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ON THE DISTRIBUTION OF CONSECUTIVE SQUARE-FREE PRIMITIVE ROOTS MODULO p

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Abstract. A positive integer n is called a square-free number if it is not divisible by a perfect square except 1. Let p be an odd prime. For n with $(n,p)=1$, the smallest positive integer f such that $n^f \equiv 1 \pmod p$ is called the exponent of n modulo p. If the exponent of n modulo p is $p-1$, then n is called a primitive root mod p.

Let $A(n)$ be the characteristic function of the square-free primitive roots modulo p. In this paper we study the distribution

$$\sum_{n\leqslant x} A(n)A(n+1),$$

and give an asymptotic formula by using properties of character sums.

Keywords: square-free; primitive root; square sieve; character sum

MSC 2010: 11N25, 11B50, 11L40

1. Introduction

Let p be an odd prime. For any integer n with $(n,p)=1$, the smallest positive integer f such that $n^f \equiv 1 \pmod p$ is called the exponent of n modulo p. If the exponent of n modulo p is $p-1$, then n is called a primitive root mod p. On the other hand, a positive integer n is called a square-free number if it is not divisible by

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a perfect square except 1. From [7] we know that the number of positive square-free primitive roots modulo p not exceeding x equals

$$\frac{p\varphi(p-1)}{(p^2-1)\zeta(2)}x + O(2^{\omega(p-1)}p^{1/4}(\log p)^{1/2}x^{1/2}), \tag{1.1}$$

where φ is Euler's totient function, ζ is the Riemann zeta function, and $\omega(q)$ denotes the number of the distinct prime factors of q.

H. Liu and W. Zhang [2] improved the error term in (1.1). They showed that the number of positive square-free primitive roots modulo p that are less or equal x is

$$\frac{p\varphi(p-1)}{(p^2-1)\zeta(2)}x + O(p^{9/44+\varepsilon}x^{1/2+\varepsilon}),$$

where ε is any fixed positive number.

In this paper we study the distribution of consecutive square-free primitive roots modulo p and give an asymptotic formula, by using properties of character sums. Our main result is the following.

Theorem 1.1. *Let p be an odd prime, and let $A(n)$ be the characteristic function of the square-free primitive roots modulo p. Then we have*

$$\sum_{n\leqslant x} A(n)A(n+1) = x\,\frac{\varphi^2(p-1)}{(p-1)^2}\frac{p(p-2)}{p^2-2}\prod_{p_1}\Big(1-\frac{2}{p_1^2}\Big)$$
$$+ O(4^{\omega(p-1)}p^{-1/2}(\log p)x + 4^{\omega(p-1)}p^{1/4}(\log p)^{1/2}x^{1/2}\log x),$$

where the O-constant is absolute, and $\prod_{p_1}$ denotes the product over all primes p_1.

From Theorem 1.1 we immediately get a corollary.

Corollary 1.1. *Let p be an odd prime, and let $x \geqslant 1$ be a real number with $p \asymp x^{2/3}$, i.e., $p \ll x^{2/3}$ and $x \ll p^{3/2}$. Then*

$$\sum_{n\leqslant x} A(n)A(n+1) = x\,\frac{\varphi^2(p-1)}{(p-1)^2}\frac{p(p-2)}{p^2-2}\prod_{p_1}\Big(1-\frac{2}{p_1^2}\Big) + O(x^{2/3+\varepsilon}).$$

We will study the distribution of consecutive square-free numbers coprime to p in Section 2, and give some estimates for character sums over consecutive square-free numbers in Section 3. Finally we will prove Theorem 1.1 in Section 4 by using the results of Section 2 and Section 3.

2. Consecutive square-free numbers coprime to p

Let $E(n)$ be the characteristic function of the sequence of square-free numbers. From [5] we know that

$$\sum_{n\leqslant x} E(n) = \frac{6}{\pi^2}x + O(x^{1/2}).$$

L. Mirsky [3] studied the frequency of pairs of square-free numbers with a given difference, and proved the asymptotic formula

$$\sum_{n\leqslant x} E(n)E(n+r) = x\prod_{p}\Big(1-\frac{2}{p^2}\Big)\prod_{p^2|r}\Big(1+\frac{1}{p^2-2}\Big) + O_r(x^{2/3}(\log x)^{4/3}).$$

D. R. Heath-Brown [1] studied the number of consecutive square-free numbers not greater than x, and obtained the following result:

$$\sum_{n\leqslant x} E(n)E(n+1) = x\prod_{p}\Big(1-\frac{2}{p^2}\Big) + O(x^{7/11}(\log x)^7).$$

From [1] we have a lemma.

