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ISOMORPHISMS AND SEVERAL CHARACTERIZATIONS OF MUSIELAK-ORLICZ-HARDY SPACES ASSOCIATED WITH SOME SCHRÖDINGER OPERATORS

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Abstract. Let $L := -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$ with $n \geqslant 3$ and $V \geqslant 0$ satisfying $\Delta^{-1}V \in L^\infty(\mathbb{R}^n)$. Assume that $\varphi\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is a function such that $\varphi(x,\cdot)$ is an Orlicz function, $\varphi(\cdot,t) \in \mathbb{A}_\infty(\mathbb{R}^n)$ (the class of uniformly Muckenhoupt weights). Let w be an L-harmonic function on $\mathbb{R}^n$ with $0 < C_1 \leqslant w \leqslant C_2$, where C_1 and C_2 are positive constants. In this article, the author proves that the mapping $H_{\varphi,L}(\mathbb{R}^n) \ni f \mapsto wf \in H_\varphi(\mathbb{R}^n)$ is an isomorphism from the Musielak-Orlicz-Hardy space associated with L, $H_{\varphi,L}(\mathbb{R}^n)$, to the Musielak-Orlicz-Hardy space $H_\varphi(\mathbb{R}^n)$ under some assumptions on φ. As applications, the author further obtains the atomic and molecular characterizations of the space $H_{\varphi,L}(\mathbb{R}^n)$ associated with w, and proves that the operator $(-\Delta)^{-1/2}L^{1/2}$ is an isomorphism of the spaces $H_{\varphi,L}(\mathbb{R}^n)$ and $H_\varphi(\mathbb{R}^n)$. All these results are new even when $\varphi(x,t) := t^p$, for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, with $p \in (n/(n+\mu_0),1)$ and some $\mu_0 \in (0,1]$.

Keywords: Musielak-Orlicz-Hardy space; Schrödinger operator; L-harmonic function; isomorphism of Hardy space; atom; molecule

MSC 2010: 42B20, 42B30, 42B35, 35J10, 42B37, 46E30

1. Introduction

Let $n \geqslant 3$. Denote by $W^{1,2}(\mathbb{R}^n)$ the usual *Sobolev space* on the Euclidean space $\mathbb{R}^n$ equipped with the norm $(\|f\|^2_{L^2(\mathbb{R}^n)} + \|\nabla f\|^2_{L^2(\mathbb{R}^n)})^{1/2}$, where ∇f denotes the *distributional gradient* of f. Let $0 \leqslant V \in L^1_{\rm loc}(\mathbb{R}^n)$ and

$$W^{1,2}_V(\mathbb{R}^n) := \Big\{u \in W^{1,2}(\mathbb{R}^n)\colon \int_{\mathbb{R}^n} |u(x)|^2 V(x)\,\mathrm{d}x < \infty\Big\}.$$

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Denote by L the *maximal-accretive operator* (see [23], page 23, Definition 1.46) on $L^2(\mathbb{R}^n)$ with largest domain $D(L) \subset W_V^{1,2}(\mathbb{R}^n)$ such that, for any $f \in D(L)$ and $g \in W_V^{1,2}(\mathbb{R}^n)$,

$$\langle Lf, g\rangle := \int_{\mathbb{R}^n} \nabla f(x)\overline{\nabla g(x)}\,\mathrm{d}x + \int_{\mathbb{R}^n} f(x)\overline{g(x)}V(x)\,\mathrm{d}x,$$

where $\langle\cdot,\cdot\rangle$ denotes the interior product in $L^2(\mathbb{R}^n)$. In this sense, for all $f \in D(L)$ we write

$$Lf := -\Delta f + Vf. \tag{1.1}$$

Denote by $\{K_t\}_{t>0}$ and $\{P_t\}_{t>0}$ the integral kernel of the heat semigroups $\{\mathrm{e}^{-tL}\}_{t>0}$ and $\{\mathrm{e}^{t\Delta}\}$, respectively, generated by $-L$ and the Laplace operator Δ on $\mathbb{R}^n$. Then it follows from the Feynman-Kac formula (see, for example, [26], Chapter V) that for all $t \in (0,\infty)$ and $x, y \in \mathbb{R}^n$,

$$0 \leqslant K_t(x,y) \leqslant P_t(x-y) := \frac{1}{(4\pi t)^{n/2}}\exp\Big\{-\frac{|x-y|^2}{4t}\Big\}. \tag{1.2}$$

We assume in this article that the potential $0 \leqslant V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies

$$\Delta^{-1}V := -c_n\int_{\mathbb{R}^n} |\cdot - y|^{2-n}V(y)\,\mathrm{d}y \in L^\infty(\mathbb{R}^n), \tag{1.3}$$

where $c_n := \Gamma(n/2)/(2\pi^{n/2}(n-2))$ and $\Gamma(\cdot)$ denotes the Gamma function. We also remark that (1.3) is equivalent to that the heat kernels $\{K_t\}_{t>0}$ satisfy the Gaussian lower bounds, namely, there exist positive constants c and C such that for all $t \in (0,\infty)$ and $x, y \in \mathbb{R}^n$,

$$\frac{C}{t^{n/2}}\exp\Big\{-\frac{c|x-y|^2}{t}\Big\} \leqslant K_t(x,y) \tag{1.4}$$

(see, for example, [25]).

Now we recall the definition of L-harmonic functions as follows.

Definition 1.1. Let L be as in (1.1). A function w on $\mathbb{R}^n$ is said to be L-harmonic, if $w \in D(L)$ and $Lw = 0$.

Remark 1.2. (i) We remark that a function w is L-harmonic if and only if $\mathrm{e}^{-tL}w = w$ for all $t \in (0,\infty)$. In fact, if w is L-harmonic, then for any $t \in (0,\infty)$,

$$\mathrm{e}^{-tL}w - w = (\mathrm{e}^{-tL} - I)w = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{-sL}w\,\mathrm{d}s = -\int_0^t \mathrm{e}^{-sL}Lw\,\mathrm{d}s = 0,$$

where I denotes the identical operator in $L^2(\mathbb{R}^n)$, which further yields that $\mathrm{e}^{-tL}w = w$.

Conversely, if $\mathrm{e}^{-tL}w = w$ for any $t \in (0,\infty)$, then by the definition of the infinitesimal generator of the semigroup $\{\mathrm{e}^{-tL}\}_{t>0}$, we see that $w \in D(L)$ and $Lw = 0$.

(ii) It follows from [7], Lemma 2.1, that (1.3) holds true if and only if there exists an L-harmonic function w such that $0 < \delta \leqslant w \leqslant 1$, where δ is a positive constant. Moreover, the function w is unique up to a multiplicative constant. Furthermore, the function w is given for all $x \in \mathbb{R}^n$ by

$$w(x) := \lim_{t\to\infty} \int_{\mathbb{R}^n} K_t(y,x)\,\mathrm{d}y \tag{1.5}$$

up to a multiplicative constant.

(iii) We also point out that, if a nonnegative function V on $\mathbb{R}^n$, with $n \geqslant 3$, satisfies that there exists $\varepsilon \in (0,\infty)$ such that $V \in L^{n/2-\varepsilon}(\mathbb{R}^n) \cap L^{n/2+\varepsilon}(\mathbb{R}^n)$, then (1.3) holds true for V (see [6] for more examples).

Let L be as in (1.1) and satisfy (1.3). Denote by $H^1_L(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, respectively, the Hardy space associated with L (see, for example, [11]) and the classical Hardy space (see, for example, [8]). Assume that w is an L-harmonic function satisfying $0 < \delta \leqslant w \leqslant C$, where δ and C are positive constants. It was proved in [7], Theorem 1.1, that the mapping $H^1_L(\mathbb{R}^n) \ni f \mapsto wf \in H^1(\mathbb{R}^n)$ is an isomorphism from $H^1_L(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. As corollaries, the atomic and molecular characterizations of $H^1_L(\mathbb{R}^n)$, associated with w, were obtained in [7], Corollary 1.2, and [6], Section 3. Moreover, it was also proved in [6], Theorem 1.10, that the operator $(-\Delta)^{-1/2}L^{-1/2}$ is an isomorphism of $H^1_L(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$.

