Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 56 (2015), No. 3, 269–285

Persistent URL: http://dml.cz/dmlcz/144344

Comment.Math.Univ.Carolin. 56,3 (2015) 269–285

A co-ideal based identity-summand graph of a commutative semiring

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Abstract. Let I be a strong co-ideal of a commutative semiring R with identity. Let $\Gamma_I(R)$ be a graph with the set of vertices $S_I(R) = \{x \in R \setminus I : x + y \in I$ for some $y \in R \setminus I\}$, where two distinct vertices x and y are adjacent if and only if $x + y \in I$. We look at the diameter and girth of this graph. Also we discuss when $\Gamma_I(R)$ is bipartite. Moreover, studies are done on the planarity, clique, and chromatic number of this graph. Examples illustrating the results are presented.

Keywords: strong co-ideal; Q-strong co-ideal; identity-summand element; identity-summand graph; co-ideal based

Classification: 16Y60, 05C62

1. Introduction

Among the most interesting graphs are the zero-divisor graphs, because these involve both ring theory and graph theory. By studying these graphs, we can gain a broader insight into the concepts and properties that involve both graphs and rings. It was Beck (see [3]) who first introduced the notion of a zero-divisor graph for commutative ring. This notion was later redefined by D. F. Anderson and P. S. Livingston in [1]. In [12], Redmond introduced the zero-divisor graph with respect to a proper ideal. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions (see [2], [11], [12] and [13]). Recently, such graphs are used to study semirings [5], [6] and [9].

Semirings have proven to be useful in theoretical computer science, in particular for studying automata and formal languages, hence, ought to be in the literature [10] and [14]. From now on let R be a commutative semiring with identity. In [8], the present authors introduced the identity-summand graph, denoted by $\Gamma(R)$, such that vertices are all non-identity identity-summands of R and two distinct vertices are joint by an edge when the sum of them is 1. We use the notation $S(R)$ to refer to the set of elements of R that are identity-summands (we use $S^*(R)$ to denote the set of non-identity identity-summands of R), we say that $r \in R$ is an identity-summand of R, if there exists $1 \neq a \in R$ such that $r + a = 1$.

DOI 10.14712/1213-7243.2015.124

In this paper we will generalize this notion by replacing elements whose sum is identity with elements whose sum lies in some strong co-ideal I of R. Indeed, we define an undirected graph $\Gamma_I(R)$ with vertices $S_I(R) = \{x \in R \setminus I : \ x + y \in I$ for some $y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $x + y \in I$. This definition was motivated by [12], [6] and [8]. Here is a brief summary of our paper. We will make an intensive study on identity-summand graph of commutative semirings based on strong co-ideals. In section 2, it is shown that $\Gamma_I(R)$ is connected with $\mathrm{diam}(\Gamma_I(R)) \leq 3$, and if I is a subtractive co-ideal, then $\Gamma_I(R)$ is not complete. We show that if $\Gamma_I(R)$ contains a cycle, then $\mathrm{gr}(\Gamma_I(R)) \leq 4$ and several characterizations of $\Gamma_I(R)$ by girth are given. Also it is proved that if I is a Q-strong co-ideal and $\Gamma_I(R)$ and $\Gamma(R/I)$ has a cycle, then $\mathrm{gr}(\Gamma_I(R)) = \mathrm{gr}(\Gamma(R/I))$. In Section 3, it is shown that for a subtractive strong co-ideal I of R, $\Gamma_I(R)$ is complete bipartite if and only if there exist two distinct prime strong co-ideals P_1 and P_2 of R such that $P_1 \cap P_2 = I$. Section 4 is devoted to study chromatic number, clique number and planar property of $\Gamma_I(R)$.

In order to make this paper easier to follow, we recall various notions which will be used in the sequel. For a graph Γ, we denote by $E(\Gamma)$ and $V(\Gamma)$ the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a, b) = \infty$, also $d(a, a) = 0$). The diameter of graph Γ, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a, b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. We denote the complete graph on n vertices by K_n. The girth of a graph Γ, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in Γ, provided Γ contains a cycle; otherwise $\mathrm{gr}(\Gamma) = \infty$. An edge for which the two ends are the same is called a loop at the common vertex. For r a nonnegative integer, an r-partite graph is one whose set of vertices can be partitioned into r subsets so that no edge has both ends in any single subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with parts of size m and n is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a star graph. We define a coloring of a graph G to be an assignment of colors (elements of some set) to vertices of G, one color to each vertex, so that distinct colors are assigned to adjacent vertices. If n colors are used, then the coloring is referred to as an n-coloring. If there exists an n-coloring of a graph G, then G is called n-colorable. The minimum n for which a graph G is n-colorable is called the chromatic number of G, and is denoted by $\chi(G)$. A clique of a graph is its maximal complete subgraph and the maximal number of vertices in any clique of graph G, denoted by $w(G)$, is called the clique number of G.

A commutative semiring R is defined as an algebraic system $(R, +, \cdot)$ such that $(R, +)$ and $(R, \cdot)$ are commutative semigroups, connected by $a(b + c) = ab + ac$ for all $a, b, c \in R$, and there exists $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$

and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings with non-zero identity.

Definition 1.1. Let R be a semiring.

(1) A non-empty subset I of R is called *co-ideal*, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$ (so $0 \in I$ if and only if $I = R$). A co-ideal I of R is called *strong* co-ideal provided that $1 \in I$ (in this case, $1 + x \in I$ for every $x \in R$).

(2) A co-ideal I of R is called *subtractive* if $x, xy \in I$ implies $y \in I$ (so every subtractive co-ideal is a strong co-ideal).

(3) If I is a co-ideal of R, then the co-rad(I) of I, is the set of all $x \in R$ for which $nx \in I$ for some positive integer n. This is a co-ideal of R containing I [7].

(4) A proper co-ideal P of R is called *prime* if $x + y \in P$ implies $x \in P$ or $y \in P$. The set of all prime co-ideals of R is denoted by co-Spec(R). A proper co-ideal I of R is called *primary* if $a + b \in I$ implies $a \in I$ or $b \in$ co-rad(I). If I is primary, then co-rad(I) is a prime co-ideal. We say that I is P-primary if I is primary and co-rad$(I) = P$ [7].

(5) If D is an arbitrary nonempty subset of R, then the set $F(D)$ consisting of all elements of R of the form $d_1 d_2 \dots d_n + r$ (with $d_i \in D$ for all $1 \leq i \leq n$ and $r \in R$) is a co-ideal of R generated by D [7], [10] and [14].

(6) A semiring R is called *co-semidomain*, if $a + b = 1$ $(a, b \in R)$ implies either $a = 1$ or $b = 1$ [7].

A strong co-ideal I of a semiring R is called a *partitioning strong co-ideal* (= *Q-strong co-ideal*) if there exists a subset Q of R such that the following hold.

(1) $R = \bigcup\{qI : q \in Q\}$, where $qI = \{qt : t \in I\}$.

(2) If $q_1, q_2 \in Q$, then $(q_1 I) \cap (q_2 I) \neq \emptyset$ if and only if $q_1 = q_2$.

(3) For each $q_1, q_2 \in Q$, there exists $q_3 \in Q$ such that $q_1 I + q_2 I \subseteq q_3 I$.

Let I be a Q-strong co-ideal of a semiring R and let $R/I = \{qI : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 I) \oplus (q_2 I) = q_3 I$, where q_3 is the unique element in Q such that $(q_1 I + q_2 I) \subseteq q_3 I$, and $(q_1 I) \odot (q_2 I) = q_3 I$, where q_3 is the unique element in Q such that $(q_1 q_2) I \subseteq q_3 I$ [7]. If q_e is the unique element in Q such that $1 \in q_e I$, then $q_e I = I$ is the identity of R/I. Note that every Q-strong co-ideal is subtractive [7]. Throughout this paper we shall assume unless otherwise stated, that $q_0 I$ (resp. $q_e I$) is the zero element (resp. the identity element) of R/I. In the following, we give an example of a Q-strong co-ideal. One can see another example of Q-strong co-ideal in [7].

Example 1.2. Let R be the set of all non-negative integers. Define $a + b = \gcd(a, b)$ and $a \times b = \operatorname{lcm}(a, b)$ (take $0 + 0 = 0$ and $0 \times 0 = 0$). Then $(R, +, \times)$ is easily checked to be a commutative semiring. Let I be the set of all non-negative odd integers. Then I is a strong co-ideal of R. Set $Q = \{0, 1, 2, 4, 8, 16, 32, 64, ...\}$. It is clear that I is a Q-strong co-ideal.

