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INEQUALITIES INVOLVING HEAT POTENTIALS AND GREEN FUNCTIONS

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Abstract. We take some well-known inequalities for Green functions relative to Laplace's equation, and prove not only analogues of them relative to the heat equation, but generalizations of those analogues to the heat potentials of nonnegative measures on an arbitrary open set E whose supports are compact polar subsets of E. We then use the special case where the measure associated to the potential has point support, in the following situation. Given a nonnegative supertemperature on an open set E, we prove a formula for the associated Riesz measure of any point of E in terms of a limit inferior of the quotient of the supertemperature and the Green function for E with a pole at that point.

Keywords: heat potential; supertemperature; Green function; Riesz measure

MSC 2010: 31C05, 31C15, 35K05

In his book, Doob [1], Theorem 1.VII.3, presented some inequalities, and their consequences, pertaining to the Green function for Laplace's equation on any open set that possesses such a function. However, he did not give a corresponding result for Green functions relative to the heat equation, but merely remarked [1], page 299, that "We shall use the fact (cf. Theorem VII.3) that $G_D(\xi,\cdot)$ and $G_D(\cdot,\eta)$ are bounded outside neighborhoods of their poles". In this note, we will prove a generalization of the analogue for the heat equation of [1], Theorem 1.VII.3, and use it to prove an analogue of [1], Theorem 1.VIII.10.

Notation and terminology will generally follow [2], but we also need the definition of a *coheat ball.* Let

$$W(x,t)=\begin{cases}(4\pi t)^{-n/2}\exp\Bigl(-\dfrac{|x|^2}{4t}\Bigr) & \text{if } t>0,\\ 0 & \text{if } t\leqslant 0\end{cases}$$

denote the fundamental temperature on $\mathbb{R}^{n+1}$. For any point $q_0 = (y_0, s_0) \in \mathbb{R}^{n+1}$ and any positive number c, the set

$$\begin{aligned}\Omega^*(q_0; c) &= \{(x,t) \in \mathbb{R}^{n+1} \colon W(x - y_0, t - s_0) > (4\pi c)^{-n/2}\} \\ &= \{(x,t) \in \mathbb{R}^{n+1} \colon |x - y_0|^2 < 2n(t - s_0)\log(c/(t - s_0)),\ s_0 < t < s_0 + c\}\end{aligned}$$

is called the *coheat ball* with *centre* q_0 and *radius* c. It is the reflection of the heat ball in the hyperplane $\mathbb{R}^n \times \{s_0\}$. In the sequel, we shall write $\tau(c)$ for $(4\pi c)^{-n/2}$.

Given a point $p_0 \in E$, we denote by $\Lambda(p_0; E)$ the set of points p for which there is a polygonal path in E that joins p_0 to p, along which the temporal variable is strictly decreasing. By a polygonal path, we mean a path which is the union of finitely many line segments. We also denote by $\Lambda^*(p_0; E)$ the set of points p for which there is a polygonal path in E joining p_0 to p, along which the temporal variable is strictly *increasing.*

Theorem 1. *Let E be an open set, and let $G_E\mu$ be the heat potential of a nonnegative measure μ whose support F is a compact polar subset of E. For any positive number c such that the closed coheat ball satisfies $\overline{\Omega}^*(q; c) \subseteq E$ for all $q \in F$, we put*

$$\Upsilon = \Upsilon(F, c) = \bigcup_{q \in F} \Omega^*(q; c).$$

(a) *If K is a compact subset of E such that $\Upsilon \subseteq K$, and v is a nonnegative supertemperature on $E \setminus K$ such that*

$$\liminf_{p \to r} v(p) \geqslant G_E\mu(r) \tag{1}$$

for quasi-every point $r \in \partial K$, then $v \geqslant G_E\mu$ on $E \setminus K$.

(b) *If v is a nonnegative supertemperature on E such that $v \geqslant G_E\mu$ quasi-everywhere on $\overline{\Upsilon}$, then $v \geqslant G_E\mu$ on E.*

(c) *If L is a subset of E that contains Υ, then $R^L_{G_E\mu} = \widehat{R}^L_{G_E\mu} = G_E\mu$ on E.*

(d) *If u is a nonnegative supertemperature on E that is positive on $\partial\Upsilon$, then there is a constant α such that $G_E\mu \leqslant \alpha u$ on $E \setminus \Upsilon$.*

(e) *The heat potential $G_E\mu$ is bounded on $E \setminus \Upsilon$.*

(f) *If $G_E\nu$ is a heat potential, and $\nu(\Lambda(q; E)) > 0$ for every point $q \in F$, then there is a constant α such that $G_E\mu \leqslant \alpha G_E\nu$ on $E \setminus \Upsilon$.*

