \beta)) \wedge ((\alpha \vee \beta) \twoheadrightarrow f^{2m-1}(f\alpha \wedge f\beta))$,

(A34) $(D_{n-1}q_1\alpha \wedge (\alpha \twoheadrightarrow \beta)) \to \beta$.

The consequence operation $C_{\mathcal{L}}$ in $\mathcal{L} = (A^0, F)$ is determined by $\mathcal{A}_l$ and by the following rules of inference:

(R1) $\dfrac{\alpha, \alpha \to \beta}{\beta}$,

(R2) $\dfrac{D_i\alpha \to D_j\beta, D_j\beta \to D_i\alpha}{D_i\alpha \twoheadrightarrow D_j\beta}$, $1 \leqslant i, j \leqslant n-1$,

(R3) $\dfrac{D_iq_s\alpha \to D_iq_s(\alpha \wedge \beta)}{D_iq_s(\alpha \vee (\alpha \wedge \beta)) \to D_iq_s(\alpha \wedge (\alpha \wedge \beta))}$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(R4) $\dfrac{\alpha}{D_{n-1}q_1\alpha}$,

(R5) $\dfrac{D_iq_s(\alpha \vee \beta) \to D_iq_s(\alpha \wedge \beta)}{D_iq_s(\beta \vee \alpha) \to D_iq_s(\beta \wedge \alpha)}$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$.

The system $\ell_n^m = (\mathcal{L}, C_{\mathcal{L}})$ thus obtained will be called the $\mathcal{L}_n^m$-propositional calculus. It is worth mentioning that the above connectives are not independent, however, we consider them for simplicity. We will denote by $\mathcal{T}$ the set of all formulas derivable in $\ell_n^m$. If $\alpha \in \mathcal{T}$, we will write $\vdash \alpha$.

Lemma 4.1 summarizes the most important rules and theorems necessary for the further development.

**Lemma 4.1.** *In $\ell_n^m$ the following rules and theorems hold:*

(R6) $\dfrac{\alpha}{\beta \to \alpha}$,

(R7) $\dfrac{\alpha \to (\beta \to \gamma)}{(\alpha \to \beta) \to (\alpha \to \gamma)}$,

(T1) $\vdash \alpha \to \alpha$,

(T2) $\vdash (\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$,

(R8) $\dfrac{\alpha \to \beta}{(\gamma \to \alpha) \to (\gamma \to \beta)}$,

(R9) $\dfrac{(\alpha \to \beta) \to (\alpha \to \gamma)}{\beta \to (\alpha \to \gamma)}$,

(R10) $\dfrac{\alpha \to (\beta \to \gamma)}{\beta \to (\alpha \to \gamma)}$,

(T3) $\vdash (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)$,

(T4) $\vdash (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$,

(R11) $\dfrac{\alpha \to \beta, \beta \to \gamma}{\alpha \to \gamma}$, (R12) $\dfrac{\alpha \to \beta}{(\beta \to \gamma) \to (\alpha \to \gamma)}$,

(R13) $\dfrac{\alpha, \beta}{\alpha \wedge \beta}$,

(T5) $\vdash \alpha \twoheadrightarrow \alpha$,

(R14) $\dfrac{D_i q_s(\alpha \vee \beta) \to D_i q_s(\alpha \wedge \beta)}{\alpha \twoheadrightarrow \beta}$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(R15) $\dfrac{\alpha, \alpha \twoheadrightarrow \beta}{\beta}$,

(R16) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \gamma}{\alpha \twoheadrightarrow \gamma}$,

(R17) $\dfrac{\beta}{\alpha \twoheadrightarrow \beta}$,

(R18) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{f\alpha \twoheadrightarrow f\beta}$,

(R19) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{D_i\alpha \twoheadrightarrow D_i\beta}$, $1 \leqslant i \leqslant n-1$,

(R20) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{(\alpha \wedge \gamma) \twoheadrightarrow (\beta \wedge \gamma)}$,

(R21) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{(\gamma \wedge \alpha) \twoheadrightarrow (\gamma \wedge \beta)}$,

(R22) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{(\alpha \vee \gamma) \twoheadrightarrow (\beta \vee \gamma)}$,

