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ON THE BOUNDS OF LAPLACIAN EIGENVALUES OF k-CONNECTED GRAPHS

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Abstract. Let $\mu_{n-1}(G)$ be the algebraic connectivity, and let $\mu_1(G)$ be the Laplacian spectral radius of a k-connected graph G with n vertices and m edges. In this paper, we prove that

$$\mu_{n-1}(G) \geqslant \frac{2nk^2}{(n(n-1)-2m)(n+k-2)+2k^2},$$

with equality if and only if G is the complete graph K_n or $K_n - e$. Moreover, if G is non-regular, then

$$\mu_1(G) < 2\Delta - \frac{2(n\Delta-2m)k^2}{2(n\Delta-2m)(n^2-2n+2k)+nk^2},$$

where Δ stands for the maximum degree of G. Remark that in some cases, these two inequalities improve some previously known results.

Keywords: k-connected graph; non-regular graph; algebraic connectivity; Laplacian spectral radius; maximum degree

MSC 2010: 05C50, 15A18

1. Introduction

In this paper we consider undirected simple graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $i \in \{1, 2, \ldots, n\}$, let $N_G(v_i)$ and $d_G(v_i)$ denote the neighborhood (the set of vertices adjacent to vertex v_i) and the degree of vertex v_i in G, respectively. We also use $\Delta(G)$ and $\delta(G)$ to denote the

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maximum and minimum degree of vertices in G, respectively. Recall that a graph G is regular if $\Delta(G) = \delta(G)$. The distance between two vertices v_i and v_j $(i \neq j)$ in G is the number of edges in a shortest path connecting v_i and v_j. The diameter of G, written $D(G)$, is the maximum distance over all pairs of vertices in G. The (vertex) connectivity of G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal disconnects G or reduces it to a single vertex. A graph G is k-connected if $\kappa(G) \geqslant k$. A 1-connected graph is precisely a nontrivial connected graph.

The Laplacian matrix of a graph G is $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G and $D(G) = \mathrm{diag}(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ is the diagonal matrix of vertex degrees in G. Observe that $L(G)$ is a positive semi-definite matrix, and so all its eigenvalues are non-negative real numbers, which are also called the Laplacian eigenvalues of G and usually ordered as

$$\mu_1(G) \geqslant \ldots \geqslant \mu_{n-1}(G) \geqslant \mu_n(G).$$

It is well known that $\mu_n(G) = 0$ and the algebraic multiplicity of zero as an eigenvalue of G is exactly the number of connected components in G, which implies that $\mu_{n-1}(G) > 0$ if and only if G is connected (see, e.g., [1]). Fiedler [4] further showed that if G a non-complete graph then $\mu_{n-1}(G) \leqslant \kappa(G)$, and based on these facts he defined $\mu_{n-1}(G)$ as the algebraic connectivity of G. It is worth pointing out that, besides connectivity, $\mu_{n-1}(G)$ has also close relations to other important graph invariants, such as isoperimetric number, maximum cut, expanding properties, independence number, genus, diameter, mean distance, and bandwidth-type parameters, for more details we refer the reader to [1], [4], [8] and the latest comprehensive review [2].

There have also been many attempts to find a lower bound for $\mu_{n-1}(G)$ based on simple properties of the graph G, such as the number of vertices, number of edges, minimum degree, and diameter. Fiedler [4] first showed that for a graph G on n vertices with $\delta(G) = \delta$,

$$\mu_{n-1}(G) \geqslant 2\delta - n + 2.$$

Later in [9], Mohar proved that if G is a connected graph on n vertices with the diameter $D(G) = D$, then

$$\mu_{n-1}(G) \geqslant \frac{4}{nD}. \tag{1.1}$$

Recently, in [7], Lu, Zhang and Tian showed that

$$\mu_{n-1}(G) \geqslant \frac{2n}{(n(n-1) - 2m)D + 2}, \tag{1.2}$$

where m is the number of edges in G. The equality holds in (1.2) if and only if $G = K_n$ or $G = P_3$ (the path on 3 vertices). Notice that the lower bounds (1.1) and (1.2) are incomparable. In fact, it is not difficult to check that for graphs with $m \geqslant n(n-2)/4+1$, the lower bound (1.2) is better than (1.1), while for graphs with $m \leqslant n(n-2)/4$, the lower bound (1.1) is better than (1.2).

