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Some Results on the Properties of Differential Polynomials Generated by Solutions of Complex Differential Equations

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Abstract

This paper is devoted to considering the complex oscillation of differential polynomials generated by meromorphic solutions of the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$

where $A_i(z)$ $(i = 0, 1, \cdots, k-1)$ are meromorphic functions of finite order in the complex plane.

Key words: Linear differential equations, finite order, hyper-order, exponent of convergence of the sequence of distinct zeros, hyper-exponent of convergence of the sequence of distinct zeros.

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1 Introduction and main results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [7], [16]). In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r,\varphi) = o(T(r,f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure, where $T(r,f)$ is the Nevanlinna characteristic function of f.

Definition 1.1 ([13], [16]) Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}.$$

Definition 1.2 ([7], [12]) The type of a meromorphic function f of order ρ $(0<\rho<\infty)$ is defined by

$$\tau(f) = \limsup_{r\to+\infty} \frac{T(r,f)}{r^{\rho}}.$$

If f is an entire function, then the type of f of order ρ $(0<\rho<\infty)$ is defined by

$$\tau_M(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^{\rho}},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

Remark 1.1 There exist entire functions f which satisfy $\tau_M(f) \neq \tau(f)$. For example, if $f(z) = e^z$, then we have $\tau_M(f) = 1$ and $\tau(f) = \frac{1}{\pi}$.

Definition 1.3 ([13], [16]) Let f be a meromorphic function. Then the hyper-exponent of convergence of zeros sequence of $f(z)$ is defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z\colon |z|\leq r\}$. Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z\colon |z|\leq r\}$.

Consider the complex differential equation

$$f^{(k)} + A(z)f = 0 \tag{1.1}$$

and the differential polynomial

$$g_f = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_0 f, \tag{1.2}$$

where $A(z)$ and $d_j(z)$ $(j = 0,1,\ldots,k)$ are meromorphic functions in the complex plane.

In 2000, Chen [5] first studied the fixed points of solutions of second-order linear differential equations and obtained some precise estimation on the number of fixed points of the solutions. After that, many authors [1, 6, 8, 9, 10, 13, 15] investigated the complex oscillation theory of solutions and differential polynomials generated by solutions of differential equations in $\mathbb{C}$. In [10], the authors investigated the growth and oscillation of differential polynomials generated by solutions of (1.1), and obtained the following results.

Theorem A ([10]) *Let $A(z)$ be a meromorphic function of finite order, and let $d_j(z)$ $(j = 0, 1, \dots, k)$ be finite order meromorphic functions that are not all vanishing identically such that*

$$h = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdot\,\cdot & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdot\,\cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot\,\cdot & \cdot \\ \cdot & \cdot & \cdot\,\cdot & \cdot \\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdot\,\cdot & \alpha_{k-1,k-1} \end{vmatrix} \not\equiv 0,$$

where the sequence of functions $\alpha_{i,j}$ $(i = 0, \dots, k-1;\ j = 0, \dots, k-1)$ are defined by

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1}, & \textit{for all } i = 1, \cdots, k-1, \\ \alpha'_{0,j-1} - A\alpha_{k-1,j-1}, & \textit{for } i = 0 \end{cases}$$

and

$$\alpha_{i,0} = \begin{cases} d_i, & \textit{for all } i = 1, \cdots, k-1, \\ d_0 - d_k A, & \textit{for } i = 0. \end{cases}$$

If $f(z)$ is an infinite order meromorphic solution of (1.1) *with $\rho_2(f) = \rho$, then the differential polynomial* (1.2) *satisfies*

$$\rho(g_f) = \rho(f) = \infty \quad \textit{and} \quad \rho_2(g_f) = \rho_2(f) = \rho.$$