(g) *Given any point $r \in \bigcap_{q \in F} \Lambda(q; E)$, there is a constant α such that $G_E\mu \leqslant \alpha G_E(\cdot; r)$ on $E \setminus \Upsilon$.*

P r o o f. (a) We first suppose that condition (1) holds for all points $r \in \partial K$. We define a nonnegative function w on E by putting

$$w = \begin{cases} (G_E\mu) \wedge v & \text{on } E \setminus K, \\ G_E\mu & \text{on } K. \end{cases}$$

In view of condition (1) and [2], Lemma 7.20, w is a supertemperature on E. Since F is compact we have $\mu(F) < \infty$, and so $G\mu$ is a heat potential on $\mathbb{R}^{n+1}$, by [2], Theorem 6.18. We denote by h the greatest thermic minorant of $G\mu$ on E. By [2], Theorem 6.31, the Riesz measure associated with $G\mu$ is μ itself, and so the Riesz Decomposition Theorem [2], Theorem 6.34, shows that $G\mu = G_E\mu + h$ on E. We put $u = G\mu - w$ on $E \setminus F$. Since $G\mu$ is a temperature on $\mathbb{R}^{n+1} \setminus F$ by [2], Theorem 6.25, the function u is a subtemperature on $E \setminus F$, and on $K \setminus F$ we have $u = G\mu - G_E\mu = h$. Since h is bounded on K, u is bounded on $K \setminus F$. Furthermore, whenever $p \in E \setminus K$ and $q \in F$ we have $G(p;q) \leqslant \tau(c)$ because $\Omega^*(q;c) \subseteq \Upsilon \subseteq K$, and hence

$$u(p) \leqslant G\mu(p) = \int_F G(p;q)\,\mathrm{d}\mu(q) \leqslant \tau(c)\mu(F) < \infty$$

for all $p \in E \setminus K$. Thus u is upper bounded on $E \setminus K$, and hence on $E \setminus F$. Since F is closed and polar, it follows from [2], Theorem 7.14, that u can be extended to a subtemperature $\overline{u}$ on E. Since $u \leqslant G\mu$ on $E \setminus F$, and F is Lebesgue null, we have $G\mu - \overline{u} \geqslant 0$ almost everywhere on E. Both sides of this last inequality are supertemperatures on E, and so the inequality holds everywhere on E by [2], Theorem 3.59. Thus $\overline{u} \leqslant G\mu$ on E, which implies that $\overline{u} \leqslant h$ on E, in view of [2], Definition 3.65. On $E \setminus K$ we therefore have

$$G\mu - ((G_E\mu) \wedge v) \leqslant G\mu - G_E\mu,$$

so that $v \geqslant G_E\mu$ as required.

We now consider the general case, where (1) holds only for every $r \in \partial K \setminus Z$, where Z is a polar set. We choose a heat potential v_0 on E such that $v_0 = \infty$ on Z. Then for each $\varepsilon > 0$, the function $v + \varepsilon v_0$ is a nonnegative supertemperature on $E \setminus K$ such that $\liminf\limits_{p \to r} (v + \varepsilon v_0)(p) \geqslant G_E\mu(r)$ for every point $r \in \partial K$. Therefore $v + \varepsilon v_0 \geqslant G_E\mu$ on $E \setminus K$ by the case proved above. Making $\varepsilon \to 0+$, we see that $v \geqslant G_E\mu$ except, possibly, on the polar subset of $E \setminus K$ where $v_0 = \infty$. Since polar sets are Lebesgue null, it follows from [2], Theorem 3.59, that $v \geqslant G_E\mu$ everywhere on $E \setminus K$.

(b) Since $\liminf\limits_{p \to r,\ p \in E \setminus \overline{\Upsilon}} v(p) \geqslant v(r) \geqslant G_E\mu(r)$ for quasi-every point $r \in \partial\Upsilon$, it follows from part (a) with $K = \overline{\Upsilon}$ that $v \geqslant G_E\mu$ on $E \setminus \overline{\Upsilon}$. Thus $v \geqslant G_E\mu$ almost everywhere on E, and hence everywhere on E by [2], Theorem 3.59.