Another important and well-studied Laplacian eigenvalue of a graph G is the largest Laplacian eigenvalue $\mu_1(G)$, which is also called the Laplacian spectral radius of G. There is a nice relation between the Laplacian spectral radius and the algebraic connectivity of G, that is, $\mu_1(G) + \mu_{n-1}(\overline{G}) = n$, where $\overline{G}$ is the complement of G. Hence, it is not surprising at all that the importance of one of these eigenvalues implies the importance of the other. For more about the Laplacian spectral radius of graphs one can refer to [1], [8].

It is well-known that $\mu_1(G) \leqslant 2\Delta(G)$ with equality if and only if G is a bipartite regular graph. Then it is natural to ask how small $2\Delta(G) - \mu_1(G)$ can be when G is non-regular. Let G be a connected non-regular graph on n vertices with $\Delta(G) = \Delta$ and $D(G) = D$. Shi [11] proved that

$$\mu_1(G) < 2\Delta - \frac{2}{(2D+1)n}.$$

Later in [6], Li, Shiu and Chang improved Shi's bound as follows:

$$\mu_1(G) < 2\Delta - \frac{1}{nD}.$$

Recently, Ning, Li and Lu [10] further showed that

$$\mu_1(G) < 2\Delta - \frac{1}{n(D-1/4)}. \tag{1.3}$$

In this paper, we continue to investigate the bounds for the algebraic connectivity and the Laplacian spectral radius of graphs. In Section 2, we present sharp lower bounds on the algebraic connectivity for k-connected graphs, which, in some cases, improve the bounds (1.1) and (1.2). In Section 3, we give upper bounds on the Laplacian spectral radius for k-connected non-regular graphs, which, in some cases, improve the bound (1.3).

2. The algebraic connectivity of k-connected graphs

In this section, we give sharp lower bounds on $\mu_{n-1}(G)$ for a k-connected graph G, using some ideas of Lu, Zhang and Tian [7].

As usual, let K_n denote the complete graph on n vertices and let $K_n - e$ denote the graph obtained from K_n by deleting its arbitrary edge e. We denote by $\overline{G}$ the complement of the graph G.

Theorem 2.1. *If G is a k-connected graph with n vertices and m edges, then*

$$\mu_{n-1}(G) \geqslant \frac{2nk^2}{(n(n-1)-2m)(n+k-2)+2k^2}, \tag{2.1}$$

with equality holding if and only if $G = K_n$ or $G = K_n - e$.

Proof. Consider first the complete graph K_n. Note that $m(K_n) = n(n-1)/2$ and $\mu_{n-1}(K_n) = n$. Then a trivial calculation shows that the statement holds for the case of the complete graph. Hence, in what follows, we only consider non-complete graphs.

Let $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^{\mathrm{T}} \in \mathbb{R}^n \setminus \{0\}$ be an eigenvector of $L(G)$ corresponding to $\mu_{n-1}(G)$. Then $\mu_{n-1}(G)x = L(G)x$, and consequently,

$$\mu_{n-1}(G) = \frac{x^{\mathrm{T}}L(G)x}{x^{\mathrm{T}}x} = \frac{\sum_{v_iv_j\in E(G)}(x_{v_i}-x_{v_j})^2}{\sum_{v_i\in V(G)} x_{v_i}^2}. \tag{2.2}$$

