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Igor Ž. Milovanović; Emina I. Milovanović; Edin Glogić

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ON LAPLACIAN EIGENVALUES OF CONNECTED GRAPHS

Igor Ž. Milovanović, Emina I. Milovanović, Niš,
Edin Glogić, Novi Pazar

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Abstract. Let G be an undirected connected graph with n, $n \geqslant 3$, vertices and m edges with Laplacian eigenvalues $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{n-1} > \mu_n = 0$. Denote by $\mu_I = \mu_{r_1} + \mu_{r_2} + \ldots + \mu_{r_k}$, $1 \leqslant k \leqslant n-2$, $1 \leqslant r_1 < r_2 < \ldots < r_k \leqslant n-1$, the sum of k arbitrary Laplacian eigenvalues, with $\mu_{I_1} = \mu_1 + \mu_2 + \ldots + \mu_k$ and $\mu_{I_n} = \mu_{n-k} + \ldots + \mu_{n-1}$. Lower bounds of graph invariants $\mu_{I_1} - \mu_{I_n}$ and μ_{I_1}/μ_{I_n} are obtained. Some known inequalities follow as a special case.

Keywords: Laplacian eigenvalues; linear spread; ratio spread

MSC 2010: 15A18, 05C50

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be an undirected connected graph with n, $n \geqslant 3$, vertices and m edges. Further, let $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{n-1} > \mu_n = 0$ and $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$, $d_i = d(v_i)$, $i = 1, 2, \ldots, n$, be the Laplacian eigenvalues and the vertex degree sequence of G, respectively. Denote by $S = \{1, 2, \ldots, n-1\}$ an index set, and by $J = \{I = (r_1, r_2, \ldots, r_k);\ 1 \leqslant r_1 < r_2 < \ldots < r_k \leqslant n-1\}$ the set of all subsets of S of cardinality k, $1 \leqslant k \leqslant n-2$. In addition, denote by $\mu_I = \mu_{r_1} + \mu_{r_2} + \ldots + \mu_{r_k}$, $1 \leqslant k \leqslant n-2$, $1 \leqslant r_1 < r_2 < \ldots < r_k \leqslant n-1$ the sum of k arbitrary Laplacian eigenvalues, where $\mu_{I_1} = \mu_1 + \mu_2 + \ldots + \mu_k$ and $\mu_{I_n} = \mu_{n-k} + \ldots + \mu_{n-1}$. It is easy to verify that $\mu_{I_n} \leqslant \mu_I \leqslant \mu_{I_1}$ for each I, $I \in J$. Many results have been obtained for invariants μ_{I_1} and μ_{I_n} (see for example [2], [4],

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[5], [7], [11], [12], [15], [18]). In [15] the following inequalities were proved

$$\mu_{I_1} \leqslant \frac{2mk + \sqrt{k(n-k-1)((n-1)(M_1+2m)-4m^2)}}{n-1}$$

and

$$\mu_{I_n} \geqslant \frac{2mk - \sqrt{k(n-k-1)((n-1)(M_1+2m)-4m^2)}}{n-1}$$

where $M_1 = \sum_{i=1}^{n} d_i^2$ is the first Zagreb index (see [9], [10]). From these inequalities, the following inequalities that determine upper bounds for $\mu_{I_1} - \mu_{I_n}$ and μ_{I_1}/μ_{I_n} can be derived

$$\mu_{I_1} - \mu_{I_n} \leqslant \frac{2\sqrt{k(n-k-1)((n-1)(M_1+2m)-4m^2)}}{n-1} \tag{1.1}$$

and

$$\frac{\mu_{I_1}}{\mu_{I_n}} \leqslant \frac{2mk + \sqrt{k(n-k-1)((n-1)(M_1+2m)-4m^2)}}{2mk - \sqrt{k(n-k-1)(n-1)(M_1+2m)-4m^2)}}. \tag{1.2}$$

In this paper we are going to prove inequalities which are reverse to (1.1) and (1.2), i.e., which set lower bounds for invariants $\mu_{I_1} - \mu_{I_n}$ and μ_{I_1}/μ_{I_n}. We will also point out some inequalities, known the literature, which are obtained as a special case of inequalities proved in the current paper.

2. Main result

We first prove an auxiliary result that will be needed in the subsequent considerations.

