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ON UPPER TRIANGULAR NONNEGATIVE MATRICES

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Abstract. We first investigate factorizations of elements of the semigroup S of upper triangular matrices with nonnegative entries and nonzero determinant, provide a formula for $\varrho(S)$, and, given $A \in S$, also provide formulas for $l(A)$, $L(A)$ and $\varrho(A)$. As a consequence, open problem 2 and problem 4 presented in N. Baeth et al. (2011), are partly answered. Secondly, we study the semigroup of upper triangular matrices with only positive integral entries, compute some invariants of such semigroup, and also partly answer open Problem 1 and Problem 3 in N. Baeth et al. (2011).

Keywords: upper triangular; nonnegative matrix; factorization; matrix semigroup

MSC 2010: 11Y05, 15A23

1. Introduction

Upper triangular matrices are an important class of matrices. This is a well-studied class in part because determinants of upper triangular matrices are easy to compute, and in part because any integer-valued matrix can be put in Hermite Normal Form. Their study leads to a broader study of all integer-valued matrices. There are many papers in the literature considering these matrices and similar topics. Note that all of the results about upper triangular matrices go through for the semigroup of lower triangular matrices.

Factoring such matrices plays an important role in the study of upper triangular matrices (see [6]). The problem of factoring matrices was studied by Cohn ([5]) as early as 1963. Later, Jacobson and Wisner ([7], [8]), Chuan and Chuan ([3], [4])

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investigated these factorization problems in the context of semigroups of matrices. Motivated by these results, Baeth et al. [2] applied the concepts of contemporary factorization theory to semigroups of integral-valued matrices, and calculated certain important invariants to give an idea of how unique or non unique factorization is in each of these semigroups. In [2], Baeth et al. presented six open problems.

In particular, we will investigate factorizations of elements of the semigroup of upper triangular matrices with nonnegative entries and study the semigroup of upper triangular matrices with only positive integral entries. Also, we will consider open Problems 1–4 presented in [2].

Throughout this paper, $\mathbb{N}$ will denote the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Also, $T_n(\mathbb{N}_0)$ and $T_n(\mathbb{N})$ will denote the semigroup of $n \times n$ upper triangular matrices with nonnegative and positive integral-valued entries, respectively.

In the following, analogously to [2] or [1], we recall some concepts and preliminaries.

A *semigroup* is a pairing $(S, \cdot)$ where S is a set and $\cdot$ is an associative binary operation on S. When the binary operation is clear from the context and $A, B \in S$, we will simply write AB instead of $A \cdot B$. If S contains an element I such that $AI = IA = A$ for all $A \in S$, then I is the *identity* of S.

Let S be a semigroup with identity I. An element $A \in S$ is a *unit* of S if there exists an element $B \in S$ such that $AB = BA = I$. A non unit $A \in S$ is called an *atom* of S if whenever $A = BC$ for some elements $B, C \in S$, either B or C is a unit of S. The semigroup S is said to be *atomic* provided each non unit element in S can be written as a product of atoms of S.

Let S denote an atomic semigroup and let A be a non unit element of S. The set

$$\mathcal{L}(A) = \{t\colon\ A = A_1 A_2 \dots A_t \text{ with each } A_i \text{ an atom of } S\}$$

is called the *set of lengths of* A.

We denote by $L(A) = \sup \mathcal{L}(A)$ the longest (if finite) factorization length of A and by $l(A) = \min \mathcal{L}(A)$ the minimum factorization length of A. The *elasticity* of A, denoted by

$$\varrho(A) = \frac{L(A)}{l(A)},$$

gives a coarse measure of how far away A is from having a unique factorization. It is not hard to see that if A has a unique factorization $A = A_1 A_2 \dots A_t$, then $\mathcal{L}(A) = \{t\}$ and so

$$l(A) = L(A) = t \quad \text{and} \quad \varrho(A) = \frac{t}{t} = 1.$$

The *elasticity of the semigroup* S, denoted by $\varrho(S)$, is given by

$$\varrho(S) = \sup\{\varrho(A)\colon\ A \in S\}.$$

If $\mathcal{L}(A) = \{t_1, t_2, \ldots\}$ with $t_i < t_{i+1}$ for each i, then the Delta set of A is given by

$$\Delta(A) = \{t_{i+1} - t_i \colon\ t_i, t_{i+1} \in \mathcal{L}(A)\}$$

and $\Delta(S) = \bigcup\limits_{A \in S} \Delta(A)$.

An atomic semigroup is called *bifurcus* provided $l(A) = 2$ for every non unit non atom A of S. By [3], we know that if S is bifurcus, then $\varrho(S) = \infty$ and $\Delta(S) = \{1\}$.

This paper will be divided into five sections. In Section 2, we will consider the semigroup S of upper triangular $n \times n$ ($n \geqslant 2$) matrices with nonnegative entries and nonzero determinant. In particular, we investigate the atoms of S, provide a formula for $\varrho(S)$, and, given $A \in S$, provide formulas for $l(A)$, $L(A)$ and $\varrho(A)$. As a consequence, open problem 4 in [2] is partly answered. In Section 3 we study the semigroup

$$S = \left\{ \begin{pmatrix} 1 & c \\ 0 & b \end{pmatrix} :\ b \in \mathbb{N},\ c \in \mathbb{N}_0 \right\}.$$

For any $A \in S$, we give a method of calculating $l(A)$, $\varrho(A)$ and $\Delta(A)$. Also, some special cases of open problem 2 in [2] are discussed and answered. In Section 4 and 5, we study the upper triangular matrices with positive entries. Section 4 investigates some special subsemigroups of $T_n(\mathbb{N})$ for $n > 2$ which are bifurcus, and we compute invariants of such semigroups. Consequently, some special cases of open Problem 3 in [2] are partly answered. In Section 5 we study a special class of matrices in $T_2(\mathbb{N})$ and also partly answer open Problem 1 in [2].