The main purpose of this article is to prove that the mapping

$$H_{\varphi,L}(\mathbb{R}^n) \ni f \mapsto wf \in H_\varphi(\mathbb{R}^n)$$

is an isomorphism from the Musielak-Orlicz-Hardy space $H_{\varphi,L}(\mathbb{R}^n)$, associated with L, to the Musielak-Orlicz-Hardy space $H_\varphi(\mathbb{R}^n)$ under some assumptions on the Musielak-Orlicz function φ. As applications, we further obtain the atomic and molecular characterizations of the space $H_{\varphi,L}(\mathbb{R}^n)$ associated with w, and prove that the operator $(-\Delta)^{-1/2}L^{1/2}$ is an isomorphism from $H_{\varphi,L}(\mathbb{R}^n)$ to $H_\varphi(\mathbb{R}^n)$. It is worth pointing out that all these results are new even when $\varphi(x,t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, with $p \in (n/(n+\mu_0),1)$ and some $\mu_0 \in (0,1]$.

Moreover, we remark that the Musielak-Orlicz-Hardy space is a function space of Hardy-type which unifies the classical Hardy space, the weighted Hardy space, the Orlicz-Hardy space and the weighted Orlicz-Hardy space, in which the spatial and the time variables may not be separable (see [8], [15], [27], [22], [24], [28], [31] for more details on the developments of Hardy-type spaces and Musielak-Orlicz spaces). Furthermore, the Musielak-Orlicz-Hardy space appears naturally in many applications (see, for example, [1], [2], [21], [19]). This kind of Musielak-Orlicz-Hardy spaces associated with operators generalizes the (Orlicz-)Hardy space and the (weighted) Hardy space associated with operators, which has attracted great interests in recent years. Such function spaces associated with operators play important roles in the study for the boundedness of singular integrals associated with some differential operators, which may not fall within the scope of the classical Calderón-Zygmund theory (see, for example, [3], [5], [11], [12], [13], [17], [16], [18], [29], [30]).

Moreover, denote by $\{\widetilde{K}_t\}_{t>0}$ the integral kernels of the semigroup $\{\mathrm{e}^{-t\sqrt{L}}\}_{t>0}$. Then $\widetilde{K}_t$ has the following property, which is just [7], Corollary 3.2.

Lemma 1.3. *Let L be as in* (1.1). *Assume that the potential V satisfies* (1.3) *and w is an L-harmonic function with $0 < C_1 \leqslant w \leqslant C_2$, where C_1 and C_2 are constants. Then there exists positive constants $C > 0$ and $\mu_0 \in (0,1]$ such that for all $t \in (0,\infty)$ and $x, y, z \in \mathbb{R}^n$ with $t > |y-z|$,*

$$\Big|\frac{\widetilde{K}_t(x,y)}{\widetilde{K}_t(x,z)} - \frac{w(y)}{w(z)}\Big| \leqslant C\Big[\frac{|y-z|}{t}\Big]^{\mu_0}.$$

Let the Musielak-Orlicz function φ, the Musielak-Orlicz-Hardy spaces $H_{\varphi,L}(\mathbb{R}^n)$ and $H_\varphi(\mathbb{R}^n)$ be, respectively, as in Definitions 2.4, 2.6 and 2.7 below. Now we give the first main result of this article.

Theorem 1.4. *Let L and φ be, respectively, as in* (1.1) *and Definition* 2.4 *below. Assume that $n + \mu_0 > nq(\varphi)/i(\varphi)$ with μ_0, $q(\varphi)$ and $i(\varphi)$, respectively, as in Lemma* 1.3, (2.2) *and* (2.1) *below, the potential V satisfies* (1.3) *and w is an L-harmonic function with $0 < C_1 \leqslant w \leqslant C_2$, where C_1 and C_2 are constants. Then the mapping*

$$H_{\varphi,L}(\mathbb{R}^n) \ni f \mapsto wf$$

is an isomorphism from the spaces $H_{\varphi,L}(\mathbb{R}^n)$ onto $H_\varphi(\mathbb{R}^n)$. Namely, there exist positive constants C_3, C_4 such that for all $f \in H_{\varphi,L}(\mathbb{R}^n)$,

$$C_3\|wf\|_{H_\varphi(\mathbb{R}^n)} \leqslant \|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \leqslant C_4\|wf\|_{H_\varphi(\mathbb{R}^n)}.$$

By using the atomic characterization of $H_{\varphi,L}(\mathbb{R}^n)$ obtained in [3], Theorem 5.4 (see also Lemma 3.2 below), the molecular characterization of $H_\varphi(\mathbb{R}^n)$ established in [14], Theorem 4.13 (see also Lemma 3.4 below), the definitions of L-harmonic functions, the radial maximal function characterization of $H_{\varphi,L}(\mathbb{R}^n)$ associated with the Poisson semigroup $\{\mathrm{e}^{-t\sqrt{L}}\}_{t>0}$ obtained in [3], Theorem 8.3 (see also Lemma 3.6 below) and Lemma 1.3, we complete the proof of Theorem 1.4.

As a corollary of Theorem 1.6, we can obtain a kind of atomic and molecular characterizations for the space $H_{\varphi,L}(\mathbb{R}^n)$. We first begin with the definitions of (φ,q,w)-atoms and $(\varphi,q,w,\varepsilon)$-molecules. In what follows, for any measurable subset $E\subset\mathbb{R}^n$ and $t\in[0,\infty)$, let $\varphi(E,t):=\int_E\varphi(x,t)\,\mathrm{d}x$.

Definition 1.5. Let L and φ be, respectively, as in (1.1) and Definition 2.4 below. Assume that $q\in(1,\infty)$, w is an L-harmonic function and $B\subset\mathbb{R}^n$ is a ball.

(I) A function $\alpha\in L^q(\mathbb{R}^n)$ is called a (φ,q,w)-*atom* associated with B, if

  (i) $\operatorname{supp}(\alpha)\subset B$;

  (ii) $\|\alpha\|_{L^q(\mathbb{R}^n)}\leqslant|B|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;

  (iii) $\int_{\mathbb{R}^n}\alpha(x)w(x)\,\mathrm{d}x=0$.

(II) For $f\in L^2(\mathbb{R}^n)$,

$$f=\sum_j\lambda_j\alpha_j \tag{1.6}$$

is called an *atomic* (φ,q,w)-*representation* of f if, for all j, α_j is a (φ,q,w)-atom associated with the ball $B_j\subset\mathbb{R}^n$, the summation (1.6) converges in $L^2(\mathbb{R}^n)$ and $\{\lambda_j\}_j\subset\mathbb{C}$ satisfies $\sum_j\varphi(B_j,|\lambda_j|\|\chi_{B_j}\|^{-1}_{L^\varphi(\mathbb{R}^n)})<\infty$. Let

$$\widetilde{H}^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n):=\{f\in L^2(\mathbb{R}^n)\colon\ f\text{ has an atomic }(\varphi,q,w)\text{-representation}\}$$

with the quasi-norm

$$\|f\|_{H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)}:=\inf\left\{\Lambda(\{\lambda_j\alpha_j\}_j)\colon\ \sum_j\lambda_j\alpha_j\text{ is a }(\varphi,q,w)\text{-representation of }f\right\},$$

where the infimum is taken over all atomic (φ,q,w)-representations of f as above and

$$\Lambda(\{\lambda_j\alpha_j\}_j):=\inf\left\{\lambda\in(0,\infty)\colon\ \sum_j\varphi\Big(B_j,\frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\Big)\leqslant1\right\}. \tag{1.7}$$

The *atomic Musielak-Orlicz-Hardy space* $H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$ is then defined as the completion of the set $\widetilde{H}^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)}$.

(III) Let $\varepsilon\in(0,\infty)$. A function $b\in L^q(\mathbb{R}^n)$ is called a $(\varphi,q,w,\varepsilon)$-*molecule* associated with B, if

  (i) $\|b\|_{L^q(S_j(B))}\leqslant2^{-j\varepsilon}|2^jB|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$, where $S_0(B):=2B$ and $S_j(B):=2^{j+1}B\setminus2^jB$ for $j\in\mathbb{N}$;

  (ii) $\int_{\mathbb{R}^n}b(x)w(x)\,\mathrm{d}x=0$.