2. Examples and basic properties of $\Gamma_I(R)$

In this section we study the diameter, girth and cut-point of $\Gamma_I(R)$, when I is a strong co-ideal of the semiring R.

Proposition 2.1. *Let I be a subtractive co-ideal of a semiring R. Then the following hold:*

(1) *if $xy \in I$, then $x, y \in I$ for all $x, y \in R$;*
(2) $I = \text{co-rad}(I)$*;*
(3) $(I : a) = \{r \in R : r + a \in I\}$ *is a subtractive co-ideal of R for all $a \in R$;*
(4) *if I is a Q-strong co-ideal of R and q_eI is the identity element in R/I, then $q_eI \oplus qI = q_eI$ and $qI \oplus qI = qI$ for all $qI \in R/I$.*

Proof: (1) Observe that $1 + x \in I$ for each $x \in R$. If $xy \in I$, then $y(1 + x) = xy + y \in I$ gives $y \in I$, since I is subtractive. Similarly, $x \in I$.

(2) It suffices to show that co-rad$(I) \subseteq I$. Let $x \in$ co-rad(I), so $nx \in I$ for some positive integer $n \in \mathbb{N}$. Thus $nx = x(\underbrace{1 + 1 + \cdots + 1}_{n\ times}) \in I$ gives $x \in I$.

(3) Clearly, $1 \in (I : a)$. If $x, y \in (I : a)$, then $x + a \in I$ and $y + a \in I$, implying $a^2 + ax + ay + xy \in I$. Since $(xy + a)(1 + a)(1 + y)(1 + x) \in I$, $xy + a \in I$ by (1). Thus $xy \in (I : a)$. As I is a co-ideal, $r + x + a \in I$ for each $r \in R$ and so $x + r \in (I : a)$ for each $r \in R$. This shows that $(I : a)$ is a co-ideal of R. Now let $xy, x \in (I : a)$. Then $xy + a + y + xa = (x + 1)(y + a) \in I$, which gives $y + a \in I$, and so $y \in (I : a)$, as desired.

(4) Let $q_eI \oplus qI = q'I$, where q' is the unique element in Q such that $q_eI + qI \subseteq q'I$. Since I is co-ideal, $qI + q_eI \subseteq q_eI \cap q'I$, which gives $q_eI = q'I$. Finally, $qI \oplus qI = qI \odot (q_eI \oplus q_eI) = qI \odot q_eI = qI$. □

Proposition 2.2. *Let I be a strong co-ideal of a semiring R. Then $S_I(R) = \emptyset$ if and only if I is a prime strong co-ideal of R.*

Proof: This follows directly from the definitions. □

Theorem 2.3. *Let I be a Q-strong co-ideal of R. Then the following are equivalent:*

(1) $S_I(R) = \emptyset$*;*
(2) *I is a prime co-ideal of R;*
(3) $S^*(R/I) = \emptyset$*;*
(4) *I is P-primary.*

Proof: (1) $\Leftrightarrow$ (2) follows from Proposition 2.2.

(2) $\Leftrightarrow$ (3) By [[7], Theorem 3.8], I is prime if and only if R/I is co-semidomain. Therefore I is prime if and only if $S^*(R/I) = \emptyset$.

(2) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (2) If I is a P-primary strong co-ideal of R, then $I = \text{co-rad}(I) = P$ by Proposition 2.1(2) and [7, Proposition 2.2]; hence I is prime. □

Redmond [12] explored the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$. He gave an example of rings R, T and ideals $I \trianglelefteq R$, $J \trianglelefteq T$, where $\Gamma(R/I) \cong \Gamma(T/J)$ but $\Gamma_I(R) \ncong \Gamma_J(T)$. Here we generalize this concept to the case of semirings.

Example 2.4. Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$ a semiring with $1_R = X$, where $P(X)$ is the set of all subsets of X. If $I = \{X, \{a, b\}\}$, then I is a Q-strong co-ideal, where $Q = \{q_1 = \{c\}, q_2 = \{a, c\}, q_3 = \{b, c\}, q_e = X\}$. An inspection will show that $q_2I \oplus q_3I = q_eI$ and $S^*(R/I) = \{q_2I, q_3I\}$. Also $S_I(R) = \{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Let $T = \{0, 1, 2, 3, 4, 6, 12\}$. Then $(T, \text{gcd}, \text{lcm})$ (take $\text{gcd}(0, 0) = 0$ and $\text{lcm}(0, 0) = 0$) is a commutative semiring. If $J = \{1, 2\}$, then it easily can be checked that J is a Q-strong co-ideal with $Q = \{0, 1, 3, 4, 12\}$, $T/J = \{0J, 1J, 3J, 4J, 12J\}$, $S^*(T/J) = \{3J, 4J\}$ and $S_J(T) = \{3, 4, 6\}$. Thus $\Gamma(R/I) \cong \Gamma(T/J)$, however $\Gamma_I(R) \ncong \Gamma_J(T)$.

The next several results investigate the relationship between $\Gamma(R/I)$ and $\Gamma_I(R)$.

Proposition 2.5. *Let I be a Q-strong co-ideal of a semiring R and let $x, y \in S_I(R)$ such that $x \in q_1I$ and $y \in q_2I$, for some $q_1, q_2 \in Q$. Then:*

1. *x is adjacent to y in $\Gamma_I(R)$ if and only if q_1I and q_2I are adjacent in $\Gamma(R/I)$ and $q_1 \neq q_2$. In particular, each elements of q_1I are adjacent to each elements of q_2I in $\Gamma_I(R)$.*
2. *If $q_1I \in S^*(R/I)$, then all the distinct elements of q_1I are not adjacent to each other in $\Gamma_I(R)$.*

PROOF: (1) Let x be adjacent to y in $\Gamma_I(R)$, so $x + y \in q_eI = I$. Let $q_1I \oplus q_2I = q_3I$, where q_3 is the unique element in Q such that $q_1I + q_2I \subseteq q_3I$. Since $x + y \in q_3I \cap q_eI$, $q_3 = q_e$. Thus q_1I is adjacent to q_2I in $\Gamma(R/I)$. We show $q_1 \neq q_2$. Suppose, on the contrary, $q_1 = q_2$. Since q_1I and q_2I are adjacent, we have $I = q_eI = q_1I \oplus q_2I = q_1I \oplus q_1I = q_1I$ by Proposition 2.1(4), a contradiction. Thus $q_1 \neq q_2$. Conversely, let q_1I be adjacent to q_2I in $\Gamma(R/I)$, so $q_1I \oplus q_2I = q_eI$, where $(q_1I + q_2I) \subseteq q_eI$. Then $x + y \in q_1I + q_2I \subseteq q_eI = I$; hence x is adjacent to y in $\Gamma_I(R)$. Now, from above discussion, it is clear that each elements of q_1I are adjacent to each elements of q_2I in $\Gamma_I(R)$.

(2) It is similar to the proof of (1). □

An edge for which the two ends are the same is called a loop at the common vertex.

Theorem 2.6. *Let I be a strong co-ideal of a semiring R.*

1. *If I is subtractive, then $\Gamma_I(R)$ has no loop.*
2. *If I is a Q-strong co-ideal and $\Gamma(R/I) \neq \emptyset$, then $\Gamma(R/I)$ has at least two vertices and has no loop.*
3. *If I is subtractive and $a \in R$ is a vertex of $\Gamma_I(R)$ which is adjacent to every other vertex, then $a + a = a$ and $(I : a)$ is a maximal element of the*

set $\Delta = \{(I : x) : x \in R \setminus I\}$ with respect to inclusion. Moreover, $(I : a)$ is a prime co-ideal of R.

Proof: (1) Suppose that $a \in R \setminus I$ with $a + a = a(1 + 1) \in I$. Since I is subtractive $a \in I$, which is a contradiction. So $\Gamma_I(R)$ has no loop.

(2) By Proposition 2.1(4), $\Gamma(R/I)$ has no loop, so it has more than one vertex.

(3) Let $a + a \neq a$. As I is subtractive and $a \notin I$, $a + a \notin I$. Since a is adjacent to every other vertex in $\Gamma_I(R)$, $a + a + x \in I$ for each $x \in S_I(R)$. Thus $a + a \in S_I(R)$. Hence $a + a + a = a(1 + 1 + 1) \in I$ gives $a \in I$, a contradiction. So $a + a = a$. Suppose, on the contrary, $(I : a)$ is not maximal. So there is $x \in R \setminus I$ such that $(I : a) \subset (I : x)$. Since a is adjacent to every other vertex in $\Gamma_I(R)$, $x + a \in I$, which gives $x \in (I : a) \subset (I : x)$. So $x + x \in I$, a contradiction by (1).