2. Subsemigroups of $T_n(\mathbb{N}_0)$

In this section we consider the semigroup S of upper triangular $n \times n$ ($n \geqslant 2$) matrices with nonnegative entries and nonzero determinant. In this case, I_n is the only unit of S.

For each pair $i, j \in \{1, 2, \ldots, n\}$, let E_{ij} denote the matrix whose only nonzero entry is $e_{ij} = 1$.

In the following theorem we characterize the atoms of S.

Theorem 2.1. *Let S denote the subsemigroup of the matrices in $T_n(\mathbb{N}_0)$ with nonzero determinant. The set of atoms of S consists of the matrices $X_{ij} = I + E_{ij}$ for each pair i and j with $1 < i < j < n$ and, for each prime p, the matrices $Y_{ii} = I + (p-1)E_{ii}$ for $1 \leqslant i \leqslant n$.*

Proof. Suppose that $X_{ij} = X_1X_2$ for some $X_1, X_2 \in S$. Since $\det(X_{ij}) = 1$, $\det(X_1) = \det(X_2) = 1$ and we can write

$$X_{ij} = X_1X_2 = \begin{pmatrix} 1 & b_{12} & \dots & b_{1n} \\ 0 & 1 & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & c_{12} & \dots & c_{1n} \\ 0 & 1 & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

where $\sum_{k=1}^{n} b_{ik}c_{kj} = 1$ and $\sum_{k=1}^{n} b_{hk}c_{kl} = 0$ if $h \neq i$ and $l \neq j$. As a result, either X_1 or X_2 is the identity and hence X_{ij} is an atom.

Suppose now that p is prime and $Y_{ii} = Y_1Y_2$ for some $i \in \{1, 2, \dots, n\}$, $Y_1, Y_2 \in S$. Since p is prime, either

$$Y_1Y_2 = \begin{pmatrix} 1 & b_{12} & \dots & \dots & \dots & b_{1\,n-1} & b_{1n} \\ 0 & 1 & \dots & \dots & \dots & b_{2\,n-1} & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & p & \dots & b_{i\,n-1} & b_{in} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 1 & b_{n-1\,n} \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_{12} & \dots & \dots & \dots & c_{1\,n-1} & c_{1n} \\ 0 & 1 & \dots & \dots & \dots & c_{2\,n-1} & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \dots & c_{i\,n-1} & c_{in} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 1 & c_{n-1\,n} \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

or

$$Y_1Y_2 = \begin{pmatrix} 1 & b_{12} & \dots & \dots & \dots & b_{1\,n-1} & b_{1n} \\ 0 & 1 & \dots & \dots & \dots & b_{2\,n-1} & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \dots & b_{i\,n-1} & b_{in} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 1 & b_{n-1\,n} \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c_{12} & \dots & \dots & \dots & c_{1\,n-1} & c_{1n} \\ 0 & 1 & \dots & \dots & \dots & c_{2\,n-1} & c_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & p & \dots & c_{i\,n-1} & c_{in} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 1 & c_{n-1\,n} \\ 0 & 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix}.$$

In either case, $b_{ij} = c_{ij} = 0$ for $1 \leqslant i < j \leqslant n$. Consequently, either Y_1 or Y_2 is the identity and hence Y_{ii} is an atom of S.

Finally, we will show that these are the only atoms of S.

For any

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1\,n-1} & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix} \in S$$

we can write

$$
\begin{aligned}
A = \prod_{j=1}^{n}\prod_{i=1}^{n-j} & [I + (a_{n+1-j,n+1-j} - 1)E_{n+1-j,n+1-j}] \\
& \times [I + (a_{n+1-j-i,n+1-j} - 1)E_{n+1-j-i,n+1-j}] \\
= & [I + (a_{nn} - 1)E_{nn}][I + a_{n-1,n}E_{n-1,n}]\dots[I + a_{1n}E_{1n}] \\
& \times [I + (a_{n-1,n-1} - 1)E_{n-1,n-1}][I + a_{n-2,n-1}E_{n-2,n-1}] \\
& \times \dots [I + a_{1,n-1}E_{1,n-1}]\dots[I + (a_{22} - 1)E_{22}][I + a_{12}E_{12}][I + (a_{11} - 1)E_{11}].
\end{aligned}
$$

Thus A is an atom if and only if $A = I + E_{ij}$ for each pair i and j with $1 < i < j < n$, or $A = I + (p-1)E_{ii}$ for some prime p and $1 \leqslant i \leqslant n$. □

Recall that a unit triangular matrix is a matrix in $T_n(\mathbb{N}_0)$ whose all diagonal elements are 1's. Denote $\Sigma(A) = \sum_{1 \leqslant i,j \leqslant n} a_{ij}$. By the proof of Theorem 2.1, we can immediately obtain the following corollary which is Corollary 4.5 in [2].

Corollary 2.2. *Let S denote the set of unit triangular matrices in $T_n(\mathbb{N}_0)$ and let $A \in S$. Then A is an atom if and only if $\Sigma(A) = 1$.*

Also, if we take either $n = 2$ or $n = 3$ in Theorem 2.1, then we have the following corollaries, where Corollary 2.3 is Lemma 4.10 in [2].