(R23) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{(\gamma \vee \alpha) \twoheadrightarrow (\gamma \vee \beta)}$,

(R24) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{(\gamma \to \alpha) \twoheadrightarrow (\gamma \to \beta)}$,

(R25) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{(\alpha \to \gamma) \twoheadrightarrow (\beta \to \gamma))}$,

(R26) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{(\gamma \twoheadrightarrow \alpha) \twoheadrightarrow (\gamma \twoheadrightarrow \beta))}$,

(R27) $\dfrac{\alpha \twoheadrightarrow \beta, \beta \twoheadrightarrow \alpha}{(\alpha \twoheadrightarrow \gamma) \twoheadrightarrow (\beta \twoheadrightarrow \gamma)}$.

P r o o f. The proof of (R6) to (R13) is routine.

(T5):

(1) $D_i q_s(\alpha \vee \alpha) \to D_i q_s(\alpha \wedge \alpha)$, [(A13)]

(2) $\bigwedge\limits_{i=1}^{n-1} \bigwedge\limits_{s=1}^{m} (D_i q_s(\alpha \vee \alpha) \to D_i q_s(\alpha \wedge \alpha))$, [(1), (R13)]

(3) $\alpha \twoheadrightarrow \alpha$. [(2), (A4), (R1), (A23)]

(R14):

(1) $D_i q_s(\alpha \vee \beta) \to D_i q_s(\alpha \wedge \beta)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$,

(2) $\bigwedge\limits_{i=1}^{n-1} \bigwedge\limits_{s=1}^{m} (D_i q_s(\alpha \vee \beta) \to D_i q_s(\alpha \wedge \beta))$, [(1), (R13)]

(3) $\bigwedge_{i=1}^{n-1}\bigwedge_{s=1}^{m}(D_iq_s(\alpha\vee\beta)\to D_iq_s(\alpha\wedge\beta))\to$
$\Big(D_{n-1}q_1\beta\vee\bigwedge_{i=1}^{n-1}\bigwedge_{s=1}^{m}(D_iq_s(\alpha\vee\beta)\to D_iq_s(\alpha\wedge\beta))\Big),$ [(A4)]

(4) $D_{n-1}q_1\beta\vee\bigwedge_{i=1}^{n-1}\bigwedge_{s=1}^{m}(D_iq_s(\alpha\vee\beta)\to D_iq_s(\alpha\wedge\beta)),$ [(2), (3), (R1)]

(5) $\alpha\twoheadrightarrow\beta.$ [(4), (A23)]

(R15): It is a consequence of (R4), (R13), (A34) and (R1).

(R16): It is routine.

(R17): It follows as a consequence of (R4), (A3), (R1) and (A23).

(R18):

(1) $\alpha\twoheadrightarrow\beta,$
(2) $\beta\twoheadrightarrow\alpha,$
(3) $\alpha\twoheadleftrightarrow\beta,$ [(1), (2), (R13)]
(4) $(D_iq_s(f\alpha\vee f\beta)\wedge(\alpha\twoheadleftrightarrow\beta))\to(D_iq_s(f\alpha\wedge f\beta)\wedge(\alpha\twoheadleftrightarrow\beta)),\ 1\leqslant i\leqslant n-1,$ $1\leqslant s\leqslant m,$ [(A24)]
(5) $(\alpha\twoheadleftrightarrow\beta)\to(D_iq_s(f\alpha\vee f\beta)\to D_iq_s(f\alpha\wedge f\beta)),\ 1\leqslant i\leqslant n-1,\ 1\leqslant s\leqslant m,$ [(A11), (4), (R1)]
(6) $D_iq_s(f\alpha\vee f\beta)\to D_iq_s(f\alpha\wedge f\beta),\ 1\leqslant i\leqslant n-1,\ 1\leqslant s\leqslant m,$ [(3), (5), (R1)]
(7) $f\alpha\twoheadrightarrow f\beta.$ [(6), (R14)]

(R19):