Lemma 2.1. *Let x and y be real numbers with $y = x^{7/11}(\log x)^6$. Then*

$$xy^{-1}\log y + y\log y + \sum_{\substack{j,k\\ jk>y}}\ \sum_{\substack{n\leqslant x\\ j^2|n\\ k^2|n+1}} 1 \ll x^{7/11}(\log x)^7.$$

Now we study the mean value

$$E(n)E(n+1),$$

by using Heath-Brown's method, and give an asymptotic formula.

Theorem 2.1. *Let p be an odd prime. Then*

$$\sum_{\substack{n\leqslant x\\ (n(n+1),p)=1}} E(n)E(n+1) = x\,\frac{p(p-2)}{p^2-2}\prod_{p_1}\Big(1-\frac{2}{p_1^2}\Big) + O(x^{7/11}(\log x)^7).$$

Proof. Let $\mu(n)$ be the Möbius function. It is not hard to show that

$$E(n) = \sum_{j^2|n}\mu(j).$$

We get

$$\sum_{\substack{n\leqslant x\\(n(n+1),p)=1}} E(n)E(n+1) = \sum_{\substack{n\leqslant x\\(n(n+1),p)=1}} \sum_{j^2|n}\mu(j)\sum_{k^2|n+1}\mu(k) \tag{2.1}$$

$$= \sum_j\sum_k \mu(j)\mu(k)\sum_{\substack{n\leqslant x\\(n(n+1),p)=1\\j^2|n\\k^2|n+1}} 1 = \sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1\\jk\leqslant y}} \mu(j)\mu(k)\sum_{\substack{n\leqslant x\\(n(n+1),p)=1\\j^2|n\\k^2|n+1}} 1$$

$$= \sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1\\jk\leqslant y}} \mu(j)\mu(k)\sum_{\substack{n\leqslant x\\(n(n+1),p)=1\\j^2|n\\k^2|n+1}} 1 + \sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1\\jk>y}} \mu(j)\mu(k)\sum_{\substack{n\leqslant x\\(n(n+1),p)=1\\j^2|n\\k^2|n+1}} 1$$

$$= \Sigma_1 + \Sigma_2.$$

Note that

$$\sum_{\substack{n\leqslant x\\(n(n+1),p)=1\\j^2|n\\k^2|n+1}} 1 = \sum_{\substack{n\leqslant x\\j^2|n\\k^2|n+1}} 1 - \sum_{\substack{n\leqslant x\\p|n\\j^2|n\\k^2|n+1}} 1 - \sum_{\substack{n\leqslant x\\p|n+1\\j^2|n\\k^2|n+1}} 1$$

$$= \frac{x}{j^2k^2} - \frac{x}{j^2k^2p} - \frac{x}{j^2k^2p} + O(1)$$

$$= \frac{x}{j^2k^2}\Big(1-\frac{2}{p}\Big) + O(1).$$

Thus we have

$$\Sigma_1 = x\Big(1-\frac{2}{p}\Big)\sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1\\jk\leqslant y}} \frac{\mu(j)\mu(k)}{j^2k^2} + O\Big(\sum_{jk\leqslant y} 1\Big)$$

$$= x\Big(1-\frac{2}{p}\Big)\sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1}} \frac{\mu(j)\mu(k)}{j^2k^2} + O\Big(x\sum_{n>y}\frac{d(n)}{n^2}\Big) + O\Big(\sum_{n\leqslant y} d(n)\Big),$$

where $d(n)$ is the divisor function.

Noting that

$$\sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1}} \frac{\mu(j)\mu(k)}{j^2k^2} = \frac{p^2}{p^2-2}\prod_{p_1}\Big(1-\frac{2}{p_1^2}\Big),$$

we get

$$\Sigma_1 = x\,\frac{p(p-2)}{p^2-2}\prod_{p_1}\Big(1-\frac{2}{p_1^2}\Big) + O(xy^{-1}\log y) + O(y\log y). \tag{2.2}$$

Now from (2.1), (2.2) and Lemma 2.1 we immediately get

$$\sum_{\substack{n\leqslant x\\(n(n+1),p)=1}} E(n)E(n+1) = x\,\frac{p(p-2)}{p^2-2}\prod_{p_1}\Big(1-\frac{2}{p_1^2}\Big) + O(x^{7/11}(\log x)^7).$$

This proves Theorem 2.1. □

3. Character sums over consecutive square-free numbers

Let $q>2$ be an integer, and let χ be a non-principal character modulo q. From the classical inequality of Pólya-Vinogradov we know that

$$\sum_{n\leqslant x}\chi(n)\leqslant 6\sqrt{q}\log q.$$

M. Munsch [4] studied character sums over square-free numbers, and gave the upper bounds

$$\sum_{n\leqslant x}E(n)\chi(n) \ll \begin{cases} x^{1/2}q^{1/4}(\log q)^{1/2}, \\ x^{1/2}(\log x)q^{3/16+\varepsilon}. \end{cases}$$

Moreover, from Lemma 3 of [6] we know the following estimate for character sums of polynomials.