Lemma 2.1. *Let G be an undirected connected graph with n, $n \geqslant 3$, vertices and m edges. Then for each k, $1 \leqslant k \leqslant n-2$, the following is valid*

$$\begin{aligned} &\sum_{I\in J} 1 = \binom{n-1}{k}, \\ &\sum_{I\in J} \mu_I = 2m\binom{n-2}{k-1}, \\ &\sum_{I\in J} \mu_I^2 = \frac{\binom{n-2}{k-1}}{n-2}((n-k-1)(M_1+2m)+4m^2(k-1)). \end{aligned} \tag{2.1}$$

P r o o f. The first two inequalities in (2.1) are obvious, so we prove only the last one.

$$\begin{aligned}\sum_{I\in J}\mu_I^2 &= \binom{n-2}{k-1}(\mu_1^2+\ldots+\mu_{n-1}^2)+\binom{n-3}{k-2}(2\mu_1\mu_2+\ldots+2\mu_{n-2}\mu_{n-1})\\ &= \binom{n-2}{k-1}(\mu_1^2+\ldots+\mu_{n-1}^2)+\binom{n-3}{k-2}(\mu_1+\ldots+\mu_{n-1})^2\\ &\quad-\binom{n-3}{k-2}(\mu_1^2+\ldots+\mu_{n-1}^2)\\ &= \binom{n-3}{k-1}(\mu_1^2+\ldots+\mu_{n-1}^2)+\binom{n-3}{k-2}(\mu_1+\ldots+\mu_{n-1})^2.\end{aligned}$$

Since (see for example [1], [13])

$$\sum_{i=1}^{n-1}\mu_i=\sum_{i=1}^{n}d_i=2m \quad\text{and}\quad \sum_{i=1}^{n-1}\mu_i^2=\sum_{i=1}^{n}d_i^2+\sum_{i=1}^{n}d_i=M_1+2m, \tag{2.2}$$

rearranging the last equality we obtain the desired result.

□

Remark 2.1. For $k=1$ from (2.1), the equalities (2.2) are obtained. Also, from the second equality in (2.1) it follows that $\mu_{I_1}\geqslant 2mk/(n-1)$ and $\mu_{I_n}\leqslant 2mk/(n-1)$, for each k, $1\leqslant k\leqslant n-2$. For $k=1$, these inequalities reduce to the well known inequalities $\mu_1\geqslant 2m/(n-1)$ and $\mu_{n-1}\leqslant 2m/(n-1)$.

In the following theorem we prove an inequality reverse to (1.1), which establishes a lower bound for the invariant $\mu_{I_1}-\mu_{I_n}$ in terms of the parameters k,n,m and M_1.

Theorem 2.1. *Let G be an undirected connected graph with n, $n\geqslant 3$, vertices and m edges. Then for each k, $1\leqslant k\leqslant n-2$, the following is valid*

$$\mu_{I_1}-\mu_{I_n}\geqslant\frac{2}{n-1}\sqrt{\frac{k(n-k-1)((n-1)(M_1+2m)-4m^2)}{n-2}}. \tag{2.3}$$

Equality holds if and only if $G\cong K_n$.

P r o o f. If in (see [14])

$$\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}b_i^2-\left(\sum_{i=1}^{n}a_ib_i\right)^2\leqslant\frac{n^2}{4}\,(R_1R_2-r_1r_2)^2$$

we substitute $n := \binom{n-1}{k}$, $1 \leqslant k \leqslant n-2$, $a_i := \mu_I$, $b_i := 1$, $i = 1, 2, \ldots, \binom{n-1}{k}$, $R_1 = \mu_{I_1}$, $r_1 = \mu_{I_n}$, $r_2 = R_2 = 1$, it transforms into

$$\sum_{I \in J} \mu_I^2 \sum_{I \in J} 1 - \Big(\sum_{I \in J} \mu_I\Big)^2 \leqslant \frac{\binom{n-1}{k}^2}{4} (\mu_{I_1} - \mu_{I_n})^2 . \tag{2.4}$$

Now, according to (2.1) the above inequality becomes

$$(\mu_{I_1} - \mu_{I_n})^2 \geqslant \frac{4}{\binom{n-1}{k}^2} \left(\binom{n-1}{k} \binom{n-3}{k} \left(M_1 + 2m + \frac{4m^2(k-1)}{n-k-1} \right) - 4m^2 \binom{n-2}{k-1} \right).$$

Rearranging the last inequality yields the desired result.

Equality in (2.4) holds if and only if $\mu_I = \mu_{I_1} = \mu_{I_n}$, for each $I \in J$, i.e. when $\mu_1 = \mu_2 = \ldots = \mu_{n-1}$. Consequently, equality in (2.3) holds if and only if $G \cong K_n$. □

The following corollary of the inequality (2.3) sets a lower bound of the invariant $\mu_{I_1} - \mu_{I_n}$ in terms of the parameters k, n and m.

Corollary 2.1. *Let G be an undirected connected graph with n, $n \geqslant 3$, vertices and m edges. Then for each k, $1 \leqslant k \leqslant n-2$,*

$$\mu_{I_1} - \mu_{I_n} \geqslant \frac{2}{n-1} \sqrt{\frac{2mk(n-k-1)(n(n-1)-2m)}{n(n-2)}}. \tag{2.5}$$

Equality holds if and only if $G \cong K_n$.