(1) $\alpha\twoheadrightarrow\beta,$
(2) $\beta\twoheadrightarrow\alpha,$
(3) $\alpha\twoheadleftrightarrow\beta,$ [(1), (2), (R13)]
(4) $(D_i\alpha\wedge(\alpha\twoheadleftrightarrow\beta))\to(D_i\beta\wedge(\alpha\twoheadleftrightarrow\beta)),\ 1\leqslant i\leqslant n-1,$ [(A25)]
(5) $(\alpha\twoheadleftrightarrow\beta)\to(D_i\alpha\to D_i\beta),\ 1\leqslant i\leqslant n-1,$ [(A11), (4), (R1)]
(6) $D_i\alpha\to D_i\beta,\ 1\leqslant i\leqslant n-1,$ [(3), (5), (R1)]
(7) $(D_i\beta\wedge(\beta\twoheadleftrightarrow\alpha))\to(D_i\alpha\wedge(\beta\twoheadleftrightarrow\alpha)),\ 1\leqslant i\leqslant n-1,$ [(A25)]
(8) $\beta\twoheadleftrightarrow\alpha,$ [(2), (1), (R13)]
(9) $(\beta\twoheadleftrightarrow\alpha)\to(D_i\beta\to D_i\alpha),\ 1\leqslant i\leqslant n-1,$ [(A11), (7), (R1)]
(10) $D_i\beta\to D_i\alpha,\ 1\leqslant i\leqslant n-1,$ [(8), (9), (R1)]
(11) $D_i\alpha\twoheadrightarrow D_i\beta,\ 1\leqslant i\leqslant n-1.$ [(6), (10), (R2)]

(R20):

(1) $\alpha\twoheadrightarrow\beta,$
(2) $\beta\twoheadrightarrow\alpha,$
(3) $\alpha\twoheadleftrightarrow\beta,$ [(1), (2), (R13)]
(4) $(D_iq_s((\alpha\wedge\gamma)\vee(\beta\wedge\gamma))\wedge(\alpha\twoheadleftrightarrow\beta))\to(D_iq_s((\alpha\wedge\gamma)\wedge(\beta\wedge\gamma))\wedge(\alpha\twoheadleftrightarrow\beta)),$ $1\leqslant i\leqslant n-1,\ 1\leqslant s\leqslant m,$ [(A26)]

(5) $(\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta) \to (D_i q_s((\alpha \wedge \gamma) \vee (\beta \wedge \gamma)) \to D_i q_s((\alpha \wedge \gamma) \wedge (\beta \wedge \gamma)))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(A11), (4), (R1)]

(6) $D_i q_s((\alpha \wedge \gamma) \vee (\beta \wedge \gamma)) \to D_i q_s((\alpha \wedge \gamma) \wedge (\beta \wedge \gamma))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(3), (5), (R1)]

(7) $(\alpha \wedge \gamma) \twoheadrightarrow (\beta \wedge \gamma)$. [(6), (R14)]

(R22):

(1) $\alpha \twoheadrightarrow \beta$, $\beta \twoheadrightarrow \alpha$,

(2) $f\alpha \twoheadrightarrow f\beta$, $f\beta \twoheadrightarrow f\alpha$, [(1), (R18)]

(3) $(f\alpha \wedge f\gamma) \twoheadrightarrow (f\beta \wedge f\gamma)$, [(2), (R20)]

(4) $(f\beta \wedge f\gamma) \twoheadrightarrow (f\alpha \wedge f\gamma)$, [(2), (R20)]

(5) $f^{2m-1}(f\alpha \wedge f\gamma) \twoheadrightarrow f^{2m-1}(f\beta \wedge f\gamma)$, [(3), (4), (R18)]

(6) $f^{2m-1}(f\beta \wedge f\gamma) \twoheadrightarrow (\beta \vee \gamma)$, [(A5), (A33), (R1)]

(7) $(\alpha \vee \gamma) \twoheadrightarrow f^{2m-1}(f\alpha \wedge f\gamma)$, [(A6), (A33), (R1)]

(8) $(\alpha \vee \gamma) \twoheadrightarrow (\beta \vee \gamma)$. [(7), (5), (6), (R16)]

(R24):

(1) $\alpha \twoheadrightarrow \beta$, [hip.]