Noting that $\mu_n(G) = 0$ with eigenvector $j = \{1,1,\ldots,1\}^{\mathrm{T}}$, we have $x \perp j$, and so $\sum\limits_{v_i\in V(G)} x_{v_i} = 0$. Using this fact and some calculation, it follows from (2.2) that

$$\mu_{n-1}(G) = \frac{2n\sum_{v_iv_j\in E(G)}(x_{v_i}-x_{v_j})^2}{\sum_{v_i\in V(G)}\sum_{v_j\in V(G)}(x_{v_i}-x_{v_j})^2}. \tag{2.3}$$

Moreover, observing that $\sum\limits_{v_iv_j\in E(G)}(x_{v_i}-x_{v_j})^2 \neq 0$ (since G is connected), from (2.3) we obtain

$$\begin{aligned}\mu_{n-1}(G) &= \frac{n\sum_{v_iv_j\in E(G)}(x_{v_i}-x_{v_j})^2}{\sum_{v_iv_j\in E(G)}(x_{v_i}-x_{v_j})^2+\sum_{v_iv_j\in E(\overline{G})}(x_{v_i}-x_{v_j})^2}\\ &= \frac{n}{1+\sum_{v_iv_j\in E(\overline{G})}(x_{v_i}-x_{v_j})^2/\sum_{v_iv_j\in E(G)}(x_{v_i}-x_{v_j})^2}.\end{aligned} \tag{2.4}$$

Suppose that u, v are two vertices satisfying $x_u = \max\limits_{1\leqslant i\leqslant n} x_{v_i}$ and $x_v = \min\limits_{1\leqslant i\leqslant n} x_{v_i}$, respectively. Clearly, $x_v < 0 < x_u$. Since G is k-connected, by Menger's Theorem (see, e.g., [3]), there are k independent (i.e., pairwise internally disjoint) paths joining u and v in G, say, $P^{(1)}, P^{(2)}, \ldots, P^{(k)}$. In particular, let

$$P^{(t)} := v_1^{(t)} v_2^{(t)} \ldots v_{l_t}^{(t)}, \quad t = 1, 2, \ldots, k,$$

where $v_1^{(t)} = u$, $v_{l_t}^{(t)} = v$, and all other vertices in these k paths are distinct. It is easy to see that

$$\sum_{t=1}^{k} |V(P^{(t)})| = \sum_{t=1}^{k} l_t \leqslant n + 2k - 2, \tag{2.5}$$

with equality holding if and only if $V(P^{(1)}) \cup V(P^{(2)}) \cup \ldots \cup V(P^{(k)}) = V(G)$. Now using Cauchy-Schwarz inequality together with this inequality, we get

$$\begin{aligned} \sum_{v_i v_j \in E(G)} (x_{v_i} - x_{v_j})^2 &\geqslant \sum_{t=1}^{k} \sum_{v_i v_j \in E(P^{(t)})} (x_{v_i} - x_{v_j})^2 \\ &\geqslant \sum_{t=1}^{k} \frac{1}{|E(P^{(t)})|} \Big(\sum_{i=1}^{l_t - 1} (x_{v_i^{(t)}} - x_{v_{i+1}^{(t)}}) \Big)^2 \\ &= \sum_{t=1}^{k} \frac{1}{|V(P^{(t)})| - 1} (x_u - x_v)^2 \\ &\geqslant \frac{k^2}{\sum_{t=1}^{k} (|V(P^{(t)})| - 1)} (x_u - x_v)^2 \\ &\geqslant \frac{k^2}{n + k - 2} (x_u - x_v)^2. \end{aligned} \tag{2.6}$$

On the other hand, we have

$$\sum_{v_i v_j \in E(\overline{G})} (x_{v_i} - x_{v_j})^2 \leqslant \Big(\frac{n(n-1)}{2} - m \Big) (x_u - x_v)^2. \tag{2.7}$$

Obviously, (2.1) follows from (2.4), (2.6) and (2.7) directly.