Corollary 2.3. *Let S denote the subsemigroup of the matrices in $T_2(\mathbb{N}_0)$ with nonzero determinant. The set of atoms of S consists of the matrix $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and, for each prime p, the matrices $Y_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $Z_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$.*

Corollary 2.4. *Let S denote the subsemigroup of the matrices in $T_3(\mathbb{N}_0)$ with nonzero determinant. The set of atoms of S consists of the matrices*

$$
X_{12} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad X_{13} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad X_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
$$

and, for each prime p, the matrices

$$
Y_{11} = \begin{pmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Y_{22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad Y_{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix}.
$$

Hereafter, for any given $A \in T_n(\mathbb{N}_0)$ with nonzero determinant, we let $r(A)$ denote the number of (not necessarily distinct) prime factors of $\det(A)$.

Proposition 2.5. *Let S denote the subsemigroup of the matrices in $T_n(\mathbb{N}_0)$ with nonzero determinant. If A can be factored as $A = A_1 A_2 \dots A_t$ with each A_i an atom of S, then $t = r(A) + k$, where*

$$k = |\{i\colon\ A_i \in \{I + E_{12}, \dots, I + E_{1n}, I + E_{23}, \dots, I + E_{2n}, \dots, I + E_{n-1,n}\}\}|.$$

P r o o f. For each i, A_i is an atom and thus $\det(A_i)$ is either 1 or is prime. Since

$$\det(A) = \det(A_1)\det(A_2)\dots\det(A_t),$$

we have

$$|\{i\colon\ \det(A_i) \text{ is prime}\}| = r(A).$$

If $k = |\{i\colon\ \det(A_i) = 1\}|$, then the length of this factorization of A is

$$t = |\{i\colon\ \det(A_i) \text{ is prime}\}| + |\{i\colon\ \det(A_i) = 1\}| = r(A) + k.$$

□

Taking either $n = 2$ or $n = 3$ in Proposition 2.5, we have the following corollaries, where Corollary 2.6 is Lemma 4.11 in [2].

Corollary 2.6. *Let S denote the subsemigroup of the matrices in $T_2(\mathbb{N}_0)$ with nonzero determinant. If A can be factored as $A = A_1 A_2 \dots A_t$ with each A_i an atom of S, then $t = r(A) + k$, where*

$$k = \left|\left\{i\colon\ A_i = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}\right|.$$

Corollary 2.7. *Let S denote the subsemigroup of the matrices in $T_3(\mathbb{N}_0)$ with nonzero determinant. If A can be factored as $A = A_1 A_2 \dots A_t$ with each A_i an atom of S, then $t = r(A) + k$, where*

$$k = \left|\left\{i\colon\ A_i \in \left\{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right\}\right\}\right|.$$

Lemma 2.8 ([2], Theorem 4.4)**.** *Let S denote the subsemigroup of $T_n(\mathbb{N}_0)$ of unit triangular matrices and let $A \in S$. Then $L(A) = \Sigma(A)$.*

Theorem 2.9. *Let S denote the subsemigroup of the matrices in $T_n(\mathbb{N}_0)$ with nonzero determinant and let*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\,n-1} & a_{1n} \\ 0 & a_{22} & \dots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix} \in S.$$

(1) $L(A) = r(A) + \Sigma(A)$.

(2) *If A is a diagonal matrix, then $l(A) = r(A) = L(A)$ and $\varrho(A) = 1$.*

(3) *If $a_{ij} > 0$, $a_{ij} \mid a_{ii}a_{jj}$ for some $i, j \in \{1, 2, \dots, n\}$ with $i < j$, and each of the other superdiagonal entries of A is 0, then*

$$l(A) = r(A) + 1, \quad \varrho(A) = \frac{r(A) + a_{ij}}{r(A) + 1} \quad \text{and} \quad \varrho(S) = \infty.$$

Proof. (1) Suppose that $A = A_1 A_2 \dots A_t$ with each A_i an atom of S. By Proposition 2.5, $t = r(A) + k$ where

$$k = |\{i\colon\ A_i \in \{I + E_{12}, \dots, I + E_{1n}, I + E_{23}, \dots, I + E_{2n}, \dots, I + E_{n-1,n}\}\}|.$$

It is not hard to see that

$$k \leqslant a_{12} + a_{13} + \dots + a_{1n} + a_{23} + \dots + a_{2n} + \dots + a_{n-1\,n} = \Sigma(A)$$

and thus

$$L(A) \leqslant r(A) + \Sigma(A). \tag{2.1}$$

Also, by the proof of Theorem 2.1, we know that

$$\begin{aligned} A = {} & \prod_{j=1}^{n} \prod_{i=1}^{n-j} [I + (a_{n+1-j,n+1-j} - 1)E_{n+1-j,n+1-j}] \\ & \times [I + (a_{n+1-j-i,n+1-j} - 1)E_{n+1-j-i,n+1-j}] \\ = {} & [I + (a_{nn} - 1)E_{nn}][I + a_{n-1,n}E_{n-1,n}] \dots [I + a_{1n}E_{1n}] \\ & \times [I + (a_{n-1,n-1} - 1)E_{n-1,n-1}][I + a_{n-2,n-1}E_{n-2,n-1}] \dots [I + a_{1,n-1}E_{1,n-1}] \\ & \times \dots [I + (a_{22} - 1)E_{22}][I + a_{12}E_{12}][I + (a_{11} - 1)E_{11}], \end{aligned}$$

and by Lemma 2.8,

$$L(A) \geqslant L(A_1) + L(A_2) + \dots + L(A_t) = r(A) + \sum_{1 \leqslant i < j \leqslant n} a_{ij} = r(A) + \Sigma(A). \tag{2.2}$$

Thus, combining (2.1) and (2.2), we have $L(A) = r(A) + \Sigma(A)$.