(2) $\beta \twoheadrightarrow \alpha$, [hip.]

(3) $\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta$, [(1), (2), (R13)]

(4) $(D_i q_s((\gamma \to \alpha) \vee (\gamma \to \beta)) \wedge (\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta)) \to (D_i q_s((\gamma \to \alpha) \wedge (\gamma \to \beta)), \wedge(\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta))$ [(A28)]

(5) $(\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta) \to ((D_i q_s((\gamma \to \alpha) \vee (\gamma \to \beta))) \to (D_i q_s((\gamma \to \alpha) \wedge (\gamma \to \beta))))$, [(A11), (4), (R1)]

(6) $D_i q_s((\gamma \to \alpha) \vee (\gamma \to \beta)) \to D_i q_s((\gamma \to \alpha) \wedge (\gamma \to \beta))$, [(3), (5), (R1)]

(7) $(\gamma \to \alpha) \twoheadrightarrow (\gamma \to \beta)$. [(6), (R14)]

(R26):

(1) $\alpha \twoheadrightarrow \beta$,

(2) $\beta \twoheadrightarrow \alpha$,

(3) $\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta$, [(1), (2), (R13)]

(4) $(D_i q_s((\gamma \twoheadrightarrow \alpha) \vee (\gamma \twoheadrightarrow \beta)) \wedge (\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta)) \to (D_i q_s((\gamma \twoheadrightarrow \alpha) \wedge (\gamma \twoheadrightarrow \beta)) \wedge (\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(A30)]

(5) $(\alpha \twoheadleftarrow\!\!\twoheadrightarrow \beta) \to ((D_i q_s((\gamma \twoheadrightarrow \alpha) \vee (\gamma \twoheadrightarrow \beta))) \to (D_i q_s((\gamma \twoheadrightarrow \alpha) \wedge (\gamma \twoheadrightarrow \beta))))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(A11), (4), (R1)]

(6) $D_i q_s((\gamma \twoheadrightarrow \alpha) \vee (\gamma \twoheadrightarrow \beta)) \to D_i q_s((\gamma \twoheadrightarrow \alpha) \wedge (\gamma \twoheadrightarrow \beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(3), (5), (R1)]

(7) $(\gamma \twoheadrightarrow \alpha) \twoheadrightarrow (\gamma \twoheadrightarrow \beta)$. [(6), (R14)]

Using a reasoning similar to that for (R20), (R22), (R24) and (R26) we infer (R21), (R23), (R25) and (R27), respectively. □

**Theorem 4.1.** *The propositional calculus $\ell_n^m$ belongs to the class of standard systems of implicative extensional propositional calculi.*

P r o o f. We have to prove that conditions (s1) to (s8) in Section 1 are verified. Clearly, (s1) and (s2) hold. Besides, (s3), (s4), (s5) and (s6) follow from (T12), (R15), (R16) and (R17), respectively. On the other hand, taking into account (R18) and (R19), we have that (s7) is satisfied. Finally, if $\alpha \twoheadrightarrow \beta$, $\beta \twoheadrightarrow \alpha$, $\delta \twoheadrightarrow \gamma$, $\gamma \twoheadrightarrow \delta \in C_{\mathcal{L}}(H)$ for every subset $H$ of formulas, then by (R20) we have that $(\alpha \wedge \delta) \twoheadrightarrow (\beta \wedge \delta) \in C_{\mathcal{L}}(H)$. Besides, by (R21) we get $(\beta \wedge \delta) \twoheadrightarrow (\beta \wedge \gamma) \in C_{\mathcal{L}}(H)$. Hence, by (R16) we infer that $(\alpha \wedge \delta) \twoheadrightarrow (\beta \wedge \gamma) \in C_{\mathcal{L}}(H)$. In an analogous manner, from (R22), (R23), (R25), (R26) and (R27) we conclude the proof of (s8). □

In what follows, our attention is focused on establishing the relationship between $L_n^m$-algebras and $\ell_n^m$-algebras which are the class of algebras determined by the system $\ell_n^m$. To this aim, Lemma 4.2 will be fundamental.