Moreover, the *molecular Musielak-Orlicz-Hardy space* $H^{q,w,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ is defined via replacing (φ,q,w)-atoms by $(\varphi,q,w,\varepsilon)$-molecules in the definition of the space $H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$.

Now we describe the atomic and molecular characterizations of $H_{\varphi,L}(\mathbb{R}^n)$ associated with an L-harmonic function w.

Theorem 1.6. *Let L, φ and w be as in Theorem 1.4. Assume that $q \in (q(\varphi)[r(\varphi)]',\infty)$ and $\varepsilon \in (nq(\varphi)/i(\varphi),\infty)$, where $q(\varphi)$, $r(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.3) and (2.1) below, and $[r(\varphi)]'$ denotes the conjugate exponent of $r(\varphi)$. Then the spaces $H_{\varphi,L}(\mathbb{R}^n)$, $H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$ and $H^{q,w,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Via Theorem 1.4 and the atomic and molecular characterizations of $H_\varphi(\mathbb{R}^n)$, respectively, obtained in [19], Theorem 1.1, and [14], Theorem 4.13 (see also Lemma 3.4 below), we prove Theorem 1.6.

Now we state another main result of this article.

Theorem 1.7. *Let L, φ and w be as in Theorem 1.4. Assume further that $n+1 > nq(\varphi)/i(\varphi)$ and $q(\varphi)[r(\varphi)]' < n/(nq(\varphi)/i(\varphi)-1)$. Then the mapping*

$$H_{\varphi,L}(\mathbb{R}^n) \ni f \mapsto (-\Delta)^{1/2}L^{-1/2}(f)$$

is an isomorphism from $H_{\varphi,L}(\mathbb{R}^n)$ onto $H_\varphi(\mathbb{R}^n)$. Namely, there exists a positive constant C such that for all $f \in H_{\varphi,L}(\mathbb{R}^n)$,

$$\|(-\Delta)^{1/2}L^{-1/2}(f)\|_{H_\varphi(\mathbb{R}^n)} \leqslant C\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \tag{1.8}$$

and

$$\|L^{1/2}(-\Delta)^{-1/2}(f)\|_{H_{\varphi,L}(\mathbb{R}^n)} \leqslant C\|f\|_{H_\varphi(\mathbb{R}^n)}. \tag{1.9}$$

By applying the atomic and molecular characterizations of $H_{\varphi,L}(\mathbb{R}^n)$ established in Theorem 1.6, the atomic characterization of $H_{\varphi,L}(\mathbb{R}^n)$ obtained in [3], Theorem 5.4 (see also Lemma 3.2 below), the molecular characterizations of $H_\varphi(\mathbb{R}^n)$ obtained in [14], Theorem 4.13 (see also Lemma 3.4 below) and [6], Lemmas 2.11 and 2.13 (see also Lemma 4.1 below), we prove Theorem 1.7.

Remark 1.8. Let L and φ be as in Theorem 1.4. Assume that $q(\varphi)$, $r(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2), (2.3) and (2.1) below.

(i) When $\varphi(x,t) := t$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, then $i(\varphi) = 1$, $q(\varphi) = 1$ and $r(\varphi) = \infty$. It is easy to see that the assumptions in Theorems 1.4 through 1.7 on φ automatically hold true in this case. Then Theorems 1.4 through 1.7 are just, respectively, [7], Theorem 1.1 and Corollary 1.1, and [6], Theorem 1.10, in this case.

(ii) Let μ_0 be as in Lemma 1.3. When $\varphi(x,t) := t^p$ for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$, with $p \in (n/(n+\mu_0), 1]$, $i(\varphi) = p$, then $q(\varphi) = 1$ and $r(\varphi) = \infty$. In this case, we can verify that the assumptions in Theorems 1.4 through 1.7 on φ hold true. Moreover, it is worth pointing out that Theorems 1.4 through 1.7 are new in this case.

(iii) Let φ be as in (2.4) below. Then $i(\varphi) = 1$, $q(\varphi) = 1$ and $r(\varphi) = \infty$ (see, for example, [4], Remark 1 (v)), which further implies that the assumptions in Theorems 1.4 through 1.7 on φ hold true in this case. Thus, Theorems 1.4 through 1.7 hold true for the spaces $H_{\varphi,L}(\mathbb{R}^n)$ and $H_\varphi(\mathbb{R}^n)$ associated with φ (see [4], Remark 1 (v), for more examples of φ satisfying the assumptions in Theorems 1.4 through 1.7).

The layout of this article is as follows. In Section 2, we first describe the growth function considered in this article; then we recall the definitions of Musielak-Orlicz-Hardy spaces $H_{\varphi,L}(\mathbb{R}^n)$ and $H_\varphi(\mathbb{R}^n)$; finally we introduce some conventions on notation. In Section 3, we give the proofs of Theorems 1.4 and 1.6. Then, in Section 4, we present the proof of Theorem 1.7.

2. Preliminaries

2.1. Musielak-Orlicz functions. In this subsection, we describe the growth function considered in this article. First we recall the definition of Orlicz functions (see, for example, [22], [24]).

Definition 2.1. A function $\Phi\colon [0,\infty) \to [0,\infty)$ is called an *Orlicz function* if it is nondecreasing, $\Phi(0) = 0$, $\Phi(t) > 0$ for any $t \in (0,\infty)$ and $\lim\limits_{t\to\infty} \Phi(t) = \infty$.

We point out that, unlike the classical definition of Orlicz functions, the *Orlicz functions in this article need not be convex.*

Now we recall the definition of upper (lower, respectively) type of functions as follows.

Definition 2.2. (i) An Orlicz function Φ is said to be of *upper* (*lower*) *type* p for some $p \in [0,\infty)$, if there exists a positive constant C such that, for all $s \in [1,\infty)$ ($s \in [0,1]$, respectively) and $t \in [0,\infty)$, $\Phi(st) \leqslant Cs^p\Phi(t)$.

(ii) For a given function $\varphi\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ such that, for any $x \in \mathbb{R}^n$, $\varphi(x,\cdot)$ is an Orlicz function, φ is said to be of *uniformly upper* (*lower*) *type* p for some $p \in (0,\infty)$, if there exists a positive constant C such that, for all $x \in \mathbb{R}^n$, $t \in [0,\infty)$ and $s \in [1,\infty)$ ($s \in [0,1]$, respectively), $\varphi(x,st) \leqslant Cs^p\varphi(x,t)$.

Let

$$i(\varphi) := \sup\{p \in (0,\infty)\colon\ \varphi \text{ is of uniformly lower type } p\}. \tag{2.1}$$

Observe that $i(\varphi)$ need not be attainable, namely, φ need not be of uniformly lower type $i(\varphi)$ (see, for example, [3], [4], [30]).

Definition 2.3. Let $\varphi\colon\ \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ satisfy that $\varphi(\cdot,t)$ is measurable for all $t \in [0,\infty)$. The function φ is said to satisfy the *uniformly Muckenhoupt condition for some* $q \in [1,\infty)$, denoted by $\varphi \in \mathbb{A}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$,

$$\mathbb{A}_q(\varphi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|^q} \int_B \varphi(x,t)\,\mathrm{d}x \bigg\{\int_B [\varphi(y,t)]^{1-q}\,\mathrm{d}y\bigg\}^{q-1} < \infty$$

or, when $q = 1$,

$$\mathbb{A}_1(\varphi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|} \int_B \varphi(x,t)\,\mathrm{d}x \Big(\operatorname*{ess\,sup}_{y\in B}[\varphi(y,t)]^{-1}\Big) < \infty.$$

Here the first suprema are taken over all $t \in (0,\infty)$ and the other ones over all balls $B \subset \mathbb{R}^n$.

The function φ is said to satisfy the *uniformly reverse Hölder condition for some* $q \in (1,\infty]$, denoted by $\varphi \in \mathbb{RH}_q(\mathbb{R}^n)$, if, when $q \in (1,\infty)$,

$$\mathbb{RH}_q(\varphi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \bigg\{\frac{1}{|B|}\int_B [\varphi(x,t)]^q\,\mathrm{d}x\bigg\}^{1/q} \bigg\{\frac{1}{|B|}\int_B \varphi(x,t)\,\mathrm{d}x\bigg\}^{-1} < \infty$$

or, when $q = \infty$,

$$\mathbb{RH}_\infty(\varphi) := \sup_{t\in(0,\infty)} \sup_{B\subset\mathbb{R}^n} \Big\{\operatorname*{ess\,sup}_{y\in B} \varphi(y,t)\Big\} \bigg\{\frac{1}{|B|}\int_B \varphi(x,t)\,\mathrm{d}x\bigg\}^{-1} < \infty.$$

Here the first suprema are taken over all $t \in (0,\infty)$ and the other ones over all balls $B \subset \mathbb{R}^n$.