Let $x + y \in (I : a)$ be such that $x \notin (I : a)$. So $x + a \notin I$. As $(I : a) \subseteq (I : x + a)$ and $(I : a)$ is maximal in Δ, we have $(I : a) = (I : x + a)$. Since $x + y \in (I : a)$, we get $y \in (I : a + x) = (I : a)$. Thus $(I : a)$ is prime. □

Note that the condition that I is subtractive is necessary in Proposition 2.6 (1) as the following example shows.

Example 2.7. Let $R = (\{0, 1, 2, 3\}, +, \times)$, where

$$a + b = \begin{cases} 3 & \text{if } a, b \neq 0, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0. \end{cases}$$

and $1 \times 1 = 1,\ 2 \times 1 = 1 \times 2 = 2,\ 3 \times 1 = 1 \times 3 = 3,\ 2 \times 2 = 1,\ 2 \times 3 = 3 \times 2 = 3,\ 3 \times 3 = 3$, moreover $r \times 0 = 0 \times r = 0$ for all $r \in R$. Then $I = \{1, 3\}$ is a strong co-ideal of R which is not subtractive because $3, 3 \times 2 \in I$ but $2 \notin I$. It is easy to see that $S_I(R) = \{2\}$ and $\Gamma_I(R)$ has loop.

Theorem 2.8. *Let I be a strong co-ideal of a semiring R. Then the following statements hold.*

(1) *$\Gamma_I(R)$ is connected with $\operatorname{diam}(\Gamma_I(R)) \leq 3$.*

(2) *If I is a subtractive co-ideal of R with $|S_I(R)| \geq 3$ then $\Gamma_I(R)$ is not a complete graph. In particular, $\operatorname{diam}(\Gamma_I(R)) = 2$ or 3.*

Proof: (1) Let $x, y \in S_I(R)$. If $x + y \in I$, then x, y are adjacent and $d(x, y) = 1$. Thus suppose that $x + y \notin I$. By Theorem 2.6(1), $x + x \notin I, y + y \notin I$. As $x, y \in S_I(R)$, $x + a \in I, y + b \in I$ for some $a, b \in R \setminus (I \cup \{x, y\})$. If $a = b$, then $x - a - y$ is a path. If $a \neq b$ and $a + b \in I$, then $x - a - b - y$ is a path. If $a \neq b$ and $a + b \notin I$, then $x - a + b - y$ is a path. Thus $\Gamma_I(R)$ is connected with $\operatorname{diam}\Gamma_I(R) \leq 3$.

(2) Assume that $\Gamma_I(R)$ is complete and let $a, b, c \in S_I(R)$ be distinct elements. Then $a + c, a + b \in I$, so $bc \in (I : a)$, since $(I : a)$ is a strong co-ideal of R by Proposition 2.1(3). If $bc \in I$, then Proposition 2.1(1) gives $b, c \in I$ that is a contradiction. So $bc \notin I$. If $bc = c$, then $c + b = bc + b = b(1 + c) \in I$, implying $b \in I$ by Proposition 2.1, a contradiction. So $bc \neq c$. Since $\Gamma_I(R)$ is complete,

$c(b+1) = bc + c \in I$; hence $c \in I$ which is a final contradiction. Thus $\Gamma_I(R)$ is not complete (so $\mathrm{diam}(\Gamma_I(R)) \neq 1$). Finally, by (1) and Proposition 2.6(1), $\mathrm{diam}(\Gamma_I(R)) = 2$ or 3. □

Note that the condition that I is subtractive is necessary in Theorem 2.8(2), as the following example shows.

Example 2.9. Assume that $R = \{0, 1, 2, 3, 4, 5\}$. Define

$$a + b = \begin{cases} 5 & \text{if } a \neq 0,\ b \neq 0, a \neq b, \\ a & \text{if } a = b, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0. \end{cases}$$

and

$$a * b = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0, \\ 3 & \text{if } a = b = 2, \\ b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ 5 & \text{otherwise.} \end{cases}$$

Then $(R, +, *)$ is easily checked to be a commutative semiring. An inspection will show that $I = \{1, 5\}$ is a co-ideal of R which is not subtractive because $5 * 2 \in I$, $5 \in I$ but $2 \notin I$. Also $S_I(R) = \{2, 3, 4\}$ and $\Gamma_I(R)$ is a complete graph.

A vertex x of a connected graph G is a cut-point of G if there are vertices y and z of G such that x is in every path from y to z (and $x \neq y, x \neq z$). Equivalently, for a connected graph G, x is a cut-point of G if $G - \{x\}$ is not connected.

Example 2.10. Let $R = (\{0, 1, 2, 4, 5, 10, 20, 25, 50, 100\}, \mathrm{gcd}, \mathrm{lcm})$ (take $\mathrm{gcd}(0, 0) = 0$ and $\mathrm{lcm}(0, 0) = 0$) and $I = \{1, 2\}$ be a strong co-ideal of R. Observe that $S_I(R) = \{4, 5, 10, 25, 50\}$. It can be easily seen that 4 is a cut-point of $\Gamma_I(R)$.

In the next theorems, we completely characterize the girth of the graph $\Gamma_I(R)$. A cycle graph or a circular graph is a graph that consists of a single cycle, or in other words, some number of vertices connected in a closed chain.

Theorem 2.11. *Let I be a strong co-ideal of a semiring R.*

(1) *If $\Gamma_I(R)$ contains a cycle, then* $\mathrm{gr}(\Gamma_I(R)) \leq 4$.
(2) *If I is a Q-strong co-ideal such that $\Gamma(R/I)$ and $\Gamma_I(R)$ contain a cycle, then* $\mathrm{gr}(\Gamma_I(R)) = \mathrm{gr}(\Gamma(R/I))$. *Moreover, If $\Gamma(R/I)$ has only two vertices $q_1 I$ and $q_2 I$ with $|q_i I| \geq 2$ ($i = 1, 2$), then* $\mathrm{gr}(\Gamma_I(R)) = 4$.
(3) *If I is a subtractive co-ideal, then the only cycle graph with respect to I is $K_{2,2}$.*

Proof: (1) It is well-known that for any connected graph G, if G contains a cycle, then $\mathrm{gr}(G) \leq 2\mathrm{diam}(G) + 1$. Suppose that $\Gamma_I(R)$ contains a cycle. Hence $\mathrm{gr}(\Gamma_I(R)) \leq 7$. Suppose that $\mathrm{gr}(\Gamma_I(R)) = n$, where $n \in \{5, 6, 7\}$ and let $x_1 - x_2 - \cdots - x_n - x_1$ be a cycle of minimum length. Since x_1 is not adjacent to x_3,

$x_1+x_3 \notin I$. If $x_1+x_3 \neq x_i$ for each $1 \leq i \leq n$, then $x_2-x_3-x_4-x_1+x_3-x_2$ is a 4-cycle, that is, a contradiction. Therefore $x_1+x_3=x_i$ for some $1 \leq i \leq n$. We split the proof into three cases.

Case 1: If $x_1+x_3=x_1$ (resp. $x_1+x_3=x_3$), then $x_1-x_2-x_3-x_4-x_1$ (resp. $x_1-x_2-x_3-x_n-x_1$) is a 4-cycle, a contradiction.

Case 2: If $x_1+x_3=x_2$ (resp. $x_1+x_3=x_4$), then $x_2-x_3-x_4-x_2$ (resp. $x_2-x_3-x_4-x_2$) is a 3-cycle, that is, a contradiction.

Case 3: If $x_1+x_3=x_n$, then $x_2-x_3-x_4-x_n-x_2$ is a 4-cycle, which is a contradiction. Thus, every case leads to a contradiction; hence $\mathrm{gr}(\Gamma_I(R)) \leq 4$.