Furthermore, if f is a finite order meromorphic solution such that

$$\rho(f) > \max\{\rho(A), \rho(d_j) \quad (j = 0, 1, \dots, k)\},$$

then

$$\rho(g_f) = \rho(f).$$

Theorem B ([10]) *Under the hypotheses of Theorem A, let $\varphi(z) \not\equiv 0$ be a meromorphic function with finite order such that*

$$\psi(z) = \frac{\begin{vmatrix} \varphi & \alpha_{1,0} & \cdot\,\cdot & \alpha_{k-1,0} \\ \varphi' & \alpha_{1,1} & \cdot\,\cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot\,\cdot & \cdot \\ \cdot & \cdot & \cdot\,\cdot & \cdot \\ \varphi^{(k-1)} & \alpha_{1,k-1} & \cdot\,\cdot & \alpha_{k-1,k-1} \end{vmatrix}}{h(z)}$$

is not a solution of (1.1)*, where* $h \not\equiv 0$ *and* $\alpha_{i,j}$ $(i = 0, \ldots, k-1;\ j = 0, \ldots, k-1)$ *are defined in Theorem A. If* $f(z)$ *is an infinite order meromorphic solution of* (1.1) *with* $\rho_2(f) = \rho$*, then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty$$

and

$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho.$$

Furthermore, if f *is a finite order meromorphic solution such that*

$$\rho(f) > \max\{\rho(A), \rho(\varphi), \rho(d_j)\ (j = 0, 1, \ldots, k)\},$$

then

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f).$$

In this paper, we continue to consider this subject and investigate the complex oscillation theory of differential polynomials generated by meromorphic solutions of differential equations in the complex plane. The main purpose of this paper is to study the controllability of solutions of the differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0. \tag{1.3}$$

In fact, we study the growth and oscillation of higher order differential polynomial (1.2) with meromorphic coefficients generated by solutions of equation (1.3). Before we state our results, we define the sequence of functions $\alpha_{i,j}$ $(i = 0, \ldots, k-1;\ j = 0, \ldots, k-1)$ by

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, & \text{for all } i = 1, \ldots, k-1, \\ \alpha'_{0,j-1} - A_0 \alpha_{k-1,j-1}, & \text{for } i = 0 \end{cases} \tag{1.4}$$

and

$$\alpha_{i,0} = d_i - d_k A_i, \quad \text{for } i = 0, \ldots, k-1. \tag{1.5}$$

We define also h by

$$h = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdot\ \cdot & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdot\ \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdot\ \cdot & \alpha_{k-1,k-1} \end{vmatrix} \tag{1.6}$$

and $\psi(z)$ by

$$\psi(z) = \frac{\begin{vmatrix} \varphi & \alpha_{1,0} & \cdot\ \cdot & \alpha_{k-1,0} \\ \varphi' & \alpha_{1,1} & \cdot\ \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \varphi^{(k-1)} & \alpha_{1,k-1} & \cdot\ \cdot & \alpha_{k-1,k-1} \end{vmatrix}}{h(z)}, \tag{1.7}$$

where $h \not\equiv 0$ and $\alpha_{i,j}$ $(i = 0, \ldots, k-1;\ j = 0, \ldots, k-1)$ are defined in (1.4) and (1.5), and $\varphi \not\equiv 0$ is a meromorphic function with $\rho(\varphi) < \infty$. The following theorems are the main results of this paper.

Theorem 1.1 *Let $A_i(z)$ $(i = 0, 1, \ldots, k-1)$ be meromorphic functions of finite order, and let $d_j(z)$ $(j = 0, 1, \ldots, k)$ be finite order meromorphic functions that are not all vanishing identically such that $h \not\equiv 0$. If $f(z)$ is an infinite order meromorphic solution of* (1.3) *with $\rho_2(f) = \rho$, then the differential polynomial* (1.2) *satisfies*

$$\rho(g_f) = \rho(f) = \infty \quad \textit{and} \quad \rho_2(g_f) = \rho_2(f) = \rho.$$