Recall that, in Definition 2.3, $\mathbb{A}_p(\mathbb{R}^n)$, with $p \in [1,\infty)$, and $\mathbb{RH}_q(\mathbb{R}^n)$, with $q \in (1,\infty]$, were introduced, respectively, in [20] and [30].

Let $\mathbb{A}_\infty(\mathbb{R}^n) := \bigcup_{q\in[1,\infty)} \mathbb{A}_q(\mathbb{R}^n)$. It is well known that

$$\mathbb{A}_\infty(\mathbb{R}^n) = \bigcup_{q\in(1,\infty]} \mathbb{RH}_q(\mathbb{R}^n),$$

$\mathbb{A}_p(\mathbb{R}^n) \subset \mathbb{A}_q(\mathbb{R}^n)$ for $1 \leqslant p \leqslant q < \infty$, and $\mathbb{RH}_p(\mathbb{R}^n) \subset \mathbb{RH}_q(\mathbb{R}^n)$ for $1 < q \leqslant p \leqslant \infty$ (see, for example, [14], Lemma 2.4, or Lemma 2.5 below). Thus, we can introduce the *critical indices* for $\varphi \in \mathbb{A}_\infty(\mathbb{R}^n)$ as follows:

$$q(\varphi) := \inf\{q \in [1,\infty)\colon\ \varphi \in \mathbb{A}_q(\mathbb{R}^n)\} \tag{2.2}$$

and

$$r(\varphi) := \sup\{q \in (1,\infty]\colon\ \varphi \in \mathbb{RH}_q(\mathbb{R}^n)\}. \tag{2.3}$$

Now we recall the notion of growth functions from Ky [20].

Definition 2.4. A function $\varphi\colon\ \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ is called a *growth function* if the following conditions hold:

(i) φ is a *Musielak-Orlicz function*, namely,
  (a) $\varphi(x,\cdot)\colon\ [0,\infty) \to [0,\infty)$ is an Orlicz function for all $x \in \mathbb{R}^n$;
  (b) $\varphi(\cdot,t)$ is a measurable function for all $t \in [0,\infty)$.

(ii) $\varphi(\cdot,t) \in \mathbb{A}_\infty(\mathbb{R}^n)$ for any $t \in (0,\infty)$.

(iii) The function φ is of uniformly lower type p for some $p \in (0,1]$ and upper type 1.

Clearly, $\varphi(x,t) := \omega(x)\Phi(t)$ is a growth function if $\omega \in A_\infty(\mathbb{R}^n)$ and Φ is an Orlicz function of lower type p for some $p \in (0,1]$ and of upper type 1. Here, $A_q(\mathbb{R}^n)$ with $q \in [1,\infty]$ denotes the class of Muckenhoupt weights (see, for example, [9], [10]). A typical example of such Orlicz function Φ is $\Phi(t) := t^p$, with $p \in (0,1]$, for all $t \in [0,\infty)$ (see, for example, [31], [30] for more examples of such Φ). Another typical example of a growth function is

$$\varphi(x,t) := \frac{t}{\ln(\mathrm{e}+|x|)+\ln(\mathrm{e}+t)} \tag{2.4}$$

for all $x \in \mathbb{R}^n$ and $t \in [0,\infty)$.

Moreover, we need some properties of φ in Definition 2.4, which are useful in the proof of Theorems 1.4, 1.6 and 1.7. Then we have the following properties for φ from [3], Lemma 2.5, based on the corresponding results of [20], [9], [10].

Lemma 2.5. *Let the function φ be as in Definition* 2.4.

(i) *There exists a positive constant C such that, for all $(x,t_j) \in \mathbb{R}^n \times [0,\infty)$ with $j \in \mathbb{N}$, $\varphi\Big(x, \sum\limits_{j=1}^{\infty} t_j\Big) \leqslant C \sum\limits_{j=1}^{\infty} \varphi(x,t_j)$.*

(ii) $\mathbb{A}_1(\mathbb{R}^n) \subset \mathbb{A}_p(\mathbb{R}^n) \subset \mathbb{A}_q(\mathbb{R}^n)$ *for* $1 \leqslant p \leqslant q < \infty$.

(iii) $\mathbb{RH}_\infty(\mathbb{R}^n) \subset \mathbb{RH}_p(\mathbb{R}^n) \subset \mathbb{RH}_q(\mathbb{R}^n)$ *for* $1 < q \leqslant p \leqslant \infty$.

(iv) $\mathbb{A}_\infty(\mathbb{R}^n) = \bigcup_{p\in[1,\infty)} \mathbb{A}_p(\mathbb{R}^n) = \bigcup_{q\in(1,\infty]} \mathbb{R}\mathbb{H}_q(\mathbb{R}^n)$.

(v) *If $\varphi \in \mathbb{A}_p(\mathbb{R}^n)$ with $p \in [1,\infty)$, then there exists a positive constant C such that, for all balls $B_1, B_2 \subset \mathbb{R}^n$ with $B_1 \subset B_2$ and $t \in (0,\infty)$, $\varphi(B_2,t)/\varphi(B_1,t) \leqslant C[|B_2|/|B_1|]^p$.*

2.2. Musielak-Orlicz-Hardy spaces. In this subsection, we recall the definitions of Musielak-Orlicz-Hardy spaces $H_\varphi(\mathbb{R}^n)$, introduced in [20], and Musielak-Orlicz-Hardy spaces $H_{\varphi,L}(\mathbb{R}^n)$ associated with Schrödinger operators L, introduced in [3], [30].

Recall that for a function φ as in Definition 2.4, a measurable function f on $\mathbb{R}^n$ is said to be in the *Musielak-Orlicz space* $L^\varphi(\mathbb{R}^n)$ if $\int_{\mathbb{R}^n} \varphi(x,|f(x)|)\,\mathrm{d}x < \infty$. Moreover, for any $f \in L^\varphi(\mathbb{R}^n)$, define

$$\|f\|_{L^\varphi(\mathbb{R}^n)} := \inf\left\{\lambda \in (0,\infty)\colon \int_{\mathbb{R}^n} \varphi\Big(x, \frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x \leqslant 1\right\}.$$

Let L and φ be, respectively, as in (1.3) and Definition 2.4. We remark that L is a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$. Moreover, the Gaussian upper bound estimate for the kernels of the semigroup $\{\mathrm{e}^{-tL}\}_{t>0}$ further implies that the semigroup $\{\mathrm{e}^{-tL}\}_{t>0}$ satisfies the reinforced $(1,\infty,1)$ off-diagonal estimates (see [3], Assumption (B), for the details). Thus, L is a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying the reinforced $(1,\infty,1)$ off-diagonal estimates. Now we recall the Musielak-Orlicz-Hardy space $H_{\varphi,L}(\mathbb{R}^n)$ associated with L introduced in [3].

Definition 2.6. For $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *Lusin area function* $S_L(f)(x)$ associated with L is defined by

$$S_L(f)(x) := \left\{\int_{\Gamma(x)} |t^2 L\mathrm{e}^{-t^2L}(f)(y)|^2 \frac{\mathrm{d}y\,\mathrm{d}t}{t^{n+1}}\right\}^{1/2},$$

where $\Gamma(x) := \{(y,t) \in \mathbb{R}^n \times (0,\infty)\colon |y-x| < t\}$. A function $f \in L^2(\mathbb{R}^n)$ is said to be in the set $\widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ if $S_L(f) \in L^\varphi(\mathbb{R}^n)$; moreover, we define $\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} := \|S_L(f)\|_{L^\varphi(\mathbb{R}^n)}$.

The *Musielak-Orlicz-Hardy space* $H_{\varphi,L}(\mathbb{R}^n)$ is defined to be the completion of $\widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H_{\varphi,L}(\mathbb{R}^n)}$.