**Lemma 4.2.** *In $\ell_n^m$ the following theorems hold:*

(T6) $\vdash (\alpha \wedge f^{2m-1}(f\alpha \wedge f\beta)) \twoheadrightarrow \alpha$,

(T7) $\vdash \alpha \twoheadrightarrow (\alpha \wedge f^{2m-1}(f\alpha \wedge f\beta))$,

(T8) $\vdash (\alpha \wedge f^{2m-1}(f\beta \wedge f\gamma)) \twoheadrightarrow f^{2m-1}(f(\gamma \wedge \alpha) \wedge f(\beta \wedge \alpha))$,

(T9) $\vdash f^{2m-1}(f(\gamma \wedge \alpha) \wedge f(\beta \wedge \alpha)) \twoheadrightarrow (\alpha \wedge f^{2m-1}(f\beta \wedge f\gamma))$,

(T10) $\vdash D_1(\alpha \twoheadrightarrow \alpha)$,

(T11) $\vdash f^2 D_1\alpha \twoheadrightarrow D_1\alpha$,

(T12) $\vdash D_1\alpha \twoheadrightarrow f^2 D_1\alpha$,

(T13) $\vdash D_i\Big(\alpha \wedge \bigvee\limits_{p=0}^{m-1} f^{2p}\beta\Big) \twoheadrightarrow (D_i\alpha \wedge D_i\beta)$, $1 \leqslant i \leqslant n-1$,

(T14) $\vdash (D_i\alpha \wedge D_i\beta) \twoheadrightarrow D_i\Big(\alpha \wedge \bigvee\limits_{p=0}^{m-1} f^{2p}\beta\Big)$, $1 \leqslant i \leqslant n-1$,

(T15) $\vdash D_j\alpha \twoheadrightarrow (D_i\alpha \wedge D_j\alpha)$, $1 \leqslant i \leqslant j \leqslant n-1$,

(T16) $\vdash (D_i\alpha \wedge D_j\alpha) \twoheadrightarrow D_j\alpha$, $1 \leqslant i \leqslant j \leqslant n-1$,

(T17) $\vdash D_i f\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}\alpha\Big) \twoheadrightarrow fD_{n-i}\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}\alpha\Big)$, $1 \leqslant i \leqslant n-1$,

(T18) $\vdash fD_{n-i}\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}\alpha\Big) \twoheadrightarrow D_i f\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}\alpha\Big)$, $1 \leqslant i \leqslant n-1$,

(T19) $\vdash D_i\alpha \twoheadrightarrow D_jD_i\alpha$, $1 \leqslant i, j \leqslant n-1$,

(T20) $\vdash D_jD_i\alpha \twoheadrightarrow D_i\alpha$, $1 \leqslant i, j \leqslant n-1$,

(T21) $\vdash (\alpha \wedge D_1\alpha) \twoheadrightarrow \alpha$,

(T22) $\vdash \alpha \twoheadrightarrow (\alpha \wedge D_1\alpha)$,

(T23) $\vdash D_i\Big(\bigvee\limits_{p=0}^{m-1} f^{2p}\alpha\Big) \twoheadrightarrow D_i\alpha$, $1 \leqslant i \leqslant n-1$, (T24) $\vdash D_i\alpha \twoheadrightarrow D_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)$, $1 \leqslant i \leqslant n-1$,

(T25) $\vdash ((\alpha \wedge f\alpha) \wedge (\beta \vee f\beta)) \twoheadrightarrow (\alpha \wedge f\alpha)$,

(T26) $\vdash (\alpha \wedge f\alpha) \twoheadrightarrow ((\alpha \wedge f\alpha) \wedge (\beta \vee f\beta))$,

(T27) $\vdash \Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge \Big(\bigvee_{p=0}^{m-1} f^{2p}\beta \vee fD_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee D_{i+1}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)\Big) \twoheadrightarrow \bigvee_{p=0}^{m-1} f^{2p}\alpha$, $1 \leqslant i \leqslant n-1$,

(T28) $\vdash \Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \twoheadrightarrow \Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge \Big(\bigvee_{p=0}^{m-1} f^{2p}\beta \vee fD_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee D_{i+1}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)\Big)$, $1 \leqslant i \leqslant n-1$.