Lemma 3.1. *Suppose that p is a prime number, χ is a non-principal character modulo p of order d, $f(x)\in\mathbb{F}_p[x]$ has s distinct zeros in $\overline{\mathbb{F}}_p$ and is not a constant multiple of the d-th power of a polynomial over $\mathbb{F}_p$. Let X, Y be real numbers with $0<Y\leqslant p$. Then we have*

$$\Big|\sum_{X<n\leqslant X+Y}\chi(f(n))\Big| < 9sp^{1/2}\log p.$$

In this section we study character sums over consecutive square-free numbers, and give some asymptotic formulas.

Theorem 3.1. *Let p be an odd prime, and let χ_1, χ_2 be non-principal characters modulo p. Then we have*

$$\sum_{n\leqslant x} E(n)\chi_1(n)E(n+1)\chi_2(n+1) \ll \frac{\log p}{p^{1/2}}x + p^{1/4}(\log p)^{1/2}x^{1/2}\log x + x^{7/11}(\log x)^7, \tag{3.1}$$

$$\sum_{\substack{n\leqslant x\\(n+1,p)=1}} E(n)\chi_1(n)E(n+1) \ll p^{1/4}(\log p)^{1/2}x^{1/2}\log x + x^{7/11}(\log x)^7, \tag{3.2}$$

$$\sum_{\substack{n\leqslant x\\(n,p)=1}} E(n)E(n+1)\chi_2(n+1) \ll p^{1/4}(\log p)^{1/2}x^{1/2}\log x + x^{7/11}(\log x)^7. \tag{3.3}$$

Proof. We only prove (3.1), since similarly we can get the other relations. Let y and z be integers with $\sqrt{x/p} < z < \sqrt{x} < y \leqslant x$. It is not hard to show that

$$\sum_{n\leqslant x} E(n)\chi_1(n)E(n+1)\chi_2(n+1) = \sum_{n\leqslant x}\chi_1(n)\chi_2(n+1)\sum_{j^2\mid n}\mu(j)\sum_{k^2\mid n+1}\mu(k)$$

$$= \sum_j\sum_k \mu(j)\mu(k)\sum_{\substack{n\leqslant x\\ j^2\mid n\\ k^2\mid n+1}}\chi_1(n)\chi_2(n+1)$$

$$= \sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1}}\mu(j)\mu(k)\sum_{\substack{n\leqslant x\\ j^2\mid n\\ k^2\mid n+1}}\chi_1(n)\chi_2(n+1)$$

$$= \sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1\\ jk\leqslant\sqrt{x/p}}}\mu(j)\mu(k)\sum_{\substack{n\leqslant x\\ j^2\mid n\\ k^2\mid n+1}}\chi_1(n)\chi_2(n+1) + \sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1\\ \sqrt{x/p}<jk\leqslant z}}\mu(j)\mu(k)\sum_{\substack{n\leqslant x\\ j^2\mid n\\ k^2\mid n+1}}\chi_1(n)\chi_2(n+1)$$

$$+ \sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1\\ z<jk\leqslant\sqrt{x}}}\mu(j)\mu(k)\sum_{\substack{n\leqslant x\\ j^2\mid n\\ k^2\mid n+1}}\chi_1(n)\chi_2(n+1)$$

$$+ \sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1\\ \sqrt{x}<jk\leqslant y}}\mu(j)\mu(k)\sum_{\substack{n\leqslant x\\ j^2\mid n\\ k^2\mid n+1}}\chi_1(n)\chi_2(n+1) + \sum_{\substack{j,k\\(j,k)=1\\(jk,p)=1\\ jk>y}}\mu(j)\mu(k)\sum_{\substack{n\leqslant x\\ j^2\mid n\\ k^2\mid n+1}}\chi_1(n)\chi_2(n+1).$$