Now we recall the definition of the Musielak-Orlicz-Hardy space $H_\varphi(\mathbb{R}^n)$ introduced in [20]. We first introduce some notions. In what follows, we denote by $\mathcal{S}(\mathbb{R}^n)$ the *space of all Schwartz functions* and by $\mathcal{S}'(\mathbb{R}^n)$ its *dual space* (namely, the *space of all tempered distributions*). Let $\mathbb{N} := \{1,\ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any $\theta := (\theta_1, \ldots, \theta_n) \in \mathbb{Z}_+^n$, let $|\theta| := \theta_1 + \ldots + \theta_n$ and $\partial_x^\theta := \partial^{|\theta|}/\partial x_1^{\theta_1} \ldots \partial x_n^{\theta_n}$. For $m \in \mathbb{N}$, we define

$$\mathcal{S}_m(\mathbb{R}^n) := \Big\{\varphi \in \mathcal{S}(\mathbb{R}^n)\colon \sup_{x\in\mathbb{R}^n} \sup_{\beta\in\mathbb{Z}_+^n, |\beta|\leqslant m+1} (1+|x|)^{(m+2)(n+1)} |\partial_x^\beta \varphi(x)| \leqslant 1\Big\}.$$

Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the *non-tangential grand maximal function* f_m^* of f is defined by setting,

$$f_m^*(x) := \sup_{\varphi\in\mathcal{S}_m(\mathbb{R}^n)} \sup_{|y-x|<t,\, t\in(0,\infty)} |f * \varphi_t(y)|,$$

where for all $t \in (0, \infty)$, $\varphi_t(\cdot) := t^{-n}\varphi(\cdot/t)$. When $m(\varphi) := \lfloor n[q(\varphi)/i(\varphi) - 1]\rfloor$, where $q(\varphi)$ and $i(\varphi)$ are, respectively, as in (2.2) and (2.1), and $\lfloor s\rfloor$ for $s \in \mathbb{R}$ denotes the *maximal integer* k such that $k \leqslant s$, we *denote* $f_{m(\varphi)}^*$ *simply by* f^*.

Definition 2.7. Let φ be as in Definition 2.4. The *Musielak-Orlicz-Hardy space* $H_\varphi(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f^* \in L^\varphi(\mathbb{R}^n)$, with the *quasi-norm* $\|f\|_{H_\varphi(\mathbb{R}^n)} := \|f^*\|_{L^\varphi(\mathbb{R}^n)}$.

It is worth noting that for such φ as in (2.4), the corresponding Musielak-Orlicz-Hardy space $H_\varphi(\mathbb{R}^n)$ or $H_{\varphi,L}(\mathbb{R}^n)$, associated with the Schrödinger operator $L := -\Delta + V$ on $\mathbb{R}^n$, appears naturally when studying the products of functions in $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$, the endpoint estimates for the div-curl lemma and the endpoint estimates for commutators of singular integrals related to the Schrödinger operator L (see [1], [2], [21], [19] for the details).

2.3. Notation. In this subsection, we make some conventions on notation. Throughout the article, we denote by C a *positive constant* which is independent of the main parameters, but may vary from line to line. We also use $C_{(\gamma,\beta,\ldots)}$ to denote a *positive constant* depending on the indicated parameters $\gamma, \beta, \ldots$ The *symbol* $A \lesssim B$ means that $A \leqslant CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. For any given (quasi-)normed spaces $\mathcal{A}$ and $\mathcal{B}$ with the corresponding norms $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{B}}$, the *symbol* $\mathcal{A} \subset \mathcal{B}$ means that for all $f \in \mathcal{A}$, we have $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \lesssim \|f\|_{\mathcal{A}}$. For any measurable subset E of $\mathbb{R}^n$, we denote by χ_E its *characteristic function* and by $E^{\complement}$ the *set* $\mathbb{R}^n \setminus E$. We also set $\mathbb{N} := \{1, \ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Moreover, for each ball $B \subset \mathbb{R}^n$, let $S_0(B) := 2B$ and $S_j(B) := 2^{j+1}B \setminus 2^jB$ for $j \in \mathbb{N}$. Finally, for $q \in [1, \infty]$ we denote by q' the *conjugate exponent* of q, namely, $1/q + 1/q' = 1$.

3. Proofs of Theorems 1.4 and 1.6

In this section we give the proofs of Theorems 1.4 and 1.6. We begin with some useful auxiliary conclusions.

We first recall the definitions of $(\varphi, q, M)_L$-atoms and the atomic Musielak-Orlicz-Hardy space $H^{M,q}_{\varphi,L,\mathrm{at}}(\mathbb{R}^n)$ introduced in [3], Definitions 5.2 and 5.3.

Definition 3.1. Let L and φ be, respectively, as in (1.1) and Definition 2.4. Assume that $q \in (1, \infty)$, $M \in \mathbb{N}$ and $B \subset \mathbb{R}^n$ is a ball. Let $\mathcal{D}(L^M)$ be the domain of L^M. A function $\alpha \in L^q(\mathbb{R}^n)$ is called a $(\varphi, q, M)_L$-*atom* associated with the ball B, if there exists a function $b \in \mathcal{D}(L^M)$ such that

(i) $\alpha = L^M b$;

(ii) for all $j \in \{0, 1, \ldots, M\}$, $\operatorname{supp}(L^j b) \subset B$;

(iii) $\|(r_B^2 L)^j b\|_{L^q(\mathbb{R}^n)} \leqslant r_B^{2M} |B|^{1/q} \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$, where r_B denotes the radius of B and $j \in \{0, 1, \ldots, M\}$.

The *atomic Musielak-Orlicz-Hardy space* $H^{M,q}_{\varphi,L,\mathrm{at}}(\mathbb{R}^n)$ is defined via replacing (φ, q, w)-atoms by $(\varphi, q, M)_L$-atoms in the definition of the space $H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$ (see Definition 1.5 (II) above).

Then we have the following atomic characterization of the space $H_{\varphi,L}(\mathbb{R}^n)$, which is just [3], Theorem 5.4.

Lemma 3.2. *Let L and φ be, respectively, as in* (1.1) *and Definition* 2.4. *Assume that $M \in \mathbb{N}$ satisfies $M > nq(\varphi)/(2i(\varphi))$ and $q \in ([r(\varphi)]', \infty)$, where $q(\varphi)$, $i(\varphi)$ and $r(\varphi)$ are, respectively, as in* (2.2), (2.1) *and* (2.3). *Then the spaces $H_{\varphi,L}(\mathbb{R}^n)$ and $H^{M,q}_{\varphi,L,\mathrm{at}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Moreover, to prove Theorem 1.4, we need the atomic and molecular characterizations of $H_\varphi(\mathbb{R}^n)$ established in [20], Theorem 1.1, and [14], Theorem 4.13. To state the atomic and molecular characterizations of the space $H_\varphi(\mathbb{R}^n)$, we first recall the definitions of (φ, ∞, s)-atoms, $(\varphi, q, s, \varepsilon)$-molecules and Hardy-type spaces defined by these atoms and molecules.

Definition 3.3. Let φ be as in Definition 2.4, $q \in (1, \infty)$, $s \in \mathbb{Z}_+$, $\varepsilon \in (0, \infty)$ and let $B \subset \mathbb{R}^n$ be a ball.

(I) A function $\alpha \in L^q(\mathbb{R}^n)$ is called a $(\varphi, q, s, \varepsilon)$-*molecule* associated with B, if

  (i) for each $j \in \mathbb{Z}_+$, $\|\alpha\|_{L^q(S_j(B))} \leqslant 2^{-j\varepsilon} |2^j B|^{1/q} \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$;

  (ii) $\int_{\mathbb{R}^n} \alpha(x) x^\beta \, \mathrm{d}x = 0$ for all $\beta \in \mathbb{Z}^n_+$ with $|\beta| \leqslant s$.

(II) The *molecular Musielak-Orlicz-Hardy space* $H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ is defined to be the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that $f = \sum_j \lambda_j \alpha_j$ in $\mathcal{S}'(\mathbb{R}^n)$, where $\{\lambda_j\}_j \subset \mathbb{C}$,

$\{\alpha_j\}_j$ is a sequence of $(\varphi, q, s, \varepsilon)$-molecules associated with the balls $\{B_j\}_j$, and

$$\sum_j \varphi(B_j, |\lambda_j| \|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}^{-1}) < \infty.$$

Moreover, we define

$$\|f\|_{H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)} := \inf\{\Lambda(\{\lambda_j \alpha_j\}_j)\},$$

where the infimum is taken over all the decompositions of f as above and $\Lambda(\{\lambda_j \alpha_j\}_j)$ is as in (1.7).