Furthermore, if f is a finite order meromorphic solution such that

$$\rho(f) > \max\left\{\rho\left(A_i\right)\ (i = 0, \ldots, k-1),\ \rho(d_j)\ (j = 0, 1, \ldots, k)\right\}, \tag{1.8}$$

then

$$\rho(g_f) = \rho(f).$$

Remark 1.2 In Theorem 1.1, if we do not have the condition $h \not\equiv 0$, then the conclusions of Theorem 1.1 can not hold. For example, if we take $d_i = d_k A_i$ $(i = 0, \ldots, k-1)$, then $h \equiv 0$. It follows that $g_f \equiv 0$ and $\rho(g_f) = 0$. So, if $f(z)$ is an infinite order meromorphic solution of (1.3), then $\rho(g_f) = 0 < \rho(f) = \infty$, and if f is a finite order meromorphic solution of (1.3) such that (1.8) holds, then $\rho(g_f) = 0 < \rho(f)$. By the proof of Theorem 1.1, we can see that the condition $h \not\equiv 0$ is equivalent to the condition g_f, g'_f, g''_f, $\ldots$, $g_f^{(k-1)}$ are linearly independent over the field of meromorphic functions of finite order.

Theorem 1.2 *Under the hypotheses of Theorem 1.1, let $\varphi(z) \not\equiv 0$ be a meromorphic function with finite order such that $\psi(z)$ is not a solution of* (1.3). *If $f(z)$ is an infinite order meromorphic solution of* (1.3) *with $\rho_2(f) = \rho$, then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty$$

and

$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho.$$

Furthermore, if f is a finite order meromorphic solution such that

$$\rho(f) > \max\left\{\rho(A_i)\ (i = 0, 1, \ldots, k-1),\ \rho(\varphi), \rho(d_j)\ (j = 0, 1, \ldots, k)\right\}, \tag{1.9}$$

then

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f).$$

Remark 1.3 The present article may be understood as an extension and improvement of the recent article of the authors [10] from equation (1.1) to equation (1.3).

Corollary 1.1 *Let $A_0, A_1, \cdots, A_{k-1}$ be entire functions satisfying $\rho(A_0) = \rho$ $(0 < \rho < \infty)$, $\tau_M(A_0) = \tau$ $(0 < \tau < \infty)$, let $\rho(A_i) \leq \rho$, $\tau_M(A_i) < \tau$ if $\rho(A_i) = \rho$ $(i = 1, \ldots, k-1)$, and let $d_j(z)$ $(j = 0, 1, \ldots, k)$ be finite order entire functions that are not all vanishing identically such that $h \not\equiv 0$. If f is a nontrivial solution of* (1.3), *then the differential polynomial* (1.2) *satisfies*

$$\rho(g_f) = \rho(f) = \infty \quad \text{and} \quad \rho_2(g_f) = \rho_2(f) = \rho(A_0).$$

Corollary 1.2 *Under the hypotheses of Corollary 1.1, let $\varphi(z) \not\equiv 0$ be an entire function with finite order such that $\psi(z) \not\equiv 0$. Then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty$$

and

$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho(A_0).$$

Corollary 1.3 ([10]) *Let $A(z)$ be a nonconstant polynomial, and let $d_j(z)$ $(j = 0, 1, \ldots, k)$ be nonconstant polynomials that are not all vanishing identically such that $h \not\equiv 0$. If f is a nontrivial solution of* (1.1), *then the differential polynomial* (1.2) *satisfies*

$$\rho(g_f) = \rho(f) = \frac{\deg(A) + k}{k}.$$

Corollary 1.4 ([10]) *Let $A(z)$ be a transcendental meromorphic function of finite order $\rho(A) > 0$ such that*

$$\delta(\infty, A) = \liminf_{r \to +\infty} \frac{m(r, A)}{T(r, A)} = \delta > 0.$$

Suppose, moreover, that either:

(i) all poles of f are uniformly bounded multiplicity or that

(ii) $\delta(\infty, f) > 0$.