(2) Suppose that A is a diagonal matrix, i.e., $a_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, n\}$ with $i < j$, and write $t = r(A) + k$ as in Proposition 2.5. If $k \geqslant 1$, then A contains at least one factor of $I + E_{12}, \ldots, I + E_{1n}, I + E_{23}, \ldots, I + E_{2n}, \ldots$, or $I + E_{n-1,n}$, and then there is at least one superdiagonal entry of A that is not 0. This contradicts the fact that A is a diagonal matrix. Thus, $t = r(A) = l(A) = L(A)$ and, in this case,

$$\varrho(A) = \frac{L(A)}{l(A)} = 1.$$

(3) Suppose that $a_{ij} > 0$, $a_{ij} \mid a_{ii}a_{jj}$ for some $i, j \in \{1, 2, \ldots, n\}$ with $i < j$, and each of the other superdiagonal entries of A is 0. We write $l(A) = r(A) + k$ as in Proposition 2.5. Notice that $r(A)$ is the number of (not necessarily distinct) prime factors of $\det(A)$ and $a_{ij} > 0$. Then $k \geqslant 1$, and so $l(A) \geqslant r(A) + 1$. Since $a_{ij} > 0$, $a_{ij} \mid a_{ii}a_{jj}$, a_{ij} can be factored as a product of one factor of a_{ii} and one factor of a_{jj}, say $a_{ij} = a'_{ii}a'_{jj}$ where $a_{ii} = m_{ii}a'_{ii}$ and $a_{jj} = n_{jj}a'_{jj}$ for some positive integers m_{ii} and n_{jj}. Factor A as

$$\begin{aligned} A = {} & [I + (a'_{ii} - 1)E_{ii}][I + (n_{jj} - 1)E_{jj}][I + E_{ij}][I + (a_{nn} - 1)E_{nn}] \\ & \times [I + (a_{n-1,n-1} - 1)E_{n-1,n-1}] \ldots [I + (a_{j+1,j+1} - 1)E_{j+1,j+1}] \\ & \times [I + (a_{j-1,j-1} - 1)E_{j-1,j-1}] \ldots [I + (a_{i+1,i+1} - 1)E_{i+1,i+1}] \\ & \times [I + (a_{i-1,i-1} - 1)E_{i-1,i-1}] \ldots [I + (a_{22} - 1)E_{22}] \\ & \times [I + (a_{11} - 1)E_{11}][I + (m_{ii} - 1)E_{ii}][I + (a'_{jj} - 1)E_{jj}]. \end{aligned}$$

From the above factorization of A and (2), it is not hard to see that

$$l(A) \leqslant l(A_1) + l(A_2) + \ldots + l(A_t) = r(A) + 1.$$

Thus, $l(A) = r(A) + 1$.

In this case we immediately get

$$\varrho(A) = \frac{r(A) + \Sigma(A)}{r(A) + 1} = \frac{r(A) + a_{ij}}{r(A) + 1},$$

and hence $\varrho(S) = \infty$. □

Specifically, if $n = 2$ or $n = 3$ in Theorem 2.9, then we have the following corollaries, where Corollary 2.10 is Theorem 4.12 in [2].

Corollary 2.10. *Let S denote the subsemigroup of the matrices in $T_2(\mathbb{N}_0)$ with nonzero determinant and let*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in S.$$

(1) $L(A) = r(A) + a_{12}$.
(2) *If* $a_{12} = 0$, *then* $l(A) = r(A) = L(A)$ *and* $\varrho(A) = 1$.
(3) *If* $a_{12} \mid a_{11}a_{22}$, *then*

$$l(A) = r(A) + 1, \quad \varrho(A) = \frac{r(A) + a_{12}}{r(A) + 1} \quad \text{and} \quad \varrho(S) = \infty.$$

Corollary 2.11. *Let S denote the subsemigroup of the matrices in $T_3(\mathbb{N}_0)$ with nonzero determinant and let*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in S.$$

(1) $L(A) = r(A) + a_{12} + a_{13} + a_{23}$.
(2) *If* $a_{12} = a_{13} = a_{23} = 0$, *then* $l(A) = r(A) = L(A)$ *and* $\varrho(A) = 1$.
(3) *If* $a_{12} \mid a_{11}a_{22}$, $a_{13} = a_{23} = 0$, *then*

$$l(A) = r(A) + 1, \varrho(A) = \frac{r(A) + a_{12}}{r(A) + 1} \quad \text{and} \quad \varrho(S) = \infty.$$

(4) *If* $a_{13} \mid a_{11}a_{33}$, $a_{12} = a_{23} = 0$, *then*

$$l(A) = r(A) + 1, \varrho(A) = \frac{r(A) + a_{13}}{r(A) + 1} \quad \text{and} \quad \varrho(S) = \infty.$$

(5) *If* $a_{23} \mid a_{22}a_{33}$, $a_{12} = a_{13} = 0$, *then*

$$l(A) = r(A) + 1, \varrho(A) = \frac{r(A) + a_{23}}{r(A) + 1} \quad \text{and} \quad \varrho(S) = \infty.$$

Remark 2.12. Theorem 2.9 not only generalizes Theorem 4.4 in [2], but also gives a formula for $\varrho(S)$, and given $A \in S$, provides formulas for $l(A)$, $L(A)$ and $\varrho(A)$. Further, open problem 4 in [2] is partly answered.

3. Subsemigroups of $T_2(\mathbb{N}_0)$

In this section we consider the subsemigroup

$$S = \left\{ \begin{pmatrix} 1 & c \\ 0 & b \end{pmatrix} : b \in \mathbb{N}, c \in \mathbb{N}_0 \right\}$$

of $T_2(\mathbb{N}_0)$.