(2) Assume that $\mathrm{gr}(\Gamma_I(R))=n$ and let $x_1-x_2-\cdots-x_n-x_1$ be a cycle in $\Gamma_I(R)$. Since I is a Q-strong co-ideal, there exist unique elements $q_i \in Q$ $(1 \leq i \leq n)$ such that $x_i \in q_iI$. By Proposition 2.5, $q_1I-q_2I-\cdots-q_nI-q_1I$ is a cycle in $\Gamma(R/I)$; thus $\mathrm{gr}(\Gamma(R/I)) \leq \mathrm{gr}(\Gamma_I(R))$. Now suppose that $\mathrm{gr}(\Gamma(R/I))=m$ and let $q_1I-q_2I-\cdots-q_mI-q_1I$ be a cycle of length m in $\Gamma(R/I)$. Then $q_1-q_2-\cdots-q_m-q_1$ is a cycle of length m in $\Gamma_I(R)$ by Proposition 2.5, so $\mathrm{gr}(\Gamma_I(R)) \leq \mathrm{gr}(\Gamma(R/I))$. Thus $\mathrm{gr}(\Gamma_I(R))=\mathrm{gr}(\Gamma(R/I))$. Let $\Gamma(R/I)$ have only two vertices q_1I and q_2I; we show that $\mathrm{gr}(\Gamma_I(R))=4$. Let $x,y \in S_I(R)$. If x,y are adjacent, then $x \in q_iI$ and $y \in q_jI$, where $i \neq j \in \{1,2\}$, and if x,y are not adjacent, then either $x,y \in q_1I$ or $x,y \in q_2I$ by Proposition 2.5. Also, as q_1I and q_2I are adjacent in $\Gamma(R/I)$, every element of q_1I and q_2I are adjacent in $\Gamma_I(R)$ by Proposition 2.5. Hence $\Gamma_I(R)$ is complete bipartite with two parts q_1I and q_2I. Since $|q_iI| \geq 2$ for $i=1,2$, $\mathrm{gr}(\Gamma_I(R))=4$.

(3) By Theorem 2.8(2), there is no 3-cycle graph. By (1), there are no cycle graph with five or more vertices. So the only cycle graph is $K_{2,2}$. □

Note that the condition that $\Gamma_I(R)$ and $\Gamma(R/I)$ contain cycle in Theorem 2.11(2) is necessary as the following example shows.

Example 2.12. Let R and I be as stated in Example 2.4. As we see $\mathrm{gr}(\Gamma(R/I))=\infty$ and $\mathrm{gr}(\Gamma_I(R)=4$.

For a graph G and vertex $x \in V(G)$, the degree of x, denoted $\deg(x)$, is the number of edges of G incident with x.

Theorem 2.13. *Let I be a subtractive co-ideal of a semiring R. Then the following assertions hold:*

(1) $\mathrm{gr}(\Gamma_I(R))=\infty$ *if and only if* $\Gamma_I(R)$ *is a star graph,*
(2) $\mathrm{gr}(\Gamma_I(R))=4$ *if and only if* $\Gamma_I(R)$ *is bipartite but not a star graph,*
(3) $\mathrm{gr}(\Gamma_I(R))=3$ *if and only if* $\Gamma_I(R)$ *contains an odd cycle,*
(4) *if* $\mathrm{gr}(\Gamma_I(R))=4$, *then there is no end vertex (i.e, vertex with degree 1) in* $\Gamma_I(R)$.

Proof: (1) First suppose that $\mathrm{gr}(\Gamma_I(R))=\infty$ and $\Gamma_I(R)$ is not a star graph. So $|S_I(R)| \geq 4$, because $\Gamma_I(R)$ is not complete by Theorem 2.8(2). Since $\Gamma_I(R)$ is connected, there exists a vertex $x \in S_I(R)$ such that $\deg(x) \geq 2$. As $\Gamma_I(R)$ is not a star graph, there exists a path of the form $a-x-b-c$ in $\Gamma_I(R)$ for some

$a, b, c \in S_I(R)$. If a is adjacent to c, then $a - x - b - c - a$ is a cycle in $\Gamma_I(R)$, a contradiction. If a is not adjacent to c, then $a + c \notin I$. Since $a + c + x \in I$, $a+c \in S_I(R)$ and $x-a+c-b-x$ is a cycle which is a contradiction. Thus $\Gamma_I(R)$ is a star graph. The other implication is clear.

(2) Let $\mathrm{gr}(\Gamma_I(R)) = 4$. So $\Gamma_I(R)$ is not a star graph by (1). It is known that a graph is bipartite if and only if it contains no odd cycle [[4], Theorem 4.7]. Thus it suffices to show that $\Gamma_I(R)$ has no odd cycle. Assume that $x_1 - x_2 - \cdots - x_n - x_1$ is an odd cycle of minimal length n in $\Gamma_I(R)$. Since $\mathrm{gr}(\Gamma_I(R)) = 4$, $n \geq 5$. As $\mathrm{gr}(\Gamma_I(R)) \neq 3$, x_2 is not adjacent to x_4, and so $x_2+x_4 \notin I$. Since $x_2+x_4+x_1 \in I$, $x_2 + x_4 \in S_I(R)$. It follows that $x_1 - x_2 + x_4 - x_5 - \cdots - x_n - x_1$ is an odd cycle of length $n - 2$ in $\Gamma_I(R)$, a contradiction. Hence $\Gamma_I(R)$ is a bipartite graph. Conversely, let $\Gamma_I(R)$ be bipartite which is not a star graph. Therefore $\Gamma_I(R)$ has no odd cycle, and so $\mathrm{gr}(\Gamma_I(R)) \neq 3$. By (1), $\mathrm{gr}(\Gamma_I(R)) \neq \infty$. Therefore $\mathrm{gr}(\Gamma_I(R)) = 4$ by Theorem 2.11(1).

(3) If $\mathrm{gr}(\Gamma_I(R)) = 3$, then we are done. Conversely, assume that $\Gamma_I(R)$ has an odd cycle. Let $\mathrm{gr}(\Gamma_I(R)) \neq 3$. If $\mathrm{gr}(\Gamma_I(R)) = 4$, then (2) implies that $\Gamma_I(R)$ is a bipartite graph which is not a star graph. Therefore, by [4, Theorem 4.7], $\Gamma_I(R)$ contains no odd cycle, a contradiction. If $\mathrm{gr}(\Gamma_I(R)) = \infty$, then $\Gamma_I(R)$ is a star graph which contradicts our assumption. Therefore $\mathrm{gr}(\Gamma_I(R)) = 3$.

(4) First we show that if $a - b - c - d$ is a path in $\Gamma_I(R)$ such that the edge $b - c$ is not contained in a 3-cycle and a, b, c, d are vertices, then the vertices a and d are distinct and are adjacent to each other. Clearly $a \neq d$. Assume that a, d are not adjacent. Since $a + b \in I$, $(a + d) + b \in I$; hence $a + d \in S_I(R)$. Thus $a + d - b - c - a + d$ is a 3-cycle, a contradiction.

Now let a be an end vertex in $\Gamma_I(R)$ and b be a vertex in $\Gamma_I(R)$ such that a and b are adjacent. Since $\mathrm{gr}(\Gamma_I(R)) < \infty$, $\Gamma_I(R)$ is not a star graph by (1). By Theorem 2.8(1), $\Gamma_I(R)$ is connected, hence there is a path $a - b - c - d$ in $\Gamma_I(R)$ with $c, d \notin \{a, b\}$, since $\Gamma_I(R))$ has at least 4 elements. As $\mathrm{gr}(\Gamma_I(R)) = 4$, the edge $b - c$ is not contained in a 3-cycle. By the above considerations, $a \neq d$ and a, d are adjacent to each other which is contradiction. □

Example 2.14. (1) Let $R = (\{0, 1, 2, 4, 5, 10, 20, 25, 50, 100\}, \mathrm{gcd}, \mathrm{lcm})$ (take $\mathrm{gcd}(0, 0) = 0$ and $\mathrm{lcm}(0, 0) = 0$) and $I = \{1, 2\}$ a strong co-ideal of R. Then $S_I(R) = \{4, 5, 10, 25, 50\}$. It can be easily seen that $\Gamma_I(R)$ is a star graph and $\mathrm{gr}(\Gamma_I(R)) = \infty$ (see Example 2.10).

(2) Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$. Then $I = \{X, \{a, b\}\}$ is a strong co-ideal of R, $\Gamma_I(R)$ is a complete bipartite graph and $\mathrm{gr}(\Gamma_I(R)) = 4$.

(3) Let $R = (\{0, 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}, \mathrm{gcd}, \mathrm{lcm})$. Then $I = \{1, 2\}$ is a strong co-ideal of R and $\Gamma_I(R)$ is a graph with odd cycle. It can be easily seen that $\mathrm{gr}(\Gamma_I(R)) = 3$.