Now, we discuss the sharpness of the bound (2.1). Suppose the equality in (2.1) holds. Then the inequalities in (2.6) and (2.7) must be equalities. In particular, by the first inequality in (2.6), we get

$$x_{v_i} - x_{v_j} = 0, \tag{2.8}$$

for any edge $v_i v_j \in E(G) \setminus (E(P^{(1)}) \cup E(P^{(2)}) \cup \ldots \cup E(P^{(k)}))$. By the second inequality in (2.6), for $t = 1, 2, \ldots, k$ and $i = 1, 2, \ldots, l_t - 1$, we have

$$x_{v_i^{(t)}} - x_{v_{i+1}^{(t)}} = c_t > 0, \tag{2.9}$$

where c_t is a constant. By the last two inequalities in (2.6), we obtain

$$|V(P^{(1)})| = |V(P^{(2)})| = \ldots = |V(P^{(k)})| = \frac{n-2}{k} + 2, \tag{2.10}$$

which also implies that (see (2.5))

$$V(G) \setminus (V(P^{(1)}) \cup V(P^{(2)}) \cup \ldots \cup V(P^{(k)})) = \emptyset. \tag{2.11}$$

In addition, from (2.7), for any edge $v_iv_j \in E(\overline{G})$, we have

$$(x_{v_i} - x_{v_j})^2 = (x_u - x_v)^2. \tag{2.12}$$

Now, using the conditions (2.8)–(2.12), we first show that $|V(P^{(t)})| = l_t = 3$ holds for each $t \in \{1, 2, \ldots, k\}$, which (together with (2.10) and (2.11)) implies that

$$k = n - 2 \quad \text{and} \quad V(G) = \{u, v, v_2^{(1)}, v_2^{(2)}, \ldots, v_2^{(n-2)}\}.$$

Indeed, by (2.10), we obtain $l_t \geqslant 3$. If $l_t \geqslant 4$, then $uv_{l_t-1}^{(t)} \in E(\overline{G})$ (because if $uv_{l_t-1}^{(t)} \in E(G)$, then from (2.8) we have $x_{v_{l_t-1}^{(t)}} = x_u$, but by (2.9) we get $x_{v_{l_t-1}^{(t)}} = x_u - (l_t - 2)c_t < x_u$, a contradiction). Thus, from (2.12) we can deduce that $x_{v_{l_t-1}^{(t)}} = x_v$, again contradicting (2.9), which implies that $x_{v_{l_t-1}^{(t)}} - x_v = c_t > 0$.

Further, we claim that $v_2^{(1)}, v_2^{(2)}, \ldots, v_2^{(n-2)}$ are pairwise adjacent, implying $G = K_n - e$. In fact, from (2.9) it follows that $x_v < x_{v_2^{(1)}}, x_{v_2^{(2)}}, \ldots, x_{v_2^{(n-2)}} < x_u$, which implies that for any $1 \leqslant i < j \leqslant n-2$,

$$\left(x_{v_2^{(i)}} - x_{v_2^{(j)}}\right)^2 < (x_u - x_v)^2.$$

Comparing this with (2.12), we have $v_2^{(i)}v_2^{(j)} \in E(G)$, the claim then holds.

Conversely, note that $\mu_{n-1}(K_n - e) = n - 2$. Then it is easy to check that the equality in (2.1) holds when $G = K_n - e$.

This completes the proof of the theorem. □

Notice that if G is k-connected, then $\delta(G) \geqslant k$, and so $2m \geqslant n\delta(G) \geqslant nk$ (with equality only if G is regular). Thus, we obtain the following corollary immediately.

Corollary 2.2. *If G is a k-connected graph with n vertices, then*

$$\mu_{n-1}(G) \geqslant \frac{2nk^2}{n(n-k-1)(n+k-2)+2k^2}, \tag{2.13}$$

with equality holding if and only if $G = K_n$.

Note also that substituting the condition that "G is a k-connected graph" for "G is a graph with $\kappa(G) = k$", Theorem 2.1 still holds (so does Corollary 2.2). Therefore, from Corollary 2.2 we may obtain an upper bound for the connectivity $\kappa(G)$ of a graph G involving its algebraic connectivity $\mu_{n-1}(G)$ as follows:

Corollary 2.3. *Let G be a non-complete graph with n vertices. Then*

$$\mu_{n-1}(G) \leqslant \kappa(G) \leqslant n\sqrt{\frac{n\mu_{n-1}(G)}{n\mu_{n-1}(G)+2(n-\mu_{n-1}(G))}}. \tag{2.14}$$

The first inequality in (2.14) *is the famous Fiedler's inequality* [4] *and the characterization for the case of equality can be found in* [5].