We will discuss factorizations of the matrices in S in the following three cases.

Case I: $A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$.

By Theorem 4.6 in [2], every factorization of A is unique up to units.

Case II: $A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix}$, where p is prime.

Write $c = kp + i$ where $i \in \{0, 1, 2, \ldots, p-1\}$. By Corollary 2.3, A can be written as

$$A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & kp+i \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \tag{3.1}$$

for some $a, b \in \mathbb{N}_0$, where the matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ need not be atoms.

It follows that

$$c = kp + i = ap + b. \tag{3.2}$$

Now write A as $A = A_1 A_2 \ldots A_t$, where each A_i is an atom. Note that for any given matrix $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in S$, the only factorization is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m$. Then, we have

$$t = a + b + r(A) = a + b + 1.$$

Hence, to calculate $l(A)$, we only need to calculate the minimum of $a + b$. By equation (3.2), we have $kp + i = ap + b$, where $i \in \{0, 1, 2, \ldots, p-1\}$. Thus, if we take $a = j \in [0, k]$, then $b = (k-j)p + i$, and then

$$a + b = j + (k-j)p + i = (1-p)j + kp + i. \tag{3.3}$$

From equation (3.3) we can see that $a + b$ has the minimum $k + i$ when $a = j = k$, $b = (k-j)p + i = (k-k)p + i = i$. Thus

$$l(A) = k + i + r(A) = k + i + 1.$$

Also, one of the factorizations of A with the minimum length $k+i+1$ is

$$A = \begin{pmatrix} 1 & kp+i \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^i.$$

Similarly, we can show that $a+b$ has the maximum $kp+i=c$ when $a=j=0$ and $b=c$.

From the above work, together with Theorem 4.12 in [2], we have the following theorem.

Theorem 3.1. *Let* $A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix} \in T_2(\mathbb{N}_0)$ *where* p *is prime and* $c = kp+i$ *for* $0 \leqslant i < p$. *Then*

$$L(A) = c+1,\ l(A) = k+i+1 \text{ and } \varrho(A) = \frac{c+1}{k+i+1}.$$

Also,

$$A = \begin{pmatrix} 1 & kp+i \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^i$$

is a factorization of A *with minimal length* $l(A) = k+i+1$.

Moreover, by equation (3.3), it is also easy to calculate that

$$\mathcal{L}(A) = \{k+i+1, k+p+i, k+i+2p-1, \ldots, (k-1)p+i+2, kp+i+1\}$$

and thus

$$\Delta(A) = \{p-1\}.$$

Therefore, we immediately have the following theorem.

Theorem 3.2. *Let* $A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix} \in T_2(\mathbb{N}_0)$ *where* p *is prime. Then* $\Delta(A) = \{p-1\}$.

Example 3.3. Let $A = \begin{pmatrix} 1 & 11 \\ 0 & 3 \end{pmatrix} \in T_2(\mathbb{N}_0)$. Since $11 = c = kp+i = 3 \cdot 3 + 2$, by Theorem 3.1 we have

$$L(A) = 11+1 = 12 \quad \text{and} \quad l(A) = k+i+1 = 3+2+1 = 6.$$

Also,

$$A = \begin{pmatrix} 1 & 11 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2$$

is a factorization of A with minimal length $l(A)$.

Remark 3.4. Theorem 3.1 and Theorem 3.2 give formulas for $L(A)$, $l(A)$, $\varrho(A)$ and $\Delta(A)$ for any $A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix} \in T_2(\mathbb{N}_0)$ where p is prime. Consequently, a special case of open problem 2 in [2] is answered.

Case III: $A = \begin{pmatrix} 1 & c \\ 0 & b \end{pmatrix}$, with b not prime.

Write $b = p_1 p_2 \ldots p_t$ where $\{p_1, p_2, \ldots, p_t\}$ is a set of primes with $t > 1$. It is not hard to see that A can be written as

$$A = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p_{i_1} \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p_{i_2} \end{pmatrix} \cdots \begin{pmatrix} 1 & a_t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p_{i_t} \end{pmatrix} \begin{pmatrix} 1 & a_{t+1} \\ 0 & 1 \end{pmatrix},$$

where none of these terms has to be an atom of S, and $\{i_1, i_2, \ldots, i_t\}$ is a permutation of $\{1, 2, \ldots, t\}$. Assume that $c = kb + j$ for $0 \leqslant j < b$. Then we have

$$c = kb + j = ba_1 + p_{i_t} \ldots p_{i_2} a_2 + \ldots + p_{i_t} a_t + a_{t+1}. \tag{3.4}$$

Clearly, there exist $(t+1)$-tuples of integers $(a_1, a_2, \ldots, a_{t+1})$ which satisfy equation (3.4). Further, the set of such $(t+1)$-tuples of integers $(a_1, a_2, \ldots, a_{t+1})$ is finite. Thus the set

$$T = \left\{ \sum_{i=1}^{t+1} a_i \colon\ c = ba_1 + p_{i_t} \ldots p_{i_2} a_2 + \ldots + p_{i_t} a_t + a_{t+1} \right\}$$

is a finite subset of $\mathbb{N}_0$. Write A as $A = A_1 A_2 \ldots A_q$, where each A_i is an atom. Then we have

$$q = \sum_{i=1}^{t+1} a_i + r(A).$$

Thus, to calculate $l(A)$, by Theorem 2.9 we only need to calculate the minimum of the set T. Now, if we denote the minimum by $m = \min T$, then we immediately get

$$l(A) = r(A) + m = t + m.$$

On the other hand, by Theorem 4.12 in [2], for any $A = \begin{pmatrix} 1 & c \\ 0 & b \end{pmatrix} \in T_2(\mathbb{N}_0)$, where $b = p_1 p_2 \ldots p_t$, $p_1, p_2, \ldots, p_t$ are prime, we have $L(A) = t + c$.