Let $d_j(z)$ $(j = 0, 1, \ldots, k)$ be finite order meromorphic functions that are not all vanishing identically such that $h \not\equiv 0$. If $f \not\equiv 0$ is a meromorphic solution of (1.1), *then the differential polynomial* (1.2) *satisfies $\rho(g_f) = \rho(f) = \infty$ and $\rho_2(g_f) = \rho_2(f) = \rho(A)$.*

Corollary 1.5 ([10]) *Under the hypotheses of Corollary 1.4, let $\varphi(z) \not\equiv 0$ be a meromorphic function with finite order such that $\psi(z) \not\equiv 0$. Then the differential polynomial* (1.2) *satisfies*

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty$$

and

$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho(A).$$

In the following we give two applications of the above results without the additional conditions $h \not\equiv 0$ and ψ is not a solution of (1.3).

Corollary 1.6 *Let $A(z)$ and $B(z)$ be entire functions such that $\rho(B) = \rho$ $(0 < \rho < \infty)$, $\tau(B) = \tau$ $(0 < \tau < \infty)$, let $\rho(A) < \rho(B)$ and $\tau(A) < \tau(B)$ if $\rho(A) = \rho(B)$, and let $d_j(z)$ $(j = 0, 1, 2)$ be finite order entire functions that are not all vanishing identically such that*

$$\max\{\rho(d_j) \colon j = 0, 1, 2\} < \rho(A).$$

If $f \not\equiv 0$ is a solution of the differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.10}$$

then the differential polynomial

$$g_f = d_2 f'' + d_1 f' + d_0 f$$

satisfies $\rho(g_f) = \rho(f) = \infty$ and $\rho_2(g_f) = \rho_2(f) = \rho(B)$.

Corollary 1.7 *Under the hypotheses of Corollary 1.6, let $\varphi(z) \not\equiv 0$ be an entire function with finite order. Then the differential polynomial*

$$g_f = d_2 f'' + d_1 f' + d_0 f$$

satisfies

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty$$

and

$$\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho(B).$$

2 Preliminary lemmas

Lemma 2.1 ([1, 4]) *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions.*

(i) If f is a meromorphic solution of the equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F \tag{2.1}$$

with $\rho(f) = +\infty$, then f satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty.$$

(ii) If f is a meromorphic solution of equation (2.1) with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty, \quad \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Here, we give a special case of the result given by T. B. Cao, Z. X. Chen, X. M. Zheng and J. Tu in [2]:

Lemma 2.2 *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of equation* (2.1) *with*

$$\max\{\rho(A_j)\ (j = 0, 1, \dots, k-1),\ \rho(F)\} < \rho(f),$$

then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f).$$

Lemma 2.3 ([11]) *Let f and g be meromorphic functions such that $0 < \rho(f)$, $\rho(g) < \infty$ and $0 < \tau(f)$, $\tau(g) < \infty$. Then we have*

(i) If $\rho(f) > \rho(g)$, then we obtain

$$\tau(f + g) = \tau(fg) = \tau(f).$$

(ii) If $\rho(f) = \rho(g)$ and $\tau(f) \neq \tau(g)$, then we get

$$\rho(f + g) = \rho(fg) = \rho(f) = \rho(g).$$

Lemma 2.4 ([7]) *Let f be a meromorphic function and let $k \in \mathbb{N}$. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f),$$

where $S(r, f) = O(\log T(r, f) + \log r)$, possibly outside a set $E_1 \subset [0, \infty)$ of a finite linear measure. If f is a finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 2.5 ([14]) *Let $A_0, A_1, \dots, A_{k-1}$ be entire functions satisfying $\rho(A_0) = \rho\ (0 < \rho < \infty)$, $\tau_M(A_0) = \tau\ (0 < \tau < \infty)$, and let $\rho(A_j) \le \rho$, $\tau_M(A_j) < \tau$ if $\rho(A_j) = \rho\ (j = 1, \dots, k-1)$. Then every solution $f \not\equiv 0$ of* (1.3) *satisfies $\rho(f) = \infty$ and $\rho_2(f) = \rho(A_0)$.*

In the following, we give a special case of the result given by T. B. Cao, J. F. Xu and Z. X. Chen in [3]. This result is a similar result to Lemma 2.5 for entire solutions f when the order and the type of the coefficients of (1.10) are defined by the Nevanlinna characteristic function $T(r, f)$.