Remark 2.4. It is easy to see that for any k-connected graph G with n vertices,

$$D(G) \leqslant \frac{n+k-2}{k}.$$

This together with (1.1) yields that

$$\mu_{n-1}(G) \geqslant \frac{4k}{n(n+k-2)}. \tag{2.15}$$

Clearly, when $k \geqslant 2$, our bound (2.13) is always better than the bound (2.15). Moreover, for graphs with large connectivity, our bound (2.13) is better than the bound (1.1). Indeed, when $k \geqslant \sqrt{n}$, it follows from (2.13) that if $D \geqslant 2$, then

$$\mu_{n-1}(G) > \frac{2k^2}{(n-k)(n+k)+2k^2/n} > \frac{2k^2}{n^2} \geqslant \frac{2}{n} \geqslant \frac{4}{nD}.$$

If $D = 1$, then $G = K_n$, and hence $\mu_{n-1}(G) = n > 4/n$.

Likewise, for graphs with large connectivity, our bound (2.1) is better than the bound (1.2). In fact, if $k \geqslant \sqrt{n}$, then $(n+k-2)/k^2 \leqslant 2 \leqslant D$, and from (2.1) we have

$$\mu_{n-1}(G) \geqslant \frac{2n}{(n(n-1)-2m)(n+k-2)/k^2+2} \geqslant \frac{2n}{(n(n-1)-2m)D+2}.$$

Moreover, we observe that there are more graphs achieving the bound in (2.1) than that in (1.2).

3. The Laplacian spectral radius of k-connected non-regular graphs

Let $Q(G) = D(G) + A(G)$ be the signless Laplacian matrix of a graph G and let $q_1(G)$ be its largest eigenvalue. It is well known that (see [12])

$$\mu_1(G) \leqslant q_1(G). \tag{3.1}$$

Moreover, if G is connected, then the equality holds if and only if G is a bipartite graph. In this section, by bounding $q_1(G)$ above, we present some upper bounds on $\mu_1(G)$ for a k-connected non-regular graph G.

Since G is k-connected, we know that $Q(G)$ is irreducible and, by the celebrated Perron-Frobenius Theorem, $q_1(G)$ is simple and has a unique positive unit eigenvector. Moreover, it is a well-known fact that $q_1(G) \leqslant 2\Delta(G)$ with equality if and only if G is regular (see, e.g., [1]). For non-regular graphs, we have the following result.

Theorem 3.1. *Let G be a k-connected non-regular graph with n vertices, m edges and maximum degree Δ. Then*

$$2\Delta - q_1(G) > \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - 2n + 2k) + nk^2}. \tag{3.2}$$

Proof. Let $x = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})^{\mathrm{T}} \in \mathbb{R}^n$ be the unique unit positive eigenvector of $Q(G)$ corresponding to $q_1(G)$. Then $q_1(G)x = Q(G)x$. Moreover, noting that $x^{\mathrm{T}}x = \sum\limits_{i=1}^{n} x_{v_i}^2 = 1$, we have

$$q_1(G) = x^{\mathrm{T}}Q(G)x = \sum_{v_iv_j \in E(G)} (x_{v_i} + x_{v_j})^2. \tag{3.3}$$