(c) Since $\widehat{R}^L_{G_E\mu} \leqslant R^L_{G_E\mu} \leqslant G_E\mu$ on E, it suffices to prove that the smoothed reduction majorizes $G_E\mu$ on E. The smoothed reduction is a nonnegative supertemperature on E, and equal to $G_E\mu$ on the open subset $\Upsilon = \Upsilon(F,c)$ of L. Therefore, for any $d < c$ we have $\widehat{R}^L_{G_E\mu} \geqslant G_E\mu$ on $\bigcup_{q\in F} \overline{\Omega}^*(q;d) \setminus F$. We now show that $\bigcup_{q\in F} \overline{\Omega}^*(q;d) = \overline{\Upsilon}(F,d)$, and because

$$\Upsilon(F,d) \subseteq \bigcup_{q\in F} \overline{\Omega}^*(q;d) \subseteq \overline{\Upsilon}(F,d),$$

it suffices to show that $\bigcup_{q\in F} \overline{\Omega}^*(q;d)$ is a closed set. Let $\{p_j\}$ be a convergent sequence of points in that union, with limit $p' = (x',t')$. For each j, we choose a point $q_j \in F$ such that $p_j \in \overline{\Omega}^*(q_j;d)$. Since F is compact, the sequence $\{q_j\}$ has a subsequence $\{q_{j_k}\}$ which converges to a point $q' = (y',s') \in F$. If $p_{j_k} = q_{j_k}$ for infinitely many values of k, then $p' = q' \in \overline{\Omega}^*(q';d)$. On the other hand, if $p_{j_k} = q_{j_k}$ for only finitely many values of k, then we choose a number k_0 such that $p_{j_k} \neq q_{j_k}$ whenever $k > k_0$. Putting $p_{j_k} = (x_k,t_k)$ and $q_{j_k} = (y_k,s_k)$, we have

$$|x_k - y_k|^2 \leqslant 2n(t_k - s_k)\log\Big(\frac{d}{t_k - s_k}\Big)$$

whenever $k > k_0$. If $t_k - s_k \to 0$ as $k \to \infty$, then $x_k - y_k \to 0$ as well, and so $p' = q' \in \overline{\Omega}^*(q';d)$. Otherwise, making $k \to \infty$ we obtain

$$|x' - y'|^2 \leqslant 2n(t' - s')\log\Big(\frac{d}{t' - s'}\Big),$$

so that again $p' \in \overline{\Omega}^*(q';d)$. Thus the union in question is a closed set, and hence is equal to $\overline{\Upsilon}(F,d)$. It follows that $\widehat{R}^L_{G_E\mu} \geqslant G_E\mu$ on $\overline{\Upsilon}(F,d) \setminus F$, and hence quasi-everywhere on $\overline{\Upsilon}(F,d)$ because F is polar. Therefore $\widehat{R}^L_{G_E\mu} \geqslant G_E\mu$ on E, by part (b).

(d) Whenever $p \in E \setminus \Upsilon$ and $q \in F$, we have $G(p;q) \leqslant \tau(c)$ because $\Omega^*(q;c) \subseteq \Upsilon$, and hence

$$G_E\mu(p) = \int_F G_E(p;q)\,\mathrm{d}\mu(q) \leqslant \int_F G(p;q)\,\mathrm{d}\mu(q) \leqslant \tau(c)\mu(F) < \infty$$

for all $p \in E \setminus \Upsilon$. Furthermore, because $u > 0$ and u is lower semicontinuous on $\partial\Upsilon$, it has a positive minimum over $\partial\Upsilon$. We can therefore find a positive constant α such that $G_E\mu \leqslant \alpha u$ on $\partial\Upsilon$. Hence

$$\alpha \liminf_{p\to r,\ p\in E\setminus\overline{\Upsilon}} u(p) \geqslant \alpha u(r) \geqslant G_E\mu(r)$$

for all points $r \in \partial\Upsilon$. Now part (a), with $K = \overline{\Upsilon}$, shows that $\alpha u \geqslant G_E\mu$ on $E \setminus \overline{\Upsilon}$, and hence on $E \setminus \Upsilon$.

(e) This follows from part (d) by taking $u = 1$.

(f) By [2], Theorem 6.11, $G_E(q;p) > 0$ if and only if $p \in \Lambda(q;E)$. Therefore the condition $\nu(\Lambda(q;E)) > 0$ implies that

$$\int_{\Lambda(q;E)} G_E(q;p)\,\mathrm{d}\nu(p) > 0.$$

Thus $G_E\nu > 0$ on F, so that the set $D = \{p \in E\colon\ G_E\nu(p) > 0\}$ is an open superset of F because $G_E\nu$ is lower semicontinuous on E. If we choose $d < c$ such that $\overline{\Upsilon}(F,d) \subseteq D$, then part (d) implies that there is a constant α such that $G_E\mu \leqslant \alpha G_E\nu$ on $E \setminus \Upsilon(F,d) \supseteq E \setminus \Upsilon(F,c)$.