Lemma 2.6 *Let $A(z)$ and $B(z)$ be entire functions such that $\rho(B) = \rho\ (0 < \rho < \infty)$, $\tau(B) = \tau\ (0 < \tau < \infty)$, and let $\rho(A) < \rho(B)$ and $\tau(A) < \tau(B)$ if $\rho(A) = \rho(B)$. If $f \not\equiv 0$ is a solution of* (1.10), *then $\rho(f) = \infty$ and $\rho_2(f) = \rho(B)$.*

3 Proof of the Theorems and the Corollaries

Proof of Theorem 1.1 Suppose that f is an infinite order meromorphic solution of (1.3) with $\rho_2(f) = \rho$. We can rewrite (1.3) as

$$f^{(k)} = -\sum_{i=0}^{k-1} A_i f^{(i)} \tag{3.1}$$

which implies

$$g_f = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_1 f' + d_0 f = \sum_{i=0}^{k-1} (d_i - d_k A_i) f^{(i)}. \tag{3.2}$$

We can rewrite (3.2) as

$$g_f = \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i)}, \tag{3.3}$$

where $\alpha_{i,0}$ are defined in (1.5). Differentiating both sides of equation (3.3) and replacing $f^{(k)}$ with $f^{(k)} = -\sum_{i=0}^{k-1} A_i f^{(i)}$, we obtain

$$\begin{aligned} g_f' &= \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,0} f^{(i)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} + \alpha_{k-1,0} f^{(k)} \\ &= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} - \sum_{i=0}^{k-1} \alpha_{k-1,0} A_i f^{(i)} \\ &= (\alpha_{0,0}' - \alpha_{k-1,0} A_0) f + \sum_{i=1}^{k-1} \left(\alpha_{i,0}' + \alpha_{i-1,0} - \alpha_{k-1,0} A_i \right) f^{(i)}. \end{aligned} \tag{3.4}$$

We can rewrite (3.4) as

$$g_f' = \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i)}, \tag{3.5}$$

where

$$\alpha_{i,1} = \begin{cases} \alpha_{i,0}' + \alpha_{i-1,0} - \alpha_{k-1,0} A_i, & \text{for all } i = 1, \ldots, k-1, \\ \alpha_{0,0}' - A_0 \alpha_{k-1,0}, & \text{for } i = 0. \end{cases} \tag{3.6}$$

Differentiating both sides of equation (3.5) and replacing $f^{(k)}$ with $f^{(k)} = -\sum_{i=0}^{k-1} A_i f^{(i)}$, we obtain

$$g_f'' = \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,1} f^{(i)}$$

$$= \alpha'_{0,1} f + \sum_{i=1}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} + \alpha_{k-1,1} f^{(k)}$$

$$= \alpha'_{0,1} f + \sum_{i=1}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} - \sum_{i=0}^{k-1} A_i \alpha_{k-1,1} f^{(i)}$$

$$= \left(\alpha'_{0,1} - \alpha_{k-1,1} A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha'_{i,1} + \alpha_{i-1,1} - A_i \alpha_{k-1,1}\right) f^{(i)} \tag{3.7}$$

which implies that

$$g''_f = \sum_{i=0}^{k-1} \alpha_{i,2} f^{(i)}, \tag{3.8}$$

where

$$\alpha_{i,2} = \begin{cases} \alpha'_{i,1} + \alpha_{i-1,1} - A_i \alpha_{k-1,1}, & \text{for all } i = 1, \dots, k-1, \\ \alpha'_{0,1} - A_0 \alpha_{k-1,1}, & \text{for } i = 0. \end{cases} \tag{3.9}$$

By using the same method as above we can easily deduce that

$$g_f^{(j)} = \sum_{i=0}^{k-1} \alpha_{i,j} f^{(i)}, \quad j = 0, 1, \dots, k-1, \tag{3.10}$$

where

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, & \text{for all } i = 1, \dots, k-1, \\ \alpha'_{0,j-1} - A_0 \alpha_{k-1,j-1}, & \text{for } i = 0 \end{cases} \tag{3.11}$$

and

$$\alpha_{i,0} = d_i - d_k A_i, \quad \text{for all } i = 0, 1, \dots, k-1. \tag{3.12}$$