3. Complete r-partite graph

In this section we state some theorems, which characterize the complete bipartite identity-summand graph $\Gamma_I(R)$ with respect to strong co-ideal I of a semiring R.

Theorem 3.1. *Let I be a strong co-ideal of a semiring R. If there exist two prime strong co-ideals P_1 and P_2 of R such that $I = P_1 \cap P_2$, then $\Gamma_I(R)$ is a complete bipartite graph, and the converse is true when I is a subtractive co-ideal of R.*

Proof: We show that $\Gamma_I(R)$ is a complete bipartite graph with two parts $V_1 = P_1 \setminus I$ and $V_2 = P_2 \setminus I$. Let $a, b \in R \setminus I$ with $a + b \in I$; so $a + b \in P_1 \cap P_2$. Since P_1, P_2 are prime and $a, b \notin I$, either $a \in P_1 \setminus I, b \in P_2 \setminus I$ or $a \in P_2 \setminus I,\ b \in P_1 \setminus I$.

Let $a, b \in S_I(R)$ be such that $a \in P_2 \setminus I, b \in P_1 \setminus I$. Then $a + b \in P_1 \cap P_2 = I$; hence a, b are adjacent. Now we show that each two elements of V_i are not adjacent. Let $c, d \in V_1$ (so $c, d \notin I$). If $c + d \in I$, then $c + d \in P_2$ gives $c \in P_2$ or $d \in P_2$. As $c, d \in V_1 \subset P_1$, $c \in I$ or $d \in I$, a contradiction. Similarly, each two elements of V_2 are not adjacent. So $\Gamma_I(R)$ is complete bipartite with two parts V_1 and V_2.

Conversely, suppose that I is a subtractive co-ideal and let V_1, V_2 be two parts of $\Gamma_I(R)$. Set $P_1 = V_1 \cup I$ and $P_2 = V_2 \cup I$. One can easily see that $I = P_1 \cap P_2$. First we show that P_1, P_2 are strong co-ideals of R. Let $a, b \in P_1$. If $a, b \in I$, then $ab \in I \subseteq P_1$. So we may assume that $a \notin I$ or $b \notin I$. If $a, b \in V_1$, we have $a + c \in I$ and $b + c \in I$ for each $c \in V_2$, since $\Gamma_I(R)$ is complete bipartite. By Proposition 2.1, $a, b \in (I : c)$ gives $ab \in (I : c)$. If $ab \in I$, then $a \in I$ and $b \in I$ by Proposition 2.1 which is a contradiction. Thus $ab \in S_I(R)$. Since $ab + c \in I$ for each $c \in V_2$, $ab \in V_1$; so $ab \in P_1$. If $a \in V_1$ and $b \in I$, then $a + c, b + c \in I$ for each $c \in V_2$ and $ab \notin I$. As I is subtractive, $ab + c \in I$ by Proposition 2.1, which gives $ab \in V_1$. Now suppose that $a \in P_1$ and $r \in R$; we show that $a + r \in P_1$. If $a \in I$, then $a + r \in I \subseteq P_1$. If $a \in V_1$, then $a + c \in I$ for each $c \in V_2$. Since I is a co-ideal of R, $(a + r) + c \in I$ for each $r \in R$. If $a + r \notin I$, then $a + r \in V_1 \subseteq P_1$(because $c \in V_2$ and $\Gamma_I(R)$ is bipartite). If $a + r \in I$, then $a + r \in P_1$. Therefore P_1 is a co-ideal of R. As I is a strong co-ideal and $1 \in I \subseteq P_1$, P_1 is a strong co-ideal of R. Similarly, P_2 is a strong co-ideal.

Now we claim that P_1 is prime. Let $a + b \in P_1$ such that $a, b \notin P_1$; so $a, b \notin I$. If $a + b \in I$, then either $a \in V_1$ and $b \in V_2$ or $a \in V_2$ and $b \in V_1$ which is a contradiction, since $a, b \notin P_1$. Thus $a + b \notin I$. If $a + b \in V_1$, then $a + b + c \in I$ for each $c \in V_2$. We claim that $b + c \notin I$. If $b + c \in I$, then $c \in V_2$ gives $b \in V_1$, a contradiction. Hence $b + c \notin I$. By the similar way, $a + c \notin I$. Since $a + (b + c) \in I$ and $a \notin V_1$, we have $a \in V_2$ and $b + c \in V_1$. Likewise, $b \in V_2$ and $a + c \in V_1$. Because $a + b + c \in I$, $(a + c) + (b + c) \in I$. It shows that two vertices $a + c$ and $b + c$ of V_1 are adjacent, a contradiction. Thus P_1 is a prime strong co-ideal of R. Similarly, P_2 is a prime strong co-ideal of R. □

Example 3.2. Let $X = \{a, b, c\}$, $R = (P(X), \cup, \cap)$ and $I = \{X, \{b, c\}\}$. Consider

$$P_1 = (I : \{a, b\}) = \{\{c\}, \{b, c\}, \{a, c\}, X\},$$
$$P_2 = (I : \{a, c\}) = \{\{b\}, \{a, b\}, \{c, b\}, X\}.$$

An inspection will show that P_1 and P_2 are prime strong co-ideals of R and $I = P_1 \cap P_2$. It is easy to see that $\Gamma_I(R)$ is a complete bipartite graph with $S_I(R) = \{\{b\}, \{c\}, \{a, b\}, \{a, c\}\}$.

Theorem 3.3. *Let I be a subtractive co-ideal of a semiring R. If $\Gamma_I(R)$ is complete r-partite, then $\Gamma_I(R)$ is a complete bipartite graph.*

Proof: Let $V_1, V_2, \ldots, V_r$ be parts of $\Gamma_I(R)$, $r \geq 3$ and $c_i \in V_i$ for each i. Since $\Gamma_I(R)$ is complete r-partite, $c_1 + c_3 \in I$ and $c_3 + c_2 \in I$. Since $c_1, c_2 \in (I : c_3)$ and $(I : c_3)$ is a strong co-ideal by Proposition 2.1, $c_1c_2 \in (I : c_3)$. As $c_1, c_2 \notin I$, $c_1c_2 \notin I$ by Proposition 2.1(1). We claim that $c_1c_2 \in V_1$. If not, then $c_1(c_2 + 1) = c_1c_2 + c_1 \in I$ because $c_1 \in V_1$. Since I is subtractive, we get $c_1 \in I$, a contradiction. Therefore c_1c_2 and c_1 are not adjacent (because $c_1, c_1c_2 \in V_1$). As $c_2 \in V_2$, $c_2(c_1 + 1) = c_1c_2 + c_2 \in I$. Since I is subtractive, $c_2 \in I$, a contradiction. Hence $r = 2$. □

The connectivity of a graph G, denoted by $k(G)$, is defined to be the minimum number of vertices that are necessary to remove from G in order to produce a disconnected graph.

Theorem 3.4. *Let I be a Q-strong co-ideal of a semiring R. If $\Gamma(R/I)$ is the graph on only two vertices q_1I, q_2I, then*

(1) *$\Gamma_I(R)$ is a complete bipartite graph and $k(\Gamma_I(R)) = \min\{| q_1I |, | q_2I |\}$;*
(2) *$I = P_1 \cap P_2$, where $P_1 = q_1I \cup I$ and $P_2 = q_2I \cup I$ are prime strong co-ideals of R.*

Proof: (1) Since q_1I and q_2I are the only vertices of $\Gamma(R/I)$ and $\Gamma(R/I)$ has no loop, $q_1I \oplus q_2I = I$; so $q_1a + q_2b \in I$ for each $a, b \in I$. Since by Proposition 2.5, all elements of q_1I and q_2I are adjacent and none of elements of q_iI are adjacent together, we get $\Gamma_I(R)$ is a complete bipartite graph. The other statement is clear.

(2) It is clear by the proof of Theorem 3.1. □

Example 3.5. Let $R = (P(X), \cup, \cap)$, where $X = \{a, b, c\}$. By Example 2.12, $I = \{X, \{a, b\}\}$ is a Q-strong co-ideal of R with $Q = \{q_1 = \{c\}, q_2 = \{a, c\}, q_3 = \{b, c\}, q_e = X\}$ and $S^*(R/I) = \{q_2I, q_3I\}$. Since $\Gamma(R/I)$ has only two vertices, $I = P_1 \cap P_2$, where $P_1 = q_2I \cup I$ and $P_2 = q_3I \cup I$. Moreover $k(\Gamma_I(R)) = 2$.