(g) If ν is a point mass at r, then $\nu(\Lambda(q;E)) = \nu(\{r\}) > 0$ for all $q \in F$, so that the result follows from part (f). □

Example. Let ω be a nonnegative Borel measure on $\mathbb{R}^n$ whose support is a Lebesgue null compact set K. Then the Gauss-Weierstrass integral w of ω exists and is a temperature on the set $D = \mathbb{R}^n \times {]0,\infty[}$. If we put $E = \mathbb{R}^{n+1}$ and

$$u = \begin{cases} w & \text{on } D, \\ 0 & \text{on } \mathbb{R}^n \times {]-\infty,0]}, \end{cases}$$

then u is the heat potential of a measure supported by the set $F = K \times \{0\}$, in view of [2], Example 6.14. Moreover, [2], Theorem 7.55, shows that the thermal capacity of F is zero, so that F is polar by [2], Theorem 7.46. Hence Theorem 1 can be used to show that:

(a) The temperature w is bounded on $D \setminus \Upsilon$.
(b) If $G\nu$ is a heat potential, and $\nu(\mathbb{R}^n \times {]-\infty,0[}) > 0$, then there is a constant α such that $w \leqslant \alpha G\nu$ on $D \setminus \Upsilon$.
(c) Given any point $r \in \mathbb{R}^n \times {]-\infty,0[}$, there is a constant α such that $w \leqslant \alpha G(\cdot;r)$ on $D \setminus \Upsilon$.

The special case of Theorem 1 where F is a singleton is analogous to a result for classical superharmonic functions given by Doob in [1], Theorem 1.VII.3. Using this special case, we now prove a result analogous to [1], Theorem 1.VIII.10. The first part was given in [1], page 307.

Theorem 2. *If v is a nonnegative supertemperature on an open set E, and ν is its associated Riesz measure, then for each point $q \in E$ we have*

$$\inf_{\Lambda^*(q;E)} \frac{v}{G_E(\cdot;q)} = \nu(\{q\}) \quad \textit{and} \quad \lim_{c\to 0+}\Big(\inf_{\Omega^*(q;c)} \frac{v}{G_E(\cdot;q)}\Big) = \nu(\{q\}).$$

Proof. We put $D = \Lambda^*(q;E)$, and note that $D = \{p \in E \colon G_E(p;q) > 0\}$ by [2], Theorem 6.7. We also put

$$\alpha = \inf_D \frac{v}{G_E(\cdot;q)}.$$

The function $v - \alpha G_E(\cdot;q)$ is a supertemperature on $E \setminus \{q\}$ which is nonnegative on D by the definition of α, and hence is nonnegative everywhere on $E \setminus \{q\}$. It therefore follows from [2], Theorem 7.14, that $v - \alpha G_E(\cdot;q)$ has a unique extension to a supertemperature u on E. Then $v = \alpha G_E(\cdot;q) + u$ on $E \setminus \{q\}$, and hence on E because both sides are supertemperatures on E. This implies that $\nu(\{q\}) \geqslant \alpha$. If $\nu(\{q\}) = \beta > \alpha$, then $v \geqslant \beta G_E(\cdot;q)$ on E, so that

$$\frac{v}{G_E(\cdot;q)} \geqslant \beta > \alpha$$

on D, contrary to the definition of α. Hence $\nu(\{q\}) = \alpha$.

We now put

$$\gamma = \lim_{c\to 0+}\Big(\inf_{\Omega^*(q;c)} \frac{v}{G_E(\cdot;q)}\Big).$$

The part just proved shows that $\gamma \geqslant \nu(\{q\})$. If $\gamma > \nu(\{q\})$, we choose δ such that $\gamma > \delta > \nu(\{q\})$. Then there is $d > 0$ such that $v > \delta G_E(\cdot;q)$ on $\Omega^*(q;d)$. If $0 < e < d$, then $\overline{\Omega}^*(q;e) \subseteq E$ and $v > \delta G_E(\cdot;q)$ on $\overline{\Omega}^*(q;e)\setminus\{q\}$, so that $v \geqslant \delta G_E(\cdot;q)$ on E, by the case $F = \{q\}$ of Theorem 1 (b). So

$$\inf_D \frac{v}{G_E(\cdot;q)} \geqslant \delta,$$

and hence $\nu(\{q\}) \geqslant \delta$ by the first part of this result. This contradicts our choice of δ. $\square$

References

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