Thus, from the above work, we have the following theorem.

Theorem 3.5. *Let* $A = \begin{pmatrix} 1 & c \\ 0 & b \end{pmatrix} \in T_2(\mathbb{N}_0)$*, where* $c = kb + i$ *for* $0 \leqslant i < b$*,* $b = p_1 p_2 \ldots p_t$ *and* $\{p_1, p_2, \ldots, p_t\}$ *is a set of primes with* $t > 1$*. Denote*

$$m = \min\left\{ \sum_{i=1}^{t+1} a_i \colon\ c = ba_1 + p_{i_t} \ldots p_{i_2} a_2 + \ldots + p_{i_t} a_t + a_{t+1} \right\},$$

where $\{i_1, i_2, \ldots, i_t\}$ *is a permutation of* $\{1, 2, \ldots, t\}$. *Then*

$$L(A) = t + c,\ l(A) = t + m,\ \textit{and}\ \varrho(A) = \frac{t+c}{t+m}.$$

Remark 3.6. Theorem 3.5 gives formulas for $L(A)$, $l(A)$ and $\varrho(A)$ for any $A = \begin{pmatrix} 1 & c \\ 0 & b \end{pmatrix} \in T_2(\mathbb{N}_0)$. Consequently, another special case of open problem 2 in [2] is answered. Further, Theorem 3.5 shows us a method how to calculate $l(A)$.

In general, we can calculate $l(A)$ by the following three steps:

Step 1: Factor $b = p_1 p_2 \ldots p_t$ as a product of primes and write $c = kb + i$, where $0 \leqslant i < p$.

Step 2: Find nonnegative integers $a_1, a_2, \ldots, a_t, a_{t+1}$ such that

$$kb + i = ba_1 + p_{i_t} \ldots p_{i_2} a_2 + \ldots + p_{i_t} a_t + a_{t+1},$$

where $\{i_1, i_2, \ldots, i_t\}$ is a permutation of $\{1, 2, \ldots, t\}$.

Step 3: Calculate the minimum m of the set

$$\left\{ \sum_{i=1}^{t+1} a_i \colon\ c = ba_1 + p_{i_t} \ldots p_{i_2} a_2 + \ldots + p_{i_t} a_t + a_{t+1} \right\},$$

where $\{i_1, i_2, \ldots, i_t\}$ is a permutation of $\{1, 2, \ldots, t\}$.

We have $l(A) = r(A) + m$.

Example 3.7. Let $A = \begin{pmatrix} 1 & 231 \\ 0 & 20 \end{pmatrix} \in S$. Clearly, $b = 20 = 2 \cdot 2 \cdot 5$ and $c = 231 = 11 \cdot 20 + 11$. Suppose that

$$A = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p_{i_1} \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p_{i_2} \end{pmatrix} \begin{pmatrix} 1 & a_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p_{i_3} \end{pmatrix} \begin{pmatrix} 1 & a_4 \\ 0 & 1 \end{pmatrix}.$$

Then we have $c = 20a_1 + p_{i_2} p_{i_3} a_2 + p_{i_3} a_3 + a_4 = 11 \cdot 20 + 11$.

Case 1: If $p_{i_1} = 5$ and $p_{i_2} = p_{i_3} = 2$, then $c = 20a_1 + 4a_2 + 2a_3 + a_4 = 11 \cdot 20 + 11$, and $m_1 = \min\left\{ \sum_{i=1}^{4} a_i \colon\ c = 20a_1 + 4a_2 + 2a_3 + a_4 \right\} = 11 + 2 + 1 + 1 = 15$.

Case 2: If $p_{i_2} = 5$ and $p_{i_1} = p_{i_3} = 2$, then $c = 20a_1 + 10a_2 + 2a_3 + a_4 = 11 \cdot 20 + 11$, and $m_2 = \min\left\{ \sum_{i=1}^{4} a_i \colon\ c = 20a_1 + 10a_2 + 2a_3 + a_4 \right\} = 11 + 1 + 0 + 1 = 13$.

Case 3: If $p_{i_3} = 5$ and $p_{i_1} = p_{i_2} = 2$, then $c = 20a_1 + 10a_2 + 5a_3 + a_4 = 11 \cdot 20 + 11$, and $m_3 = \min\left\{ \sum_{i=1}^{4} a_i \colon\ c = 20a_1 + 10a_2 + 5a_3 + a_4 \right\} = 11 + 1 + 0 + 1 = 13$.

Hence we have

$$m = \min\left\{\sum_{i=1}^{4} a_i\colon\ c = ba_1 + p_{i_t} \dots p_{i_2} a_2 + p_{i_3} a_3 + a_4\right\} = 13,$$

and

$$l(A) = r(A) + m = 3 + 13 = 16.$$

Further, the factorizations of the minimum length of A are

$$A = \begin{pmatrix} 1 & 231 \\ 0 & 20 \end{pmatrix} = \begin{pmatrix} 1 & 11 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 1 & 231 \\ 0 & 20 \end{pmatrix} = \begin{pmatrix} 1 & 11 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Also, note that

$$L(A) = r(A) + c = 3 + 231 = 234$$

and thus

$$\varrho(A) = \frac{t+c}{t+m} = \frac{234}{16} = \frac{117}{8}.$$

Remark 3.8. The above example also shows that the factorizations of the matrix $A \in T_2(\mathbb{N}_0)$ with the minimum length are not unique in general.