By (3.3)–(3.12) we obtain the system of equations

$$\begin{cases} g_f = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g'_f = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g''_f = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots \\ g_f^{(k-1)} = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}. \end{cases} \tag{3.13}$$

By Cramer's rule, and since $h \not\equiv 0$, then we have

$$f = \frac{\begin{vmatrix} g_f & \alpha_{1,0} & . \; . & \alpha_{k-1,0} \\ g'_f & \alpha_{1,1} & . \; . & \alpha_{k-1,1} \\ . & . & . \; . & . \\ . & . & . \; . & . \\ g_f^{(k-1)} & \alpha_{1,k-1} & . \; . & \alpha_{k-1,k-1} \end{vmatrix}}{h}. \tag{3.14}$$

It follows that

$$f = C_0 g_f + C_1 g_f' + \cdots + C_{k-1} g_f^{(k-1)}, \tag{3.15}$$

where C_j are finite order meromorphic functions depending on $\alpha_{i,j}$, where $\alpha_{i,j}$ are defined in (3.11) and (3.12).

If $\rho(g_f) < +\infty$, then by (3.15) we obtain $\rho(f) < +\infty$, which is a contradiction. Hence $\rho(g_f) = \rho(f) = +\infty$.

Now, we prove that $\rho_2(g_f) = \rho_2(f) = \rho$. By (3.2), we get $\rho_2(g_f) \le \rho_2(f)$ and by (3.15) we have $\rho_2(f) \le \rho_2(g_f)$. This yield $\rho_2(g_f) = \rho_2(f) = \rho$.

Furthermore, if f is a finite order meromorphic solution of equation (1.3) such that

$$\rho(f) > \max\{\rho(A_i), \rho(d_j) \colon i = 0, \ldots, k-1,\ j = 0, 1, \ldots, k\}, \tag{3.16}$$

then

$$\rho(f) > \max\{\rho(\alpha_{i,j}) \colon i = 0, \ldots, k-1,\ j = 0, \ldots, k-1\}. \tag{3.17}$$

By (3.2) and (3.16) we have $\rho(g_f) \le \rho(f)$. Now, we prove $\rho(g_f) = \rho(f)$. If $\rho(g_f) < \rho(f)$, then by (3.15) and (3.17) we get

$$\rho(f) \le \max\{\rho(C_j)\ (j = 0, \ldots, k-1),\ \rho(g_f)\} < \rho(f)$$

which is a contradiction. Hence $\rho(g_f) = \rho(f)$.

Proof of Theorem 1.2 Suppose that f is an infinite order meromorphic solution of equation (1.3) with $\rho_2(f) = \rho$. Set $w(z) = g_f - \varphi$. Since $\rho(\varphi) < \infty$, then by Theorem 1.1 we have $\rho(w) = \rho(g_f) = \infty$ and $\rho_2(w) = \rho_2(g_f) = \rho$. To prove $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho$ we need to prove $\overline{\lambda}(w) = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho$. By $g_f = w + \varphi$, and using (3.15), we get

$$f = C_0 w + C_1 w' + \cdots + C_{k-1} w^{(k-1)} + \psi(z), \tag{3.18}$$

where

$$\psi(z) = C_0 \varphi + C_1 \varphi' + \cdots + C_{k-1} \varphi^{(k-1)}.$$

Substituting (3.18) into (1.3), we obtain

$$C_{k-1} w^{(2k-1)} + \sum_{j=0}^{2k-2} \phi_j w^{(j)} = -\left(\psi^{(k)} + A_{k-1}(z)\psi^{(k-1)} + \cdots + A_0(z)\psi\right) = H,$$

where ϕ_j $(j = 0, \ldots, 2k-2)$ are meromorphic functions with finite order. Since $\psi(z)$ is not a solution of (1.3), it follows that $H \not\equiv 0$. Then by Lemma 2.1, we obtain $\overline{\lambda}(w) = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho$, i.e., $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho$.