(III) Let $s \in \mathbb{Z}_+$ satisfy that $s \geqslant \lfloor n[q(\varphi)/i(\varphi) - 1]\rfloor$. A function a on $\mathbb{R}^n$ is said to be a (φ, ∞, s)-*atom*, if there exists a ball $B \subset \mathbb{R}^n$ such that

(i) $\mathrm{supp}(a) \subset B$;

(ii) $\|a\|_{L^\infty(B)} \leqslant \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}$;

(iii) $\int_{\mathbb{R}^n} a(x) x^\alpha \, \mathrm{d}x = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leqslant s$.

The *atomic Musielak-Orlicz-Hardy space* $H^{\varphi,\infty,s}(\mathbb{R}^n)$ is defined via replacing $(\varphi, q, s, \varepsilon)$-molecules by (φ, ∞, s)-atoms in the definition of the space $H^{q,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$.

Then we have the following conclusion, which is just a corollary of [20], Theorem 1.1, and [14], Theorem 4.13.

Lemma 3.4. *Let φ be as in Definition* 2.4. *Assume that $s \in \mathbb{Z}_+$ with $s \geqslant \lfloor n(q(\varphi)/i(\varphi) - 1)\rfloor$, $\varepsilon \in (\max\{n+s, nq(\varphi)/i(\varphi)\}, \infty)$ and $p \in (q(\varphi)[r(\varphi)]', \infty)$, where $i(\varphi)$, $q(\varphi)$ and $r(\varphi)$ are, respectively, as in* (2.1), (2.2) *and* (2.3). *Then the spaces $H_\varphi(\mathbb{R}^n)$, $H^{\varphi,\infty,s}(\mathbb{R}^n)$ and $H^{p,s,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Remark 3.5. (i) Let $H_{\varphi,L}(\mathbb{R}^n)$ and $H^{M,q}_{\varphi,L,\mathrm{at}}(\mathbb{R}^n)$ be as in Lemma 3.2. By the proof of [3], Theorem 5.4, we know that, if $f \in H_{\varphi,L}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then the decomposition $f = \sum_j \lambda_j \alpha_j$ holds true in $L^2(\mathbb{R}^n)$, where $\{\lambda_j\}_j \subset \mathbb{C}$ and $\{\alpha_j\}_j$ is a sequence of $(\varphi, q, M)_L$-atoms.

(ii) Let $H_\varphi(\mathbb{R}^n)$ and $H^{\varphi,\infty,s}(\mathbb{R}^n)$ be as in Lemma 3.4. Then from the proof of [20], Theorem 5.2, it follows that, if $f \in H_\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then the decomposition $f = \sum_j \lambda_j a_j$ holds true in $L^2(\mathbb{R}^n)$, where $\{\lambda_j\}_j \subset \mathbb{C}$ and $\{a_j\}_j$ is a sequence of (φ, ∞, s)-atoms.

To prove Theorem 1.4, we need the following maximal function characterization of $H_{\varphi,L}(\mathbb{R}^n)$. We first recall the definition of maximal functions.

For $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the *radial maximal function* of f, associated with the semigroup $\{\mathrm{e}^{-t\sqrt{L}}\}_{t>0}$ generated by $-\sqrt{L}$, by setting

$$\mathcal{N}_P(f)(x) := \sup_{t\in(0,\infty)} |\mathrm{e}^{-t\sqrt{L}}(f)(x)|.$$

Let

$$\widetilde{H}_{\varphi,\mathcal{N}_P}(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n)\colon\ \mathcal{N}_P(f) \in L^{\varphi}(\mathbb{R}^n)\}$$

with $\|f\|_{H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)} := \|\mathcal{N}_P(f)\|_{L^{\varphi}(\mathbb{R}^n)}$. Then the space $H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)$ is defined as the *completion* of the set $\widetilde{H}_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)}$.

Then we have the following conclusion, which is just [3], Theorem 8.3.

Lemma 3.6. *Let L and φ be, respectively, as in* (1.1) *and Definition* 2.4. *Then the spaces $H_{\varphi,L}(\mathbb{R}^n)$ and $H_{\varphi,\mathcal{N}_P}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.*

Now we give the proof of Theorem 1.4 by applying Lemmas 3.4 through 2.5 and Lemma 1.3.

P r o o f of Theorem 1.4. To prove Theorem 1.4, it suffices to show that

$$\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \|wf\|_{H_{\varphi}(\mathbb{R}^n)} \tag{3.1}$$

holds true for all $f \in H_{\varphi,L}(\mathbb{R}^n)$.

First let $f \in \widetilde{H}_{\varphi,L}(\mathbb{R}^n)$, $q \in (q(\varphi)[r(\varphi)]', \infty)$ and let $M \in \mathbb{N}$ satisfy $M > nq(\varphi)/(2i(\varphi))$. By this, Lemma 3.2 and Remark 3.5 (i), we see that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of $(\varphi, q, M)_L$-atoms, associated with the balls $\{B_j\}_j$ such that

$$f = \sum_j \lambda_j \alpha_j \quad \text{in } L^2(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j \alpha_j\}_j). \tag{3.2}$$

Moreover, from the definition of $(\varphi, q, M)_L$-atoms, we deduce that, for any $(\varphi, q, M)_L$-atom α associated with the ball B, there exists $b \in D(L)$ such that $\alpha = Lb$, which, combined with the fact that w is an L-harmonic function and L is a self-adjoint operator on $L^2(\mathbb{R}^n)$, further implies that

$$\int_{\mathbb{R}^n} \alpha(x)w(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} Lb(x)w(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} b(x)Lw(x)\,\mathrm{d}x = 0. \tag{3.3}$$

Furthermore, by the assumptions $0 < C_1 \leqslant w \leqslant C_2$ and $\operatorname{supp}(\alpha) \subset B$, we conclude that $\operatorname{supp}(\alpha w) \subset B$ and

$$\|\alpha w\|_{L^q(\mathbb{R}^n)} \leqslant C_2\|\alpha\|_{L^q(\mathbb{R}^n)} \leqslant C_2|B|^{1/q}\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)},$$

which, together with (3.3), implies that αw is a $(\varphi, q, 0, \varepsilon)$-molecule for any $\varepsilon \in (0,\infty)$ up to a harmless constant multiple. From this and (3.2), it follows that, for any j, $\alpha_j w$ is a constant multiple of a $(\varphi, q, 0, \varepsilon)$-molecule with any $\varepsilon \in (0,\infty)$,

$$wf = \sum_j \lambda_j(\alpha_j w) \quad \text{in } L^2(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j\alpha_j\}_j) \sim \Lambda(\{\lambda_j(\alpha_j w)\}_j),$$

which, together with Lemma 3.4, further implies that $wf \in H_\varphi(\mathbb{R}^n)$ and

$$\|wf\|_{H_\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H_{\varphi,L}(\mathbb{R}^n)}.$$

This, combined with the arbitrariness of $f \in \widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ and the fact that $\widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ is dense in $H_{\varphi,L}(\mathbb{R}^n)$, yields that, for any $f \in H_{\varphi,L}(\mathbb{R}^n)$, we have $wf \in H_\varphi(\mathbb{R}^n)$ and

$$\|wf\|_{H_\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H_{\varphi,L}(\mathbb{R}^n)}. \tag{3.4}$$

Now let $wf \in H_\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then by Lemma 3.4 and Remark 3.5 (ii) we see that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $(\varphi, \infty, 0)$-atoms, supported in the balls $\{B_j\}_j$, such that

$$wf = \sum_j \lambda_j a_j \ \text{ in } L^2(\mathbb{R}^n) \quad \text{and} \quad \|wf\|_{H_\varphi(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j a_j\}_j). \tag{3.5}$$