For every nonnegative integer r, the graph G is called r-regular if the degree of each vertex of G is equal to r.

Theorem 3.6. *Let I be a subtractive co-ideal of a semiring R, and let $\Gamma_I(R)$ be a finite regular graph. Then $\Gamma_I(R)$ is $K_{n,n}$ for some $n \in \mathbb{N}$.*

Proof: The proof is similar to [8, Theorem 4.8]. □

4. Chromatic number, clique number and planar property

In this section we collect some basic properties concerning chromatic number and clique number of the graph $\Gamma_I(R)$.

Proposition 4.1. *Let I be a co-ideal of a semiring R.*

(1) *If I is a Q-strong co-ideal, then $w(\Gamma_I(R)) \leq |Q| - 2$.*
(2) *If I is a subtractive co-ideal with $w(\Gamma_I(R))$ being finite, then R has a.c.c on co-ideals of the form $(I : a)$, where $a \in R$. Moreover, if $(I : a_i)$ and $(I : a_j)$ are distinct maximal elements of $\Delta = \{(I : a) : a \in R \setminus I\}$, then a_i is adjacent to a_j in $\Gamma_I(R)$.*

Proof: (1) If $w(\Gamma_I(R)) = \infty$, then Q must be infinite by Proposition 2.5. Assume that $w(\Gamma_I(R)) = n$ and let $x_1, x_2, \ldots, x_n$ be the vertices of the greatest complete subgraph of $\Gamma_I(R)$. Since I is a Q-strong co-ideal, there exist unique elements $q_i \in Q$ such that $x_i \in q_i I$ $(1 \leq i \leq n)$. By Proposition 2.5, $q_i \neq q_0, q_e$ and $q_i \neq q_j$ for each $1 \leq i \neq j \leq n$. Thus $w(\Gamma_I(R)) \leq |Q| - 2$.

(2) The proof of the first statement is similar to [8, Lemma 5.1]. Now, if $(I : a_i)$ and $(I : a_j)$ are distinct maximal elements of $\Delta = \{(I : a) : a \in R \setminus I\}$ (partially ordered by inclusion), then by the usual argument, one can show that $(I : a_i)$ and $(I : a_j)$ are prime. We show $a_i + a_j \in I$. If not, then $(I : a_i) \subseteq (I : a_i + a_j)$ and $(I : a_j) \subseteq (I : a_i + a_j)$, and hence $(I : a_i) = (I : a_i + a_j) = (I : a_j)$, a contradiction. □

Note that the condition that $w(\Gamma_I(R))$ is finite is necessary in Proposition 4.1 as the following example shows:

Example 4.2. Let $X = \{x_i : i \in \mathbb{N}\}$ and $R = (P(X), \cup, \cap)$. Let $X_2 = \{x_i : i \geq 2\}$ and $I = \{X_2, X\}$. It is clear that I is a subtractive co-ideal of R. Set $Y_j = X - \{x_j\}$. Then $A = \{Y_i : i \in \mathbb{N}\}$ is an infinite clique in $\Gamma_I(R)$. An inspection shows that the following chain is infinite:

$$(I : \{x_2\}) \subseteq (I : \{x_2, x_3\}) \subseteq (I : \{x_2, x_3, x_4\}) \subseteq \ldots$$

The next theorem does establish a relation between the clique numbers of $\Gamma_I(R)$ and $\Gamma(R/I)$.

Theorem 4.3. *Let I be a Q-strong co-ideal of a semiring R. Then $w(\Gamma_I(R)) = w(\Gamma(R/I))$.*

Proof: Assume that $\{x_i\}_{i \in J}$ is a clique in $\Gamma_I(R)$ and let q_i be the unique element of Q such that $x_i \in q_i I$ $(i \in J)$. Then $\{q_i I\}_{i \in J}$ is a clique in $\Gamma(R/I)$ by Proposition 2.5. Hence $w(\Gamma(R/I)) \geq w(\Gamma_I(R))$. Now, let $\{q_i I\}_{i \in K}$ be a clique in $\Gamma(R/I)$, then $\{q_i\}_{i \in K}$ is a clique in $\Gamma_I(R)$, by Proposition 2.5. Thus $w(\Gamma_I(R)) \geq w(\Gamma(R/I))$. Therefore, $w(\Gamma_I(R)) = w(\Gamma(R/I))$. □

The next several results investigate the relationship between the chromatic number and clique number of the graph $\Gamma_I(R)$.

Theorem 4.4. *Let R be a semiring and I be a subtractive co-ideal of R. Then the following are equivalent:*

(1) $\chi(\Gamma_I(R))$ *is finite;*
(2) $w(\Gamma_I(R))$ *is finite;*
(3) *the subtractive co-ideal I is a finite intersection of prime co-ideals.*

PROOF: The proof is similar to [8, Theorem 5.2]. □

Remark 4.5. Let P, I be strong co-ideals of a semiring R with P prime and $I \subseteq P$. Then the non-empty set $\Delta = \{P' \in \text{co-Spec}(R) : I \subseteq P' \subseteq P\}$ has a minimal element P_1 with respect to inclusion (by partially ordering Δ by reverse inclusion and using Zorn's Lemma), so P_1 is an element of $\min(I)$, the set of minimal prime strong co-ideals of R containing I. Thus if P is a prime strong co-ideal of the commutative semiring R and P contains the strong co-ideal I of R, then there exists a minimal prime strong co-ideal P' of R with $I \subseteq P' \subseteq P$.

Theorem 4.6. *Let I be a subtractive co-ideal of a semiring R.*

(1) *If $\{P_\alpha\}_{\alpha\in\Lambda}$ is the set of all prime strong co-ideals of R containing I, then $I = \bigcap_{\alpha\in\Lambda} P_\alpha$.*
(2) *If $P_1, \dots, P_n$ are the only distinct minimal prime strong co-ideals of R containing I, then $\bigcap_{i=1}^{n} P_i = I$ and $I \neq \cap_{1\le i\le n, i\neq j} P_i$, for each $1 \le j \le n$.*

PROOF: (1) We need to show that $\cap_{\alpha\in\Lambda} P_\alpha \subseteq I$. Let $x \in \cap_{\alpha\in\Lambda} P_\alpha$ with $x \notin I$. Set $\sum = \{J : J \textit{ is a subtractive co-ideal of } R \textit{ containing } I, x \notin J\}$. Since $I \in \sum$, $\sum \neq \emptyset$. An inspection will show that the partially ordered set $(\sum, \subseteq)$ has a maximal element by Zorn's Lemma, say K. Since $x \notin K$, $K \neq R$. We show that K is prime. Let $a + b \in K$ such that $a \notin K$. Hence $a \in (K : b)$ and $a \notin K$. As $K \subsetneq (K : b)$ and $(K : b)$ is subtractive by Proposition 2.1, $x \in (K : b)$. Hence $b \in (K : x)$. It is clear that $K \subseteq (K : ax)$. If $(K : ax) \neq K$, then $x \in (K : ax)$. Hence $x(1 + a) = x + ax \in K$. Since K is subtractive, $x \in K$, a contradiction. Therefore $K = (K : ax)$. We claim that $(K : ax) = (K : a) \cap (K : x)$. Let $r \in (K : ax)$. Then $ax \in (K : r)$. By Proposition 2.1(3) and 2.1(1), $a, x \in (K : r)$. Thus $r \in (K : x) \cap (K : a)$ and $(K : ax) \subseteq (K : a) \cap (K : x)$. For the reverse of inclusion let $r \in (K : a) \cap (K : x)$. Then $a, x \in (K : r)$. By Proposition 2.1(3), $ax \in (K : r)$ and so $r \in (K : ax)$. Hence the equality holds. As $b \in (K : a) \cap (K : x)$, $b \in K$. Thus K is prime, which implies $x \in K$, a contradiction, as needed.