P r o o f. The proofs of (T6) through (T18) are routine.

(T19):

(1) $(D_i\alpha \to D_jD_i\alpha) \wedge (D_jD_i\alpha \to D_i\alpha)$, $1 \leqslant i, j \leqslant n-1$, [(A9)]

(2) $((D_i\alpha \to D_jD_i\alpha) \wedge (D_jD_i\alpha \to D_i\alpha)) \to (D_i\alpha \to D_jD_i\alpha)$, $1 \leqslant i, j \leqslant n-1$, [(A5)]

(3) $D_i\alpha \to D_jD_i\alpha$, $1 \leqslant i, j \leqslant n-1$, [(1), (2), (R1)]

(4) $((D_i\alpha \to D_jD_i\alpha) \wedge (D_jD_i\alpha \to D_i\alpha)) \to (D_jD_i\alpha \to D_i\alpha)$, $1 \leqslant i, j \leqslant n-1$, [(A6)]

(5) $D_jD_i\alpha \to D_i\alpha$, $1 \leqslant i, j \leqslant n-1$, [(1), (4), (R1)]

(6) $D_i\alpha \twoheadrightarrow D_jD_i\alpha$, $1 \leqslant i, j \leqslant n-1$. [(3), (5), (R2)]

(T21): It follows as a consequence of (A14), (R3), (R5) and (R14).

(T23):

(1) $\Big(D_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \to D_i\alpha\Big) \wedge \Big(D_i\alpha \to D_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)$, $1 \leqslant i \leqslant n-1$, [(A10)]

(2) $D_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \to D_i\alpha$, $1 \leqslant i \leqslant n-1$, [(A5), (1), (R1)]

(3) $D_i\alpha \to D_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)$, $1 \leqslant i \leqslant n-1$, [(A6), (1), (R1)]

(4) $D_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \twoheadrightarrow D_i\alpha$, $1 \leqslant i \leqslant n-1$. [(2), (3), (R2)]

(T25):

(1) $D_iq_s(\alpha \wedge f\alpha) \to D_iq_s((\alpha \wedge f\alpha) \wedge (\beta \vee f\beta))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(A19)]

(2) $D_iq_s((\alpha \wedge f\alpha) \vee ((\alpha \wedge f\alpha) \wedge (\beta \vee f\beta))) \to D_iq_s((\alpha \wedge f\alpha) \wedge ((\alpha \wedge f\alpha) \wedge (\beta \vee f\beta)))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(1), (R3)]

(3) $D_iq_s(((\alpha \wedge f\alpha) \wedge (\beta \vee f\beta)) \vee (\alpha \wedge f\alpha)) \to D_iq_s(((\alpha \wedge f\alpha) \wedge (\beta \vee f\beta)) \wedge (\alpha \wedge f\alpha))$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(2), (R5)]

(4) $((\alpha \wedge f\alpha) \wedge (\beta \vee f\beta)) \twoheadrightarrow (\alpha \wedge f\alpha)$. [(3), (R14)]

(T27):

(1) $D_i q_s\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \rightarrow D_i q_s\Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge \Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee fD_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee D_{i+1}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)\Big)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(A20)]

(2) $D_i q_s\Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \vee \Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge \Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee fD_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee D_{i+1}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)\Big)\Big) \rightarrow D_i q_s\Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge \Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge \Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee fD_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee D_{i+1}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)\Big)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(1), (R3)]

(3) $D_i q_s\Big(\Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge \Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee fD_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee D_{i+1}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)\Big) \vee \Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big) \rightarrow D_i q_s\Big(\Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge \Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee fD_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee D_{i+1}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)\Big) \wedge \Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big)$, $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant m$, [(2), (R5)]

(4) $\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big) \wedge \Big(\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee fD_i\Big(\bigvee_{p=0}^{m-1} f^{2p}\beta\Big) \vee D_{i+1}\Big(\bigvee_{p=0}^{m-1} f^{2p}\alpha\Big)\Big) \twoheadrightarrow \bigvee_{p=0}^{m-1} f^{2p}\alpha$, $1 \leqslant i \leqslant n-1$. [(3), (R14)]