Suppose that $n_0 = n_0(j,k)$ is the solution of the congruence equations

$$n \equiv 0 \pmod{j^2}, \quad n \equiv -1 \pmod{k^2}$$

satisfying $1 \leqslant n_0 \leqslant j^2k^2$. We have

$$\sum_{\substack{n \leqslant x \\ j^2 \mid n \\ k^2 \mid n+1}} \chi_1(n)\chi_2(n+1) = \sum_{\substack{n \leqslant x \\ n \equiv n_0 \pmod{j^2k^2}}} \chi_1(n)\chi_2(n+1)$$
$$= \sum_{0 \leqslant m \leqslant (x-n_0)/(j^2k^2)} \chi_1(mj^2k^2+n_0)\chi_2(mj^2k^2+n_0+1).$$

Let χ^* be a character modulo p of order $p-1$. Supposing that $\mathrm{ord}(\chi_1) = d_1$ and $\mathrm{ord}(\chi_2) = d_2$, we have $\chi_1 = (\chi^*)^{a_1(p-1)/d_1}$ for some a_1 with $(a_1, d_1) = 1$ and $\chi_2 = (\chi^*)^{a_2(p-1)/d_2}$ for some a_2 with $(a_2, d_2) = 1$. Hence,

$$\sum_{0 \leqslant m \leqslant (x-n_0)/j^2k^2} \chi_1(mj^2k^2+n_0)\chi_2(mj^2k^2+n_0+1)$$
$$= \sum_{0 \leqslant m \leqslant (x-n_0)/j^2k^2} \chi^*((mj^2k^2+n_0)^{a_1(p-1)/d_1}(mj^2k^2+n_0+1)^{a_2(p-1)/d_2})$$
$$= \sum_{0 \leqslant m \leqslant (x-n_0)/(j^2k^2)} \chi^*(f(m)),$$

where $f(m) = (mj^2k^2+n_0)^{a_1(p-1)/d_1}(mj^2k^2+n_0+1)^{a_2(p-1)/d_2}$. Therefore

$$\sum_{n \leqslant x} E(n)\chi_1(n)E(n+1)\chi_2(n+1)$$
$$\ll \sum_{\substack{j,k \\ (j,k)=1 \\ (jk,p)=1 \\ jk \leqslant \sqrt{x/p}}} \left| \sum_{0 \leqslant m \leqslant (x-n_0)/j^2k^2} \chi^*(f(m)) \right| + \sum_{\substack{j,k \\ (j,k)=1 \\ (jk,p)=1 \\ \sqrt{x/p} < jk \leqslant z}} \left| \sum_{0 \leqslant m \leqslant (x-n_0)/j^2k^2} \chi^*(f(m)) \right|$$
$$+ \sum_{\substack{j,k \\ z < jk \leqslant \sqrt{x}}} \frac{x}{j^2k^2} + \sum_{\substack{j,k \\ (j,k)=1 \\ (jk,p)=1 \\ jk \leqslant y}} 1 + \sum_{\substack{j,k \\ jk > y}} \sum_{\substack{n \leqslant x \\ j^2 \mid n \\ k^2 \mid n+1}} 1.$$

It is obvious that $f(m)$ has two distinct zeros in $\overline{\mathbb{F}}_p$ and is not a constant multiple of the $(p-1)$-st power of a polynomial over $\mathbb{F}_p$. By Lemma 3.1 we have

$$
\begin{aligned}
(3.4)\qquad \sum_{n\leqslant x} E(n)\chi_1(n)E(n+1)\chi_2(n+1) &\ll \sum_{\substack{j,k\\ jk\leqslant\sqrt{x/p}}} \frac{x}{pj^2k^2}p^{1/2}\log p \\
&+ \sum_{\substack{j,k\\ \sqrt{x/p}<jk\leqslant z}} p^{1/2}\log p + xz^{-1}\log z + y\log y + \sum_{\substack{j,k\\ jk>y}}\sum_{\substack{n\leqslant x\\ j^2\mid n\\ k^2\mid n+1}} 1 \\
&\ll \frac{x}{p^{1/2}}\log p + z(\log z)p^{1/2}\log p + xz^{-1}\log z + y\log y + \sum_{\substack{j,k\\ jk>y}}\sum_{\substack{n\leqslant x\\ j^2\mid n\\ k^2\mid n+1}} 1.
\end{aligned}
$$

Taking $z = x^{1/2}/(p^{1/4}(\log p)^{1/2})$, $y = x^{7/11}(\log x)^6$ and applying Lemma 2.1 we get

$$
\begin{aligned}
&\sum_{n\leqslant x} E(n)\chi_1(n)E(n+1)\chi_2(n+1) \\
&\qquad \ll \frac{\log p}{p^{1/2}}x + p^{1/4}(\log p)^{1/2}x^{1/2}\log x + x^{7/11}(\log x)^7.
\end{aligned}
$$