Analogously to the discussions of Theorem 3.5, we obtain the following theorem.

Theorem 3.9. *Let* $A = \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \in T_2(\mathbb{N}_0)$ *where* $c = ka + i$ *for* $0 \leqslant i < a$, $a = p_1 p_2 \dots p_t$ *and* $\{p_1, p_2, \dots, p_t\}$ *is a set of primes with* $t > 1$. *Denote*

$$m = \min\left\{\sum_{i=1}^{t+1} a_i\colon\ c = aa_{t+1} + p_{i_1} \dots p_{i_2} a_t + \dots + p_{i_1} a_2 + a_1\right\}$$

where $\{i_1, i_2, \dots, i_t\}$ *is a permutation of* $\{1, 2, \dots, t\}$. *Then we have*

$$L(A) = t + c,\ l(A) = t + m \text{ and } \varrho(A) = \frac{t+c}{t+m}.$$

4. Subsemigroups of $T_n(\mathbb{N})$

In this section we will study the subsemigroup of upper triangular matrices with only positive integral entries.

Let $n \geqslant 3$ and consider

$$S = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\,n-1} & a_{1n} \\ 0 & a_{22} & \dots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix} \in T_n(\mathbb{N})\colon\ a_{ii} < a_{i\,i+1} < \ldots < a_{in} \text{ for } 2 \leqslant i \leqslant n-1 \right\}.$$

It is not hard to check that S is a subsemigroup of $T_n(\mathbb{N})$.

Theorem 4.1. *Let $n \geqslant 3$ and*

$$S = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\,n-1} & a_{1n} \\ 0 & a_{22} & \dots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix} \in T_n(\mathbb{N})\colon\ a_{ii} < a_{i\,i+1} < \ldots < a_{in} \text{ for } 2 \leqslant i \leqslant n-1 \right\}.$$

Then $A \in S$ is an atom if and only if one of the following conditions holds:

($\mathbf{C}_{i\,i+1}$) $\quad a_{i\,i+1} = 1$ *for* $1 \leqslant i \leqslant n-1$;

($\mathbf{C}_{i\,i+2}$) $\quad 1 \leqslant a_{i\,i+2} \leqslant 2$ *for* $1 \leqslant i \leqslant n-2$;

$\vdots$

($\mathbf{C}_{1n}$) $\quad 1 \leqslant a_{1n} \leqslant n-1$.

P r o o f. Note that the form of the superdiagonal entries of the product A of two elements of S must satisfy $a_{i,i+1} \geqslant 2$ for $1 \leqslant i \leqslant n-1$, $a_{i,i+2} \geqslant 3$ for $1 \leqslant i \leqslant n-2,\ldots$, $a_{1n} \geqslant n$. Thus, if $A \in S$ satisfies any of the above conditions, then it cannot be factored as two elements of S, and A is an atom.

Conversely, suppose that

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\,n-1} & a_{1n} \\ 0 & a_{22} & \dots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1\,n-1} & a_{n-1\,n} \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix} \in S$$

satisfies $a_{i,i+1} \geqslant 2$ for $1 \leqslant i \leqslant n-1$, $a_{i,i+2} \geqslant 3$ for $1 \leqslant i \leqslant n-2$, …, $a_{1n} \geqslant n$. Then we can factor A as $A = BC$, where

$$B = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 0 & a_{22} & a_{23} - a_{22} & \dots & a_{2\,n-1} - a_{2\,n-2} & a_{2n} - a_{2\,n-1} \\ 0 & 0 & a_{33} & \dots & a_{3\,n-1} - a_{3\,n-2} & a_{3n} - a_{3\,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1\,n-1} & a_{n-1\,n} - a_{n-1\,n-1} \\ 0 & 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix}$$

and

$$C = \begin{pmatrix} a_{11} & a_{12} - 1 & a_{13} - 2 & \dots & a_{1\,n-1} - (n-2) & a_{1\,n} - (n-1) \\ 0 & 1 & 1 & \dots & 1 & 1 \\ 0 & 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

By the previous proof of sufficiency condition of the theorem, we can see that B and C are atoms of S. Thus, A is not an atom of S. □

From the proof of Theorem 4.1, for any $A \in S$ which is not an atom, the superdiagonal entries of A must satisfy the conditions $a_{i,i+1} \geqslant 2$ for $1 \leqslant i \leqslant n-1$, $a_{i,i+2} \geqslant 3$ for $1 \leqslant i \leqslant n-2$,…, $a_{1n} \geqslant n$. And then A can be written as a product of two atoms of S. Thus, we obtain the following theorem.

Theorem 4.2. *Let $n \geqslant 3$ and*

$$S = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1\,n-1} & a_{1n} \\ 0 & a_{22} & \dots & a_{2\,n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{nn} \end{pmatrix} \in T_n(\mathbb{N}) \colon\ a_{ii} < a_{i,i+1} < \ldots < a_{in} \text{ for } 2 \leqslant i \leqslant n-1 \right\}.$$

Then S is bifurcus.

In particular, if $n = 3$ in Theorem 4.2, then we have

Corollary 4.3. *Let*

$$S = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in T_3(\mathbb{N}) \colon\ a_{22} < a_{23} \right\}.$$

Then S is bifurcus.

Remark 4.4. The above theorem gives some invariants of two classes of subsemigroups of $T_n(\mathbb{N})$. Note that if S is bifurcus, then $\varrho(S) = \infty$, and $\Delta(S) = \{1\}$. Consequently, some special cases of open problem 3 in [2] are partly answered.