Suppose that f is a finite order meromorphic solution of equation (1.3) such that (1.9) holds. Set $w(z) = g_f - \varphi$. Since $\rho(\varphi) < \rho(f)$, then by Theorem 1.1

we have $\rho(w) = \rho(g_f) = \rho(f)$. To prove $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f)$ we need to prove $\overline{\lambda}(w) = \lambda(w) = \rho(f)$. Using the same reasoning as above, we get

$$C_{k-1}w^{(2k-1)} + \sum_{j=0}^{2k-2} \phi_j w^{(j)} = -\left(\psi^{(k)} + A_{k-1}(z)\psi^{(k-1)} + \cdots + A_0(z)\psi\right) = H,$$

where ϕ_j $(j = 0, \ldots, 2k-2)$ are meromorphic functions with finite order $\rho(\phi_j) < \rho(f)$ $(j = 0, \ldots, 2k-2)$ and

$$\psi(z) = C_0\varphi + C_1\varphi' + \cdots + C_{k-1}\varphi^{(k-1)}, \quad \rho(H) < \rho(f).$$

Since $\psi(z)$ is not a solution of (1.3), it follows that $H \not\equiv 0$. Then by Lemma 2.2, we obtain $\overline{\lambda}(w) = \lambda(w) = \rho(f)$, i.e., $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f)$.

Proof of Corollary 1.1 Suppose $f \not\equiv 0$ is a solution of (1.3). Then by Lemma 2.5, we have $\rho(f) = \infty$ and $\rho_2(f) = \rho(A_0)$. Thus, by Theorem 1.1 we obtain $\rho(g_f) = \rho(f) = \infty$ and $\rho_2(g_f) = \rho_2(f) = \rho(A_0)$.

Proof of Corollary 1.2 Suppose $f \not\equiv 0$ is a solution of (1.3). Then by Lemma 2.5, we have $\rho(f) = \infty$ and $\rho_2(f) = \rho(A_0)$. Since $\varphi(z) \not\equiv 0$ is entire function with finite order such that $\psi(z) \not\equiv 0$, then $\rho(\psi) < \infty$ and ψ is not a solution of (1.3). Thus, by Theorem 1.2 we obtain $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \rho(f) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho_2(f) = \rho(A_0)$.

Proof of Corollary 1.6 Suppose that f is a nontrivial solution of (1.10). Then by Lemma 2.6, we have

$$\rho(f) = \infty, \quad \rho_2(f) = \rho(B).$$

On the other hand, we have

$$g_f = d_2 f'' + d_1 f' + d_0 f. \tag{3.19}$$

It follows that

$$\begin{cases} g_f = \alpha_{0,0} f + \alpha_{1,0} f', \\ g_f' = \alpha_{0,1} f + \alpha_{1,1} f'. \end{cases} \tag{3.20}$$

By (1.5) we obtain

$$\alpha_{i,0} = \begin{cases} d_1 - d_2 A, & \text{for } i = 1, \\ d_0 - d_2 B, & \text{for } i = 0. \end{cases} \tag{3.21}$$

Now, by (1.4) we get

$$\alpha_{i,1} = \begin{cases} \alpha_{1,0}' + \alpha_{1,0} - A\alpha_{1,0}, & \text{for } i = 1, \\ \alpha_{0,0}' - B\alpha_{1,0}, & \text{for } i = 0. \end{cases}$$

Hence

$$\begin{cases} \alpha_{0,1} = d_2 BA - (d_2 B)' - d_1 B + d_0', \\ \alpha_{1,1} = d_2 A^2 - (d_2 A)' - d_1 A - d_2 B + d_0 + d_1' \end{cases} \tag{3.22}$$

and

$$\begin{aligned} h = & -d_2^2B^2 - d_0d_2A^2 + \left(-d_2d_1 + d_1'd_2 + 2d_0d_2 - d_1^2\right) B \\ & + \left(d_2'd_0 - d_2d_0' + d_0d_1\right) A + d_1d_2AB - d_1d_2B' + d_0d_2A' \\ & + d_2^2B'A - d_2^2BA' + d_0'd_1 - d_0d_1' - d_0^2. \end{aligned}$$