(2) By (1) and Remark 4.5, $\bigcap_{i=1}^{n} P_i = I$. To see the other statement, suppose $I = \bigcap_{1\le i\le n, i\neq j} P_i$ for some $1 \le j \le n$. Since for each $i \neq j$, $P_i \not\subseteq P_j$, there is $x_i \in P_i$ such that $x_i \notin P_j$. As $\sum_{i\neq j} x_i \in \bigcap_{1\le i\le n, i\neq j} P_i \subseteq P_j$, it is clear that $x_i \in P_j$ for some $i \neq j$, that is a contradiction. Thus $I \neq \bigcap_{1\le i\le n, i\neq j} P_i$ for each $1 \le j \le n$. □

Theorem 4.7. *Let I be a co-ideal of a semiring R.*

(1) *If I is a subtractive co-ideal of R which is not prime, then $w(\Gamma_I(R)) = |\min(I)|$.*

(2) *If I is a Q-strong co-ideal with $|\min(I)|$ finite, then each $P \in \min(I)$ is of the form $P = (I : q)$ for some $q \in Q$.*

(3) *If I is a subtractive co-ideal of a semiring R, then $\chi(\Gamma_I(R)) = w(\Gamma_I(R))$.*

Proof: (1) First, we prove that $|\min(I)| = \infty$ if and only if $w(\Gamma_I(R)) = \infty$. It suffices to show that $|\min(I)|$ is finite if and only if $w(\Gamma_I(R))$ is finite. Let $|\min(I)|$ be finite. Let $|\min(I)|$ be finite. Then I is a finite intersection of prime co-ideals by Theorem 4.6; so by Theorem 4.4, $w(\Gamma_I(R))$ is finite. Now assume that $w(\Gamma_I(R))$ is finite. Hence by Theorem 4.4, $I = \bigcap_{i=1}^{n} P_i$ for some prime strong co-ideals P_i of R. Let $\{Q_\alpha\}_{\alpha\in\Lambda} = \min(I)$. For each $\alpha \in \Lambda$, $I \subseteq Q_\alpha$, so $\bigcap_{i=1}^{n} P_i \subseteq Q_\alpha$ for each $\alpha \in \Lambda$. This implies that $P_i \subseteq Q_\alpha$ for some $1 \le i \le n$. Since Q_α is minimal, $P_i = Q_\alpha$. This gives Λ is finite, and so $|\min(I)|$ is finite.

Let $|\min(I)| = n$. By Theorem 4.6(2), there exists $x_j \in (\bigcap_{1\le i\le n, i\ne j} P_i) \setminus P_j$ for each $1 \le j \le n$. Since each P_i is a strong co-ideal of R, $x_i + x_j \in I$; hence $X = \{x_1, x_2, \dots, x_n\}$ is a clique in $\Gamma_I(R)$. Hence $w(\Gamma_I(R)) \ge n$. Now we show that $w(\Gamma_I(R)) \le n$. We do this by induction on n. If $n = 2$, then $\Gamma_I(R)$ is a complete bipartite graph by Theorem 3.1; hence $w(\Gamma_I(R)) = 2$. Suppose $n > 2$ and the result is true for any integer less than n. Let $\{x_1, x_2, \dots, x_m\}$ be a clique in $\Gamma_I(R)$. Thus $x_1 + x_j \in I = \bigcap_{1\le i\le n} P_i$. Without loss of generality, suppose that $x_1 \notin P_1$ and $x_2, x_3, \dots, x_m \in P_1$ and $x_2, \dots, x_m \notin \bigcap_{2\le i\le n} P_i$. Let $J = \bigcap_{2\le i\le n} P_i$. Hence $\{x_2, x_3, \dots, x_m\}$ is a clique in $\Gamma_J(R)$. By induction hypothesis $m-1 \le n-1$ and so $m \le n$.

(2) Let I be a Q-strong co-ideal of R with $|\min(I)| = n$. By Theorem 4.6(2), $I = \bigcap_{i=1}^{n} P_i$ where $\min(I) = \{P_1, \dots, P_n\}$. Then by (1), $w(\Gamma_I(R)) = n$. Let $\{x_1, x_2, \dots, x_n\}$ be a clique in $\Gamma_I(R)$ where $x_j \in (\bigcap_{1\le i\le n, i\ne j} P_i) \setminus P_j$. Since I is a Q-strong co-ideal of R, there exists unique element $q_j \in Q$ such that $x_j \in q_j I$ for each $1 \le j \le n$. As $\{x_1, x_2, \dots, x_n\}$ is a clique in $\Gamma_I(R)$, $\{q_1, q_2, \dots, q_n\}$ is a clique in $\Gamma_I(R)$ by Proposition 2.5. Let $x_j = q_j a_j$ for some $a_j \in I$. We show that $q_j \in \bigcap_{1\le i\le n, i\ne j} P_i \setminus P_j$. It suffices to show that $q_j \notin P_j$ and there is no $i \ne j$ such that $q_j \notin P_i$. If $q_j \in P_j$, then $x_j = q_j a_j \in P_j$, a contradiction (because $a_j \in I \subseteq P_j$). So $q_j \notin P_j$. Also if $q_j \notin P_i$ for some $i \ne j$, then $q_i + q_j \notin P_i$ and hence $q_i + q_j \notin I$, a contradiction (similarly, as $x_i \notin P_i$, $q_i \notin P_i$). Therefore $q_j \in \bigcap_{1\le i\le n, i\ne j} P_i \setminus P_j$. We claim that $(I : q_j) = P_j$. Let $x \in (I : q_j)$. Then $x + q_j \in I$, and so $x + q_j \in P_j$. Since $q_j \notin P_j$, $x \in P_j$. Hence $(I : q_j) \subseteq P_j$. For the reverse of inclusion, let $x \in P_j$. Then $q_j \in \bigcap_{1\le i\le n, i\ne j} P_i \setminus P_j$ gives $x + q_j \in I$. Therefore $P_j \subseteq (I : q_j)$ and we have equality.

(3) By Theorem 4.4, $w(\Gamma_I(R)) = \infty$ if and only if $\chi(\Gamma_I(R)) = \infty$. Hence, we assume that $\chi(\Gamma_I(R))$ is finite. It is known that $w(\Gamma_I(R)) \le \chi(\Gamma_I(R))$. Let $w(\Gamma_I(R)) = n$. By Theorem 4.6, $I = P_1 \cap \cdots \cap P_n$, where for each i, P_i is a minimal prime strong co-ideal. By an argument like that in Theorem 4.4 ((3) $\Rightarrow$ (1)), $\chi(\Gamma_I(R)) \le n$. Therefore $\chi(\Gamma_I(R)) = w(\Gamma_I(R))$. □

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained by replacing edges of this graph with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ [4]. It is natural to ask for which strong co-ideal I of R the $\Gamma_I(R)$ is planar.

Proposition 4.8. *Let I be a Q-strong co-ideal of R.*

(1) *If $\Gamma_I(R)$ is planar, then for each edge q_1I-q_2I of $\Gamma(R/I)$, $|q_iI| \leq 2$ for some $1 \leq i \leq 2$.*
(2) *If $\Gamma_I(R)$ is planar, then $\Gamma(R/I)$ is planar.*

Proof: (1) Assume that $\Gamma_I(R)$ is planar and q_1I and q_2I are two vertices of $\Gamma(R/I)$ such that $|q_iI| \geq 3$ for each $i = 1, 2$. Let $V_1 = \{x_1, x_2, x_3\} \subseteq q_1I$ and $V_2 = \{y_1, y_2, y_3\} \subseteq q_2I$. As q_1I and q_2I are adjacent in $\Gamma(R/I)$, x_i and y_j are adjacent in $\Gamma_I(R)$ by Proposition 2.5. Then V_1 and V_2 are two parts of a complete bipartite graph as a subgraph of $\Gamma_I(R)$. Hence $\Gamma_I(R)$ is not planar.

(2) Let $\Gamma_I(R)$ be planar. By Proposition 2.5, two vertices q_1I and q_2I are adjacent in $\Gamma(R/I)$ if and only if q_1 and q_2 are adjacent in $\Gamma_I(R)$. Hence we can take $\Gamma(R/I)$ as a subgraph of $\Gamma_I(R)$. If $\Gamma(R/I)$ is not planar, then $\Gamma_I(R)$ is not planar, a contradiction. So $\Gamma(R/I)$ is planar. □

The following example shows that the converse of Proposition 4.8 is not true.

Example 4.9. Let $X = \{a, b, c, d\}$ and $R = (P(X), \cup, \cap)$. An inspection will show that

$$P_1 = \{Y \subseteq X | b \in Y\},$$
$$P_2 = \{Y \subseteq X | a \in Y\}$$

are prime strong co-ideals of R. Let $I = P_1 \cap P_2$. By Theorem 3.1, $\Gamma_I(R)$ is a complete bipartite graph with parts V_1 and V_2 and $|V_1| = |P_1 \setminus I| = 4$ and $|V_2| = |P_2 \setminus I| = 4$. Hence $K_{3,3}$ is a subgraph of $\Gamma_I(R)$, and so $\Gamma_I(R)$ is not planar.