5. $T_2(\mathbb{N})$

In this section we investigate factorizations of matrices in $T_2(\mathbb{N})$ of the form $\begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix}$ where p is prime.

First, it is not hard to see that A can be written as

$$A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & kp+i \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ for } a, b \geqslant 0, \tag{5.1}$$

where the matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ need not be atoms.

(1) Assume that $c = kp$ for $k > 0$. Then by equation (5.1), we have

$$c = kp = ap + b + 1. \tag{5.2}$$

Now write A as $A = A_1 A_2 \dots A_t$ where each A_i is an atom. Note that for any given matrix $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in T_2(\mathbb{N})$, its only factorization is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m$. Then, together with equation (5.1), we have

$$t = a + b + r(A) = a + b + 1.$$

Hence, to calculate $l(A)$, we only need to calculate the minimum of $a + b$. By equation (5.2), we get

$$a + b = j + (k - j)p - 1 = (1 - p)j + kp - 1 \tag{5.3}$$

where j is any integral value in the interval $[0, k]$. By equation (5.3), we can see that $a + b$ has the minimum $k + p - 1$ when $a = j = k - 1$, $b = (k - j)p - 1 = p - 1$; $a + b$ has the maximum $kp - 1 = c - 1$ when $a = j = 0, b = kp - 1$. Thus,

$$l(A) = k + p - 2 + r(A) = k + p - 1 \text{ and } L(A) = kp = c.$$

Moreover, one of the factorizations of A with minimum length is as follows:

(i) If $k = 1$, then

$$A = \begin{pmatrix} 1 & p \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & p-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{p-1}.$$

(ii) If $k > 1$, then

$$A = \begin{pmatrix} 1 & kp \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & k-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & p-1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k-1} \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{p-1}.$$

(2) Assume that $0 < c = kp + i$ for $1 \leqslant i < p$. Then by equation (5.1), we have

$$c = kp + i = ap + b + 1. \tag{5.4}$$

Write A as $A = A_1 A_2 \dots A_t$, where each A_i is an atom. Then, if A can be also factored (not necessarily as a product of atoms) as (5.1), we have

$$t = a + b + r(A) = a + b + 1.$$

Hence, to calculate $l(A)$, we only need to calculate the minimum of $a + b$. By equation (5.4), we get

$$a + b = j + (k - j)p + i - 1 = (1 - p)j + kp + i - 1, \tag{5.5}$$

where j is any integral value in the interval $[0, k]$. By equation (5.5), $a + b$ has the minimum $k+i-1$ when $a = j = k, b = (k-j)p+i-1 = i-1$; $a+b$ has the maximum $kp+i-1 = c-1$ when $a = j = 0$, $b = (k-j)p+i-1 = (k-0)p+i-1 = kp+i-1$. Thus,

$$l(A) = k + i - 1 \text{ and } L(A) = kp + i = c.$$

Further, one of the factorizations of A with minimum length is as follows:

$$A = \begin{pmatrix} 1 & kp+i \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & i-1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k} \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{i-1}.$$

In particular, when $k = 0, i > 1$, then

$$A = \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & i-1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{i-1};$$

when $k > 0, i = 1$, then

$$A = \begin{pmatrix} 1 & kp+1 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k} \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix}.$$

From the above work, we have the following theorem.

Theorem 5.1. *Let* $A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix} \in T_2(\mathbb{N})$ *where* p *is prime.*

(i) *If* $c = kp$ *for* $k > 0$, *then* $L(A) = c$, $l(A) = k + p - 1$ *and* $\varrho(A) = c/(k + p - 1)$.

(ii) *If* $c = kp + i$ *for* $1 \leqslant i < p$, *then* $L(A) = c$, $l(A) = k + i$ *and* $\varrho(A) = c/(k + i)$.

Example 5.2. Let $A = \begin{pmatrix} 1 & 11 \\ 0 & 3 \end{pmatrix} \in T_2(\mathbb{N})$. Since $11 = c = kp + i = 3 \cdot 3 + 2$, by Theorem 5.1

$$L(A) = c = 11 \text{ and } l(A) = k + i = 3 + 2 = 5.$$

Also,

$$A = \begin{pmatrix} 1 & 11 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^3 \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a factorization of A with minimum length $l(A) = 5$.

Moreover, for any $A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix} \in T_2(\mathbb{N})$ where p is prime we have:
(i) If $c = kp$ for $k > 0$, then

$$\mathcal{L}(A) = \{k + p - 1, k + 2p - 2, k + 3p - 3, \ldots, (k - 1)p + 1, kp\}$$

and

$$\Delta(A) = \{p - 1\}.$$

(ii) If $c = kp + i$ for $1 \leqslant i < p$, then

$$\mathcal{L}(A) = \{k + i, k + i + p - 1, k + i + 2p - 1, \ldots, (k - 1)p + i + 1, kp + i\}$$

and

$$\Delta(A) = \{p - 1\}.$$

Hence, the following theorem is obtained.

Theorem 5.3. *Let* $A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix} \in T_2(\mathbb{N})$ *where* p *is prime. Then* $\Delta(A) = \{p-1\}$.

Remark 5.4. Theorem 5.1 and Theorem 5.3 give formulas for $l(A)$, $\varrho(A)$ and $\Delta(A)$ for any $A = \begin{pmatrix} 1 & c \\ 0 & p \end{pmatrix} \in T_2(\mathbb{N})$, where p is prime. As a consequence, a special case of open Problem 1 in [2] is answered.

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