Proof. In [5] it was proved that for each connected simple (n, m)-graph, $n \geqslant 3$, holds

$$M_1 \geqslant \frac{4m^2}{n}. \tag{2.6}$$

The inequality (2.5) is obtained from the inequalities (2.3) and (2.6). □

Corollary 2.2. *Let G be an undirected connected r-regular, $2 \leqslant r \leqslant n-1$, graph with n, $n \geqslant 3$, vertices and m edges. Then for each k, $1 \leqslant k \leqslant n-2$*

$$\mu_{I_1} - \mu_{I_n} \geqslant \frac{2}{n-1} \sqrt{\frac{nkr(n-k-1)(n-r-1)}{n-2}}. \tag{2.7}$$

Equality holds if and only if $r = n-1$, i.e. if $G \cong K_n$.

Remark 2.2. For $k = 1$ from (2.3) the inequality proved in [6], Theorem 4, and [17], Theorem 3.4, is obtained. Also, for $k = 1$ from (2.7) the inequality proved in [17], Corollary 3.2 (see also [8]) is obtained.

The following theorem establishes a lower bound for the invariant μ_{I_1}/μ_{I_n} in terms of the parameters k, n, m and M_1.

Theorem 2.2. *Let G be an undirected connected graph with n, $n \geqslant 3$, vertices and m edges. Then for each k, $1 \leqslant k \leqslant n-2$,*

$$\frac{\mu_{I_1}}{\mu_{I_n}} \geqslant \frac{1}{4m^2}\left(\sqrt{\frac{(n-1)((n-k-1)(M_1+2m)+4m^2(k-1))}{k(n-2)}} + \sqrt{\frac{(n-k-1)((n-1)(M_1+2m)-4m^2)}{k(n-2)}}\right)^2. \tag{2.8}$$

Equality holds if and only if $G \cong K_n$.

Proof. For $n := \binom{n-1}{k}$, $1 \leqslant k \leqslant n-2$, $a_i := 1$, $b_i := \mu_I$, $i = 1, 2, \ldots, \binom{n-1}{k}$, $r = \mu_{I_n}$, $R = \mu_{I_1}$, the inequality (see [3])

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leqslant (r+R)\sum_{i=1}^{n} a_i b_i$$

becomes

$$\sum_{I\in J}\mu_I^2 + \mu_{I_1}\mu_{I_n}\sum_{I\in J} 1 \leqslant (\mu_{I_1}+\mu_{I_n})\sum_{I\in J}\mu_I. \tag{2.9}$$

Using the AG-inequality, i.e., the inequality between arithmetic and geometric means, from (2.9) we obtain

$$2\sqrt{\mu_{I_1}\mu_{I_n}\sum_{I\in J} 1\sum_{I\in J}\mu_I^2} \leqslant (\mu_{I_1}+\mu_{I_n})\sum_{I\in J}\mu_I. \tag{2.10}$$

Bearing in mind the equality (2.1), the above inequality transforms into

$$\sqrt{\frac{\mu_{I_1}}{\mu_{I_n}}} + \sqrt{\frac{\mu_{I_n}}{\mu_{I_1}}} \geqslant \frac{1}{m}\sqrt{\frac{(n-1)((n-k-1)(M_1+2m)+4m^2(k-1))}{k(n-2)}}. \tag{2.11}$$

Since

$$\left(\sqrt{\frac{\mu_{I_1}}{\mu_{I_n}}} + \sqrt{\frac{\mu_{I_n}}{\mu_{I_1}}}\right)^2 = \left(\sqrt{\frac{\mu_{I_1}}{\mu_{I_n}}} - \sqrt{\frac{\mu_{I_n}}{\mu_{I_1}}}\right)^2 + 4,$$

based on (2.11) we have that

$$\sqrt{\frac{\mu_{I_1}}{\mu_{I_n}}} - \sqrt{\frac{\mu_{I_n}}{\mu_{I_1}}} \geqslant \frac{1}{m}\sqrt{\frac{(n-k-1)((n-1)(M_1+2m)-4m^2)}{k(n-2)}}. \tag{2.12}$$

The inequality (2.8) is obtained according to (2.11) and (2.12).

Equality in (2.10) holds if and only if $\mu_I = \mu_{I_1} = \mu_{I_n}$, for each $I \in J$, i.e. for $\mu_1 = \mu_2 = \ldots = \mu_{n-1}$. Consequently, equality in (2.8) holds if and only if $G \cong K_n$. □

Remark 2.3. It is easy to verify that the inequality obtained by rearranging (2.9), using (2.1), is stronger than (2.11) and (2.12).

Remark 2.4. For $k = 1$ from (2.8), the inequality obtained in [16], Theorem 2.3, is recovered. For $k = 1$, from (2.11) the inequality proved in [6], Theorem 3, and [16], Theorem 2.1, is obtained. Also, for $k = 1$ from (2.12) we arrive at the inequality obtained in [16], Theorem 2.2.

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Authors' addresses: Igor Ž. Milovanović, Emina I. Milovanović, Faculty of Electronic Engineering, A. Medvedeva 14, P.O.Box 73, 18000 Niš, Serbia, e-mail: igor@elfak.ni.ac.rs, ema@elfak.ni.ac.rs; Edin Glogić, State University of Novi Pazar, Vuka Karadžića bb, 36300 Novi Pazar, Serbia, e-mail: edin_gl@hotmail.com.