To prove $f \in H_{\varphi,L}(\mathbb{R}^n)$ via Lemma 3.6, we only need to show that for any $(\varphi, \infty, 0)$-atom a supported in the ball $B := B(x_0, r_0)$, and $\lambda \in \mathbb{C}$,

$$\int_{\mathbb{R}^n} \varphi(x, \mathcal{N}_P(\lambda a/w)(x))\, \mathrm{d}x \lesssim \varphi(B, |\lambda| \|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}). \tag{3.6}$$

In fact, if (3.6) holds true, from (3.6), (3.5) and Lemma 2.5 (i) we deduce that for any $\lambda \in (0, \infty)$,

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi\Big(x, \frac{\mathcal{N}_P(f)(x)}{\lambda}\Big)\, \mathrm{d}x &\lesssim \sum_j \int_{\mathbb{R}^n} \varphi\Big(x, \mathcal{N}_P\Big(\frac{\lambda_j a_j}{\lambda w}\Big)(x)\Big)\, \mathrm{d}x \\ &\lesssim \sum_j \varphi\Big(B_j, \frac{|\lambda_j|}{\lambda \|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\Big),\end{aligned}$$

which, together with (3.5) and Lemma 3.6, further implies that $f \in H_{\varphi,L}(\mathbb{R}^n)$ and

$$\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \|\mathcal{N}_P(f)\|_{L^\varphi(\mathbb{R}^n)} \lesssim \|wf\|_{H_\varphi(\mathbb{R}^n)}.$$

This, combined with the arbitrariness of $wf \in H_\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and the fact that $H_\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $H_\varphi(\mathbb{R}^n)$, concludes that, for any $wf \in H_\varphi(\mathbb{R}^n)$, $f \in H_{\varphi,L}(\mathbb{R}^n)$ and

$$\|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \lesssim \|wf\|_{H_\varphi(\mathbb{R}^n)},$$

which, together with (3.4), yields that (3.1) holds true.

Now we prove (3.6). From the assumption $n+\mu_0>nq(\varphi)/i(\varphi)$, we deduce that there exist $\widetilde{q}\in(q(\varphi),\infty)$ and $p_0\in(0,i(\varphi))$ such that $n+\mu_0>n\widetilde{q}/p_0$, $\varphi\in\mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$ and φ is of uniformly lower type p_0. By the well-known subordination formula

$$\mathrm{e}^{-t\sqrt{L}}=\frac{1}{\pi^{1/2}}\int_0^\infty \mathrm{e}^{-u}\mathrm{e}^{-t^2L/(4u)}\frac{\mathrm{d}u}{u^{1/2}} \tag{3.7}$$

associated with L (see, for example, [11], (4.22)), (1.2) and (1.4), we conclude that, for all $x,y\in\mathbb{R}^n$,

$$\widetilde{K}_t(x,y)\sim\frac{t}{(t+|x-y|)^{n+1}}. \tag{3.8}$$

Indeed, from (3.7) and (1.2) we deduce that for all $t\in(0,\infty)$ and x, $y\in\mathbb{R}^n$,

$$\begin{aligned}\widetilde{K}_t(x,y)&=\frac{1}{\pi^{1/2}}\int_0^\infty \mathrm{e}^{-u}K_{t^2/4u}(x,y)\frac{\mathrm{d}u}{u^{1/2}}\\ &\lesssim\int_0^\infty \mathrm{e}^{-u}\frac{u^{(n-1)/2}}{t^n}\mathrm{e}^{-|x-y|^2/t^2u}\,\mathrm{d}s\\ &\sim t^{-n}\int_0^\infty u^{(n-1)/2}\mathrm{e}^{-(1+|x-y|^2/t^2)u}\,\mathrm{d}s\\ &\sim t^{-n}\Big[1+\frac{|x-y|^2}{t^2}\Big]^{-(n+1)/2}\int_0^\infty \mathrm{e}^{-s}s^{(n-1)/2}\,\mathrm{d}s\\ &\sim\frac{t}{(t+|x-y|)^{n+1}}.\end{aligned} \tag{3.9}$$

Moreover, via (3.7) and (1.4), repeating the proof of (3.9), we see that for all $t\in(0,\infty)$ and $x,y\in\mathbb{R}^n$,

$$\widetilde{K}_t(x,y)\gtrsim\frac{t}{(t+|x-y|)^{n+1}},$$

which, together with (3.9), implies that (3.8) holds true.

Furthermore, let $x\in 2B$. Then by the uniformly upper type 1 property of φ, we know that, if $\mathcal{N}_P(a/w)(x)\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\geqslant 1$, then

$$\begin{aligned}\varphi(x,\mathcal{N}_P(\lambda a/w)(x))&=\varphi(x,|\lambda|\mathcal{N}_P(a/w)(x))\\ &\lesssim\varphi(x,|\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)})\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\mathcal{N}_P(a/w)(x).\end{aligned} \tag{3.10}$$

Similarly to the proof of (3.10), from the uniformly lower type p_0 property of φ it follows that if $\mathcal{N}_P(a/w)(x)\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\in(0,1)$, then

$$\varphi(x,\mathcal{N}_P(\lambda a/w)(x))\lesssim\varphi(x,|\lambda|\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)})[\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\mathcal{N}_P(a/w)(x)]^{p_0},$$

which, combined with (3.10), implies that

$$\varphi(x,\mathcal{N}_P(\lambda a/w)(x)) \lesssim \varphi(x,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})\{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\mathcal{N}_P(a/w)(x) + [\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\mathcal{N}_P(a/w)(x)]^{p_0}\}. \tag{3.11}$$

Denote by $\mathcal{M}$ the Hardy-Littlewood maximal operator on $\mathbb{R}^n$. Then it follows, from (3.8), that $\mathcal{N}_P(a/w) \lesssim \mathcal{M}(a/w)$. By this, (3.11), Hölder's inequality, $L^p(\mathbb{R}^n)$-boundedness of $\mathcal{M}$ with $p \in (1,\infty)$, $0 < C_1 \leqslant w \leqslant C_2$ and Lemma 2.5 (v), we conclude that

$$\begin{aligned}
&\int_{2B}\varphi(x,\mathcal{N}_P(\lambda a/w)(x))\,\mathrm{d}x \\
&\quad\lesssim \int_{2B}\varphi(x,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})\{\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\mathcal{M}(a/w)(x) \\
&\qquad + \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_0}[\mathcal{M}(a/w)(x)]^{p_0}\}\,\mathrm{d}x \\
&\quad\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|\mathcal{M}(a/w)\|_{L^q(2B)}\|\varphi(\cdot,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})\|_{L^{q'}(2B)} \\
&\qquad + \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_0}\|\mathcal{M}(a/w)\|_{L^q(2B)}^{p_0}\|\varphi(\cdot,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})\|_{L^{(q/p_0)'}(2B)} \\
&\quad\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}\|a\|_{L^q(\mathbb{R}^n)}|2B|^{-1/q}\varphi(2B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}) \\
&\qquad + \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{p_0}\|a\|_{L^q(\mathbb{R}^n)}^{p_0}|2B|^{-p_0/q}\varphi(2B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}) \\
&\quad\lesssim \varphi(B,|\lambda|\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}),
\end{aligned} \tag{3.12}$$

where $q \in (1,\infty)$ is large enough such that $\varphi \in \mathbb{RH}_{q'}(\mathbb{R}^n)$.

Moreover, for $x \in \mathbb{R}^n \setminus 2B$, we consider the following two cases for $t \in (0,\infty)$.