An argument similar to that for (T19), (T21), (T23), (T25) and (T27) allows us to prove (T20), (T22), (T24), (T26) and (T28), respectively. □

**Proposition 4.1.** *If $\alpha$ is a formula derivable in $\ell_n^m$, then $v(\alpha) = 1$ for every valuation $v$ of $\mathcal{L}$ in every $\ell_n^m$-algebra $\mathcal{U}$.*

P r o o f. Since $\alpha$ is a formula derivable in $\ell_n^m$ if and only if $\vdash \alpha$, then by (a1) and (a2) we conclude that $v(\alpha) = 1$ for every valuation $v$ of $\mathcal{L}$ in every $\ell_n^m$-algebra $\mathcal{U}$. □

**Proposition 4.2.** *Let $\langle L, \vee, \wedge, f, D_1, \ldots, D_{n-1}, 0, 1\rangle \in \mathcal{L}_n^m$. Then $\langle L, \rightarrow, \twoheadrightarrow, \vee, \wedge, f, D_1, \ldots, D_{n-1}, 1\rangle$ is an $\ell_n^m$-algebra, where $\rightarrow$ and $\twoheadrightarrow$ are defined as in Section* 1 *and Section* 2*, respectively.*

P r o o f. We will prove that conditions (a1) to (a4) in Section 1 hold. Indeed, taking into account the definitions of $\rightarrow$ and $\twoheadrightarrow$ we have that (a1) and (a2) are satisfied. On the other hand, let $a, b \in L$ be such that $a \twoheadrightarrow b = b \twoheadrightarrow c = 1$. Then, by (S6) and ($W_3$) we conclude (a3). Besides, if $a \twoheadrightarrow b = b \twoheadrightarrow a = 1$, hence (S5) allows us to infer (a4). □

**Proposition 4.3.** *Let* $\langle A, \to, \twoheadrightarrow, \vee, \wedge, f, D_1, \ldots, D_{n-1}, 1\rangle$ *be an* $\ell_n^m$*-algebra. Then* $\langle A, \vee, \wedge, f, D_1, \ldots, D_{n-1}, 0, 1\rangle \in \mathcal{L}_n^m$*, where* $0 = f1$.

P r o o f. From (T11), (T12) and (a4) we infer that $f^2 D_1(\alpha \twoheadrightarrow \alpha) = D_1(\alpha \twoheadrightarrow \alpha)$. Besides, from (T10) we have that $D_1(\alpha \twoheadrightarrow \alpha) = 1$ and so, we conclude that $f^2 1 = 1$. This assertion and the fact that $f1 = 0$ imply that $f0 = 1$. Moreover, from (T6), (T7), (T8) and (T9) we have that conditions (m1) and (m2) in Theorem 3.2 hold. Therefore, ($GL_1$) is verified. Besides, by (a4) and taking into account (T13) through (T28) we infer ($GL_2$), ($GL_3$) and ($GL_5$) through ($GL_{10}$). Furthermore, from (A12) and (a1) in Section 1 we get ($GL_4$) and so, the proof is complete. □

From Propositions 4.2 and 4.3 we conclude:

**Theorem 4.2.** *The notions of the* $\ell_n^m$*-algebra and the* $\mathcal{L}_n^m$*-algebra are equivalent.*

Let $\equiv$ be the binary relation on $F$ defined as follows:

$\alpha \equiv \beta$ if and only if $\vdash \alpha \twoheadrightarrow \beta$ and $\vdash \beta \twoheadrightarrow \alpha$ in $\ell_n^m$.

Then $\equiv$ is a congruence relation on $\langle F, \to, \twoheadrightarrow, \wedge, \vee, f, D_1, \ldots, D_{n-1}\rangle$ and $\mathcal{T}$ determines an equivalence class. On the other hand, it is easy to verify that the relation $\leqslant$ defined on $F/\!\equiv$ by

$$[\alpha] \leqslant [\beta] \quad \text{if and only if} \quad \vdash \alpha \twoheadrightarrow \beta,$$

is a preorder on $F/\!\equiv$.