Now we assume that u, v are two vertices satisfying $x_u = \max\limits_{1\leqslant i\leqslant n} x_{v_i}$ and $x_v = \min\limits_{1\leqslant i\leqslant n} x_{v_i}$, respectively. Clearly, $x_u > 1/\sqrt{n} > x_v$ (since G is non-regular). Then it follows from (3.3) that

$$\begin{aligned} 2\Delta - q_1(G) &= 2\Delta - \sum_{v_iv_j \in E(G)} (x_{v_i} + x_{v_j})^2 \\ &= 2\Delta \sum_{i=1}^{n} x_{v_i}^2 - 2\sum_{v_iv_j \in E(G)} (x_{v_i}^2 + x_{v_j}^2) + \sum_{v_iv_j \in E(G)} (x_{v_i} - x_{v_j})^2 \\ &= 2\sum_{i=1}^{n} (\Delta - d_G(v_i))x_{v_i}^2 + \sum_{v_iv_j \in E(G)} (x_{v_i} - x_{v_j})^2 \\ &\geqslant 2(n\Delta - 2m)x_v^2 + \sum_{v_iv_j \in E(G)} (x_{v_i} - x_{v_j})^2. \end{aligned} \tag{3.4}$$

Noting that G is k-connected, as in the proof of Theorem 2.1, we may obtain (see (2.6)),

$$\sum_{v_iv_j \in E(G)} (x_{v_i} - x_{v_j})^2 \geqslant \frac{k^2}{n+k-2}(x_u - x_v)^2. \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$\begin{aligned} 2\Delta - q_1(G) &\geqslant 2(n\Delta - 2m)x_v^2 + \frac{k^2}{n+k-2}(x_u - x_v)^2 \\ &\geqslant \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n+k-2) + k^2} x_u^2. \end{aligned} \tag{3.6}$$

The second inequality in (3.6) follows from the fact that if $a, b > 0$, then

$$a(x-y)^2 + by^2 \geqslant \frac{abx^2}{a+b}. \tag{3.7}$$

For convenience, set

$$C := \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - 2n + 2k) + nk^2}.$$

We shall show that

$$2\Delta - q_1(G) > C. \tag{3.8}$$

Consider the following two cases:

Case 1: $k = 1$. If $x_v^2 \geqslant C/(2(n\Delta - 2m))$, then from (3.4) we have

$$2\Delta - q_1(G) > 2(n\Delta - 2m)x_v^2 \geqslant C,$$

and (3.8) holds. Now suppose that $x_v^2 < C/(2(n\Delta - 2m))$. Recalling that $x_{v_1}^2 + x_{v_2}^2 + \ldots + x_{v_n}^2 = 1$, we get

$$x_u^2 \geqslant \frac{1 - x_v^2}{n-1} > \frac{1 - \dfrac{C}{2(n\Delta - 2m)}}{n-1}.$$

Then from (3.6) we obtain

$$\begin{aligned} 2\Delta - q_1(G) &\geqslant \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n-1) + 1} x_u^2 \\ &> \frac{2(n\Delta - 2m)}{(2(n\Delta - 2m)(n-1) + 1)(n-1)} \Big(1 - \frac{C}{2(n\Delta - 2m)}\Big) \\ &= \frac{2(n\Delta - 2m)}{2(n\Delta - 2m)(n^2 - 2n + 2) + n} \, \frac{2(n\Delta - 2m)(n^2 - 2n + 2) + n - 1}{2(n\Delta - 2m)(n - 2n + 1) + n - 1} > C, \end{aligned}$$

as desired, completing the proof of Case 1.

Case 2: $k \geqslant 2$. Notice that there are at least k vertices in $N_G(v)$ since $k \leqslant \delta(G) \leqslant d_G(v)$. We choose k of them, say $w_1, w_2, \ldots, w_k$. Here u may be w_t for some $t \in \{1, 2, \ldots, k\}$; if this is the case, without loss of generality, assume that $u = w_k$. As in Case 1, if $x_v^2 \geqslant C/(2(n\Delta - 2m))$, then (3.8) holds. If $\sum\limits_{t=1}^{k-1} x_{w_t}^2 > C(1 + (k-1)/(2(n\Delta - 2m)))$, then from (3.4) and (3.7), we get