□

4. Proof of Theorem 1.1

Let p be an odd prime, $A(n)$ be the characteristic function of the square-free primitive roots modulo p, and let $E(n)$ be the characteristic function of the square-free numbers. Noting that

$$
\frac{\varphi(p-1)}{p-1}\sum_{d\mid p-1}\frac{\mu(d)}{\varphi(d)}\sum_{\substack{\chi \bmod p\\ \operatorname{ord}(\chi)=d}}\chi(n) = \begin{cases} 1, & \text{if } n \text{ is a primitive root modulo } p,\\ 0, & \text{otherwise.}\end{cases}
$$

We have

$$
\begin{aligned}
\sum_{n\leqslant x} A(n)A(n+1) &= \frac{\varphi^2(p-1)}{(p-1)^2} \\
&\times \sum_{d_1\mid p-1}\frac{\mu(d_1)}{\varphi(d_1)}\sum_{\substack{\chi_1 \bmod p\\ \operatorname{ord}(\chi_1)=d_1}}\sum_{d_2\mid p-1}\frac{\mu(d_2)}{\varphi(d_2)}\sum_{\substack{\chi_2 \bmod p\\ \operatorname{ord}(\chi_2)=d_2}}\sum_{n\leqslant x} E(n)\chi_1(n)E(n+1)\chi_2(n+1)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\varphi^2(p-1)}{(p-1)^2} \sum_{\substack{n\leqslant x\\ (n(n+1),p)=1}} E(n)E(n+1)\\
&\quad + \frac{\varphi^2(p-1)}{(p-1)^2} \sum_{\substack{d_1|p-1\\ d_1>1}} \frac{\mu(d_1)}{\varphi(d_1)} \sum_{\substack{\chi_1 \bmod p\\ \mathrm{ord}(\chi_1)=d_1}} \sum_{\substack{n\leqslant x\\ (n+1,p)=1}} E(n)\chi_1(n)E(n+1)\\
&\quad + \frac{\varphi^2(p-1)}{(p-1)^2} \sum_{\substack{d_2|p-1\\ d_2>1}} \frac{\mu(d_2)}{\varphi(d_2)} \sum_{\substack{\chi_2 \bmod p\\ \mathrm{ord}(\chi_2)=d_2}} \sum_{\substack{n\leqslant x\\ (n,p)=1}} E(n)E(n+1)\chi_2(n+1)\\
&\quad + \frac{\varphi^2(p-1)}{(p-1)^2} \sum_{\substack{d_1|p-1\\ d_1>1}} \frac{\mu(d_1)}{\varphi(d_1)} \sum_{\substack{\chi_1 \bmod p\\ \mathrm{ord}(\chi_1)=d_1}} \sum_{\substack{d_2|p-1\\ d_2>1}} \frac{\mu(d_2)}{\varphi(d_2)}\\
&\qquad\qquad \times \sum_{\substack{\chi_2 \bmod p\\ \mathrm{ord}(\chi_2)=d_2}} \sum_{n\leqslant x} E(n)\chi_1(n)E(n+1)\chi_2(n+1).
\end{aligned}
$$

Then from Theorem 2.1 and Theorem 3.1 we get

$$
\begin{aligned}
&\sum_{n\leqslant x} A(n)A(n+1) = x\,\frac{\varphi^2(p-1)}{(p-1)^2}\,\frac{p(p-2)}{p^2-2} \prod_{p_1} \Big(1-\frac{2}{p_1^2}\Big)\\
&\quad + O(4^{\omega(p-1)}p^{-1/2}(\log p)x + 4^{\omega(p-1)}p^{1/4}(\log p)^{1/2}x^{1/2}\log x + 4^{\omega(p-1)}x^{7/11}(\log x)^7).
\end{aligned}
$$

Noting that

$$
p^{-1/2}(\log p)x + p^{1/4}(\log p)^{1/2}x^{1/2}\log x \gg x^{2/3}(\log x)^{2/3}(\log p)^{2/3},
$$

we immediately conclude

$$
\begin{aligned}
\sum_{n\leqslant x} A(n)A(n+1) &= x\,\frac{\varphi^2(p-1)}{(p-1)^2}\,\frac{p(p-2)}{p^2-2} \prod_{p_1} \Big(1-\frac{2}{p_1^2}\Big)\\
&\quad + O(4^{\omega(p-1)}p^{-1/2}(\log p)x + 4^{\omega(p-1)}p^{1/4}(\log p)^{1/2}x^{1/2}\log x).
\end{aligned}
$$

This completes the proof of Theorem 1.1. □

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