**Proposition 4.4.** $\mathcal{F} = \langle F/\!\equiv, \to, \twoheadrightarrow, \wedge, \vee, f, D_1, \ldots, D_{n-1}, 1\rangle$ *is an* $\ell_n^m$*-algebra, and* $1 = \mathcal{T}$.

P r o o f. Let $v$ be a valuation of $\mathcal{L}$ in $\mathcal{F}$ and let $\varrho$ be a substitution from $\mathcal{L}$ into $\mathcal{L}$ such that $v(x) = [\varrho(x)]$ for every propositional variable $x$ in $\mathcal{L}$ and so, we have that (1) $v(\alpha) = [\varrho(\alpha)]$ for every formula $\alpha$ in $\mathcal{L}$. Hence, conditions (a1)–(a4) are verified. Indeed, if $\alpha \in \mathcal{A}$, then by (s1), $\varrho(\alpha) \in \mathcal{A}$. Thus, $[\varrho(\alpha)] = 1$ and consequently (a1) holds.

Suppose that a rule of inference (r) assigns to premises $\alpha_1, \ldots, \alpha_n$ a formula $\beta$ as the conclusion and let $v(\alpha_i) = 1$ for all $i$, $1 \leqslant i \leqslant n$. Thus, by (1), $[\varrho(\alpha_i)] = 1$ for all $i$, $1 \leqslant i \leqslant n$. Hence, by (s2) it follows that $[\varrho(\beta)] = 1$ and so, by (1) we have that $v(\beta) = 1$, which proves that (a2) holds.

Taking into account that $\vdash \alpha \twoheadrightarrow \beta$ if and only if $1 = [\alpha \twoheadrightarrow \beta] = [\alpha] \twoheadrightarrow [\beta]$ we obtain that $[\alpha] \leqslant [\beta]$ if and only if $[\alpha] \twoheadrightarrow [\beta] = 1$. From this last assertion the proofs of (a3) and (a4) are straightforward. □

From Proposition 4.3 and Proposition 4.4 we conclude the following theorem.

**Theorem 4.3.** $\mathcal{F} = \langle F/\equiv, \wedge, \vee, f, D_1, \ldots, D_{n-1}, 0, 1\rangle \in \mathcal{L}_n^m$.

On the other hand, since $\ell_n^m$ is consistent, from [9], VIII 7, and Theorem 4.2 we have that the completeness theorem for $\ell_n^m$ holds, which is included in Theorem 4.4.

**Theorem 4.4.** *Let $\alpha$ be a formula of $\ell_n^m$. Then the following conditions are equivalent:*

(i) *$\alpha$ is derivable in $\ell_n^m$,*
(ii) *$\alpha$ is valid in every $L_n^m$-algebra,*
(iii) *$v_0(\alpha) = 1$, where $v_0$ is the canonical valuation* ([9], VIII 3.4), *in the algebra $\mathcal{F}$.*

P r o o f. (i) $\Rightarrow$ (ii): It follows from the assertions given in Section 1.

(ii) $\Rightarrow$ (iii): It is straightforward.

(iii) $\Rightarrow$ (i): From the hypothesis we have that $[\alpha] = 1 = \mathcal{T}$. Hence, $\alpha$ is derivable in $\ell_n^m$. □

R e m a r k 4.1. In case $m = 1$, we conclude that the propositional calculus $\ell_n^1$ has $n$-valued Łukasiewicz-Moisil algebras as the algebraic counterpart.

## 5. Conclusions

In this paper we have presented new results about the congruence lattice of $L_n^m$-algebras as well as the principal congruences by means of the standard implication. Furthermore, we have established a characterization of $k_{2m}$-lattices which has provided an easy way to prove that $L_n^m$-algebras are the algebraic counterpart of a propositional calculus. Finally, we have described a standard implicative extensional propositional calculus $\ell_n^m$ and proved that $L_n^m$-algebras and $\ell_n^m$-algebras are equivalent.

On the other hand, it would be interesting to find a sequent calculus, along with a proper notion of validity, sound and complete with respect to $L_n^m$-algebras, which has the desirable property of cut-elimination. Another interesting problem would be to present a Gentzen-style system using the tool of hypersequents.

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