$$\begin{aligned}
2\Delta - q_1(G) &\geqslant 2(n\Delta - 2m)x_v^2 + \sum_{t=1}^{k-1}(x_{w_t} - x_v)^2 \\
&= \sum_{t=1}^{k-1}\Big(\frac{2(n\Delta - 2m)}{k-1}x_v^2 + (x_{w_t} - x_v)^2\Big) \\
&\geqslant \sum_{t=1}^{k-1}\frac{2(n\Delta - 2m)}{2(n\Delta - 2m) + k - 1}x_{w_t}^2 > C,
\end{aligned}$$

and (3.8) holds as well. Now suppose that

$$x_v^2 < \frac{C}{2(n\Delta - 2m)} \quad \text{and} \quad \sum_{t=1}^{k-1} x_{w_t}^2 \leqslant C\Big(1 + \frac{k-1}{2(n\Delta - 2m)}\Big).$$

Then

$$x_u^2 \geqslant \frac{1 - x_v^2 - \sum\limits_{t=1}^{k-1} x_{w_t}^2}{n-k} > \frac{1 - \dfrac{2(n\Delta - 2m) + k}{2(n\Delta - 2m)}C}{n-k}.$$

Thus, from (3.6) we have

$$\begin{aligned}
2\Delta - q_1(G) &\geqslant \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n+k-2) + k^2}x_u^2 \\
&> \frac{2(n\Delta - 2m)k^2}{(2(n\Delta - 2m)(n+k-2) + k^2)(n-k)}\Big(1 - \frac{2(n\Delta - 2m) + k}{2(n\Delta - 2m)}C\Big) \\
&= \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - 2n + 2k) + nk^2} = C.
\end{aligned}$$

This completes the proof of the theorem. $\square$

Using (3.1) and (3.2), we easily obtain the following upper bound on $\mu_1(G)$ for a k-connected non-regular graph G.

Corollary 3.2. *Let G be a k-connected non-regular graph with n vertices, m edges and maximum degree Δ. Then*

$$\mu_1(G) < 2\Delta - \frac{2(n\Delta - 2m)k^2}{2(n\Delta - 2m)(n^2 - 2n + 2k) + nk^2}. \tag{3.9}$$

It is not difficult to check that the upper bound in (3.9) is decreasing strictly with respect to $(n\Delta - 2m)$. On the other hand, for a k-connected non-regular graph G, we observe that $2m \leqslant (n-1)\Delta(G) + \delta(G)$. Thus, from Corollary 3.2, we have the following result immediately.

Corollary 3.3. *Let G be a k-connected non-regular graph with n vertices, maximum degree Δ and minimum degree δ. Then*

$$\mu_1(G) < 2\Delta - \frac{2(\Delta - \delta)k^2}{2(\Delta - \delta)(n^2 - 2n + 2k) + nk^2}. \tag{3.10}$$

Further, using the fact that $\Delta - \delta \geqslant 1$, from Corollary 3.3 we obtain the following simplified version of Corollary 3.2.

Corollary 3.4. *Let G be a k-connected non-regular graph with n vertices and maximum degree Δ. Then*

$$\mu_1(G) < 2\Delta - \frac{2k^2}{2(n^2 - 2n + 2k) + nk^2}. \tag{3.11}$$

Remark 3.5. It is easy to verify that when $k \geqslant \sqrt{n}$ $(n \geqslant 6)$, our bound (3.11) is always better than the bound (1.3). Indeed, when $k \geqslant \sqrt{n}$, it follows from Corollary 3.4 that if $D \geqslant 2$ then

$$\mu_1(G) < 2\Delta - \frac{1}{n^2/k^2 + n/2} < 2\Delta - \frac{1}{7n/4} \leqslant 2\Delta - \frac{1}{n(D - 1/4)};$$

if $D = 1$ then $G = K_n$, and consequently,

$$\mu_1(G) < 2\Delta - \frac{1}{(n^2-2)/(n-1)^2 + n/2} < 2\Delta - \frac{1}{3n/4} = 2\Delta - \frac{1}{n(D - 1/4)}.$$

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