By $d_2 \not\equiv 0$, $B \not\equiv 0$ and Lemma 2.3 we have $\rho(h) = \rho(B) > 0$. Hence $h \not\equiv 0$. Now suppose $d_2 \equiv 0$, $d_1 \not\equiv 0$ or $d_2 \equiv 0, d_1 \equiv 0$ and $d_0 \not\equiv 0$. Then, by using a similar reasoning as above we get $h \not\equiv 0$. By $h \not\equiv 0$ and (3.20), we obtain

$$f = \frac{\alpha_{1,0}g_f' - \alpha_{1,1}g_f}{h}. \tag{3.23}$$

By (3.19) we have $\rho(g_f) \leq \rho(f)$ $(\rho_2(g_f) \leq \rho_2(f))$ and by (3.23) we have $\rho(f) \leq \rho(g_f)$ $(\rho_2(f) \leq \rho_2(g_f))$. Then $\rho(g_f) = \rho(f) = \infty$ and $\rho_2(g_f) = \rho_2(f) = \rho(B)$.

Proof of Corollary 1.7 Set $w(z) = d_2f'' + d_1f' + d_0f - \varphi$. Then, by $\rho(\varphi) < \infty$, we have $\rho(w) = \rho(g_f) = \rho(f) = \infty$ and $\rho_2(w) = \rho_2(g_f) = \rho_2(f) = \rho(B)$. To prove $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho(B)$ we need to prove $\overline{\lambda}(w) = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho(B)$. Using $g_f = w + \varphi$, we get from (3.23)

$$f = \frac{\alpha_{1,0}w' - \alpha_{1,1}w}{h} + \psi, \tag{3.24}$$

where

$$\psi(z) = \frac{\alpha_{1,0}\varphi' - \alpha_{1,1}\varphi}{h}. \tag{3.25}$$

Substituting (3.24) into equation (1.10), we obtain

$$\frac{\alpha_{1,0}}{h}w''' + \phi_2w'' + \phi_1w' + \phi_0w = -\left(\psi'' + A(z)\psi' + B(z)\psi\right) = C, \tag{3.26}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions with $\rho(\phi_j) < \infty$ $(j = 0, 1, 2)$. First, we prove that $\psi \not\equiv 0$. Suppose that $\psi \equiv 0$. Then by (3.25) we obtain

$$\alpha_{1,1} = \alpha_{1,0}\frac{\varphi'}{\varphi}. \tag{3.27}$$

Hence, by Lemma 2.4

$$m(r, \alpha_{1,1}) \leq m(r, \alpha_{1,0}) + O(\log r). \tag{3.28}$$

(i) If $d_2 \not\equiv 0$, then we obtain the contradiction $\rho(B) \leq \rho(A)$.

(ii) If $d_2 \equiv 0$ and $d_1 \not\equiv 0$, then we obtain the contradiction $\rho(A) \leq \rho(d_1)$.

(iii) If $d_2 = d_1 \equiv 0$ and $d_0 \not\equiv 0$, then we have by (3.27)

$$d_0 = 0 \times \frac{\varphi'}{\varphi} \equiv 0,$$

which is a contradiction.

It is clear now that $\psi \not\equiv 0$ can not be a solution of (1.10) because $\rho(\psi) < \infty$. Hence $C \not\equiv 0$. By Lemma 2.1, we obtain $\overline{\lambda}(w) = \lambda(w) = \infty$ and $\overline{\lambda}_2(w) = \lambda_2(w) = \rho(B)$, i.e., $\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \infty$ and $\overline{\lambda}_2(g_f - \varphi) = \lambda_2(g_f - \varphi) = \rho(B)$.

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