Set $Q = \{q_0 = \{d, c\}, q_e = X, q_1 = \{b, c, d\}, q_2 = \{d, c, a\}\}$, then $q_0I = \{\{d, c\}, \{c\}, \{d\}, \emptyset\}$, $q_eI = I$, $q_1I = \{\{b, c, d\}, \{b, c\}, \{b, d\}, \{b\}\}$ and $q_2I = \{\{d, c, a\}, \{d, a\}, \{a, c\}, \{a\}\}$. By usual argument, I is a Q-strong co-ideal of R and $R/I = \{q_0I, q_eI, q_1I, q_2I\}$. Since $q_1 + q_2 \in I$, $q_1I \oplus q_2I = I$ by Proposition 2.5. Hence $S^*(R/I) = \{q_1I, q_2I\}$. Therefore $\Gamma(R/I)$ is planar.

Theorem 4.10. *Let I be a subtractive co-ideal of semiring R. If $|\min(I)| \geq 4$, then $\Gamma_I(R)$ is not planar.*

Proof: If $|\min(I)| \geq 5$, then by Theorem 4.7(1), $w(\Gamma_I(R)) \geq 5$. Hence $\Gamma_I(R)$ is not planar.

If $|\min(I)| = 4$, then Theorem 4.7(1) implies that $w(\Gamma_I(R)) = 4$. Hence there exists $\{x_1, x_2, x_3, x_4\} \subseteq S_I(R)$ such that $\{x_1, \ldots, x_4\}$ forms a clique in $\Gamma_I(R)$. Let $x_{ij} = x_ix_j$, where $1 \leq i, j \leq 4$, $i \neq j$. Suppose that $k \in \{1, 2, 3, 4\} \setminus \{i, j\}$. Since $x_i, x_j \in (I : x_k)$, $x_{ij} \in (I : x_k)$. If $x_{ij} \in I$, then $x_i(x_j + 1) = x_{ij} + x_i \in I$, hence $x_i \in I$, which is a contradiction. This implies that $x_{ij} \in S_I(R)$. We claim that

$x_{ij} \notin \{x_1, x_2, x_3, x_4\}$. Assume that $x_{ij} = x_s$ for some $1 \leq s \leq 4$. If $s = i$, then $x_{ij} + x_j \in I$. This implies that $x_i \in I$ which is a contradiction. Similarly, for $s = j$. If $s \neq j$ and $s \neq i$, then $x_{ij} + x_s \in I$; hence $x_s + x_s \in I$. It follows that $x_s \in I$ by Proposition 2.1(1), a contradiction. Therefore $x_{ij} \notin \{x_1, x_2, x_3, x_4\}$. Let $s \neq k$ and $s, k \in \{1, 2, 3, 4\} - \{i, j\}$. Since $x_{ij} + x_s \in I$ and $x_{ij} + x_k \in I$, we have $x_s, x_k \in (I : x_{ij})$; thus $x_{sk} \in (I : x_{ij})$. Set $V_1 = \{x_1, x_{13}, x_3\}$ and $V_2 = \{x_2, x_{24}, x_4\}$. Then V_1 and V_2 are two parts of a complete 2-partite subgraph of $\Gamma_I(R)$. Therefore $\Gamma(R)$ is not planar. □

In the following example, it is shown that if $|\min(I)| = 3$, then $\Gamma_I(R)$ may be planar.

Example 4.11. (1) Let $R = \{p_1^i p_2^j p_3^k p_4^t : \ i \in \{0,1,2,3\},\ j \in \{0,1,2,3\},\ k \in \{0,1\},\ t \in \{0,1\}\ \} \cup \{0\}$ where p_i's are prime integer. Then $(R, \gcd, \mathrm{lcm})$ is a semiring and $I = \{1, p_4\}$ is a subtractive strong co-ideal of R. Since for each $1 \leq m, n \leq 3$ where $m \neq n$, $\gcd(p_m, p_n) = 1 \in I$, $\{p_1, p_2, p_3\}$ is a clique in $\Gamma_I(R)$ and $w(\Gamma_I(R)) = 3$. Hence $|\min(I)| = 3$ by Theorem 4.7. Set $V_1 = \{p_1, p_1^2, p_1^3\}$ and $V_2 = \{p_2, p_2^2, p_2^3\}$. Then $K_{3,3}$ is a subgraph of $\Gamma_I(R)$ with two parts V_1 and V_2. Hence $\Gamma_I(R)$ is not planar.

(2) Let $R = (\{0, 1, 2, 3, 5, 6, 10, 15, 30\}, \gcd, \mathrm{lcm})$. Then $I = \{1\}$ is a subtractive strong co-ideal of R and $S_I(R) = \{2, 3, 5, 6, 10, 15\}$. By drawing $\Gamma_I(R)$, one can see that $w(\Gamma_I(R)) = 3$. Hence $|\min(I)| = 3$ by Theorem 4.7. Also $\Gamma_I(R)$ is planar.

Remark 4.12. Let I be a subtractive strong co-ideal of a semiring R.

(1) If $|\min(I)| = 1$, then by Theorem 4.6(2), I is a prime strong co-ideal of R. Hence $\Gamma_I(R) = \emptyset$ by Proposition 2.3.

(2) If $|\min(I)| = 2$, then $I = P_1 \cap P_2$ for some prime strong co-ideals P_1 and P_2 by Theorem 4.6. Hence by Theorem 3.1, $\Gamma_I(R)$ is $K_{n,m}$ for some integer n and m, where $|P_1 \setminus I| = n$ and $|P_2 \setminus I| = m$. If $n, m \geq 3$, then $K_{3,3}$ is a subgraph of $\Gamma_I(R)$ and so $\Gamma_I(R)$ is not planar.

(3) If $|\min(I)| \geq 4$, then by Theorem 4.10, $\Gamma_I(R)$ is not planar.

(4) If R and I are the semiring and co-ideal as in Example 4.11(2) , then $|\min(I)| = 3$ and $\Gamma_I(R)$ is planar. However there exist a semiring R and a strong co-ideal I of R that have only three minimal prime co-ideals and $\Gamma_I(R)$ is not planar as Example 4.11(1) shows. It is not entirely clear for us for which strong co-ideals I with $|\min(I)| = 3$, the $\Gamma_I(R)$ is planar.

Acknowledgment. The authors express their deep gratitude to the referee for her/his helpful suggestions for the improvement of this work.

References

[1] Anderson D.F., Livingston P.S., *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434–447.

[2] Akbari S., Maimani H.R., Yassemi S., *When a zero-divisor graph is planar or a complete r-partite graph*, J. Algebra **270** (2003), 169–180.

[3] Beck I., *Coloring of commutative rings*, J. Algebra **116** (1988), 208–226.

[4] Bondy J. A., Murty U.S.R., *Graph Theory*, Graduate Texts in Mathematics, 244, Springer, New York, 2008.

[5] Ebrahimi Atani S., *The zero-divisor graph with respect to ideals of a commutative semiring*, Glas. Mat. **43(63)** (2008), 309–320.

[6] Ebrahimi Atani S., *An ideal-based zero-divisor graph of a commutative semiring*, Glas. Mat. **44(64)** (2009), 141–153.

[7] Ebrahimi Atani S., Dolati Pish Hesari S., Khoramdel M., *Strong co-ideal theory in quotients of semirings*, J. Adv. Res. Pure Math. **5(3)** (2013), 19–32.

[8] Ebrahimi Atani S., Dolati Pish Hesari S., Khoramdel M., *The identity-summand graph of commutative semirings*, J. Korean Math. Soc. **51** (2014), 189-202.

[9] Dolžan D., Oblak P., *The zero-divisor graphs of rings and semirings*, Internat. J. Algebra Comput. **22** (2012), no. 4, 1250033, 20 pp.

[10] Golan J.S., *Semirings and Their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[11] Maimani H.R., Pournaki M.R, Yassemi S., *Zero-divisor graph with respect to an ideal*, Comm. Algebra **34** (2006), 923–929.

[12] Redmond P., *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003), 4425–4443.

[13] Spiroff S., Wickham C., *A zero divisor graph determined by equivalence classes of zero divisors*, Comm. Algebra **39** (2011), 2338–2348.

[14] Wang H., *On rational series and rational language*, Theoret. Comput. Sci. **205** (1998), 329–336.

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(*Received* March 23, 2014, *revised* January 31, 2015)