Case 1: $t \in (0,r_0]$. In this case, by (3.8), $C_1 \leqslant w \leqslant C_2$ and the fact that $|x-y| \sim |x-x_0|$ for any $y \in B$, we conclude that for all $x \in \mathbb{R}^n \setminus 2B$,

$$\begin{aligned}
|\mathrm{e}^{-t\sqrt{L}}(a/w)(x)| &\lesssim \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\int_B \frac{t}{(t+|x-y|)^{n+1}}\,\mathrm{d}y \\
&\lesssim r_0^{n+1}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}|x-x_0|^{-(n+1)},
\end{aligned}$$

which implies that for all $x \in \mathbb{R}^n \setminus 2B$,

$$\sup_{t\in(0,r_0]}|\mathrm{e}^{-t\sqrt{L}}(a/w)(x)| \lesssim r_0^{n+1}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}|x-x_0|^{-(n+1)}. \tag{3.13}$$

Case 2: $t \in (r_0, \infty)$. In this case, from Lemma 1.3, $\int_{\mathbb{R}^n} a(x)\,\mathrm{d}x = 0$ and (3.8), it follows that for all $x \in \mathbb{R}^n \setminus 2B$,

$$
\begin{aligned}
|\mathrm{e}^{-t\sqrt{L}}(a/w)(x)| &= \left|\int_{\mathbb{R}^n} \Big[\frac{\widetilde{K}_t(x,y)}{w(y)} - \frac{\widetilde{K}_t(x,x_0)}{w(x_0)}\Big] a(y)\,\mathrm{d}y\right| \\
&\lesssim \int_{\mathbb{R}^n} \Big[\frac{|y-x_0|}{t}\Big]^{\mu_0} \widetilde{K}_t(x,x_0)|a(y)|\,\mathrm{d}y \\
&\lesssim r_0^{\mu_0} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \int_B \frac{t^{1-\mu_0}}{(t+|x-y|)^{n+1}}\,\mathrm{d}y \\
&\lesssim r_0^{n+\mu_0} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} |x-x_0|^{-(n+\mu_0)},
\end{aligned}
$$

which further implies that

$$
\sup_{t\in(r_0,\infty)} |\mathrm{e}^{-t\sqrt{L}}(a/w)(x)| \lesssim r_0^{n+\mu_0} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} |x-x_0|^{-(n+\mu_0)}.
$$

By this and (3.13), we see that for all $x \in \mathbb{R}^n \setminus 2B$,

$$
\mathcal{N}_P(a/w)(x) \lesssim r_0^{n+\mu_0} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} |x-x_0|^{-(n+\mu_0)},
$$

which, combined with the uniformly lower type p_0 property of φ, Lemma 2.5 (v) and $n + \mu_0 > n\widetilde{q}/p_0$, further implies that

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2B} \varphi(x, \mathcal{N}_P(\lambda a/w)(x))\,\mathrm{d}x \\
&\qquad \lesssim \sum_{j=1}^\infty 2^{-(n+\mu_0)jp_0} \int_{S_j(B)} \varphi(x, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1})\,\mathrm{d}x \\
&\qquad \lesssim \sum_{j=1}^\infty 2^{-(n+\mu_0-n\widetilde{q}/p_0)jp_0} \varphi(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}) \lesssim \varphi(B, |\lambda| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}).
\end{aligned}
$$

This, together with (3.12), completes the proof of (3.6) and hence of Theorem 1.4. $\square$

Now we give the proof of Theorem 1.6 by using Theorem 1.4.

Proof of Theorem 1.6. We first prove that the spaces $H_{\varphi,L}(\mathbb{R}^n) = H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$ coincide with equivalent quasi-norms. Let $f \in \widetilde{H}_{\varphi,L}(\mathbb{R}^n)$. Then by Theorem 1.4 we see that $wf \in H_\varphi(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. From this, Lemma 3.4, Remark 3.5 (ii) and Theorem 1.4 again, we deduce that there exist $\{\lambda_j\}_j \subset \mathbb{C}$ and a sequence $\{a_j\}_j$ of $(\varphi, \infty, 0)$-atoms such that

$$
f = \sum_j \lambda_j \frac{a_j}{w} \quad \text{in } L^2(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \|wf\|_{H_\varphi(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j a_j\}_j).
$$

It is easy to see that for any $j\in\mathbb{N}$, a_j/w is a (φ,∞,w)-atom and hence a (φ,q,w)-atom, up to a harmless constant multiple. Thus $f\in\widetilde{H}^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$ and $\|f\|_{H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)}\lesssim\|f\|_{H_{\varphi,L}(\mathbb{R}^n)}$.

Let $f\in\widetilde{H}^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$. Then there exist $\{\lambda_j\}_j\subset\mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of (φ,q,w)-atoms such that

$$f=\sum_j\lambda_j\alpha_j\quad\text{in } L^2(\mathbb{R}^n)\quad\text{and}\quad\|f\|_{H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)}\sim\Lambda(\{\lambda_j\alpha_j\}_j),$$

which implies that $wf=\sum_j\lambda_j(w\alpha_j)$ in $L^2(\mathbb{R}^n)$ and for each $j\in\mathbb{N}$, $w\alpha_j$ is a $(\varphi,q,0,\varepsilon)$-molecule with any $\varepsilon\in(0,\infty)$ up to a harmless constant multiple. By this and Lemma 3.4 we know that $wf\in H_\varphi(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, which, together with Theorem 1.4, implies that $f\in\widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ and $\|f\|_{\varphi,L}(\mathbb{R}^n)\lesssim\|f\|_{H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)}$.

From the above argument, it follows that $\widetilde{H}^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)=\widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ with equivalent quasi-norms, which, combined with the fact that $\widetilde{H}^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$ and $\widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ are, respectively, dense in $H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$ and $H_{\varphi,L}(\mathbb{R}^n)$, and a density argument, implies that the spaces $H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)=H_{\varphi,L}(\mathbb{R}^n)$ coincide with equivalent quasi-norms.

To complete the proof of Theorem 1.6, we still need to prove that for some $\varepsilon\in(nq(\varphi)/i(\varphi),\infty)$, $H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)=H^{q,w,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$ with equivalent quasi-norms. First, by an obvious observation that for any (φ,q,w)-atom a, a is also a $(\varphi,q,w,\varepsilon)$-molecule for any $\varepsilon\in(0,\infty)$, we see that $H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)\subset H^{q,w,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)$.

Conversely, to prove $H^{q,w,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)\subset H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$, we only need to prove that for any $(\varphi,q,w,\varepsilon)$-molecule b associated with B, with $\varepsilon\in(nq(\varphi)/i(\varphi),\infty)$, there exist $\{\lambda_j\}_j\subset\mathbb{C}$ and a sequence $\{a_j\}_j$ of (φ,q,w)-atoms, supported in the balls $\{B_j\}_j$, such that

$$b=\sum_j\lambda_j a_j,$$

and for any $\lambda\in(0,\infty)$,

$$\sum_j\varphi\Big(B_j,\frac{|\lambda_j|}{\lambda\|\chi_{B_j}\|_{L^\varphi(\mathbb{R}^n)}}\Big)\lesssim\varphi\Big(B,\frac{1}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big).\tag{3.14}$$

For $k\in\mathbb{Z}_+$, let $\chi_k:=\chi_{S_k(B)}$,

$$\widetilde{\chi}_k:=\left|\int_{S_k(B)}w(x)\,\mathrm{d}x\right|^{-1}\chi_k,$$

$m_k:=\int_{S_k(B)}b(x)w(x)\,\mathrm{d}x$ and $M_k:=b\chi_k-m_k\widetilde{\chi}_k$. Then we have

$$b=\sum_{k=0}^\infty M_k+\sum_{k=0}^\infty m_k\widetilde{\chi}_k.\tag{3.15}$$

For $i \in \mathbb{Z}_+$, let $N_i := \sum\limits_{j=i}^{\infty} m_j$. By $\int_{\mathbb{R}^n} b(x)w(x)\,\mathrm{d}x = 0$ and (3.15), we conclude that

$$b = \sum_{k=0}^{\infty} M_k + \sum_{k=0}^{\infty} N_{k+1}(\widetilde{\chi}_{k+1} - \widetilde{\chi}_k) =: \sum_{k=0}^{\infty} b_{1,k} + \sum_{k=0}^{\infty} b_{2,k}.$$

Then similarly to [3], Theorem 8.5 (ii), we can prove that for each $k \in \mathbb{Z}_+$, both $b_{1,k}$ and $b_{2,k}$ are multiples of a (φ, q, w)-atom and (3.14) also holds true. Indeed, it is easy to see that for all $k \in \mathbb{Z}_+$,

$$\operatorname{supp}(b_{1,k}) \subset 2^{k+1}B \quad \text{and} \quad \int_{\mathbb{R}^n} b_{1,k}(x)w(x)\,\mathrm{d}x = 0. \tag{3.16}$$

For any $k \in \mathbb{Z}_+$, by Hölder's inequality and $0 < C_1 \leqslant w \leqslant C_2$, we conclude that

$$\begin{aligned}
\|b_{1,k}\|_{L^q(\mathbb{R}^n)} &\leqslant \|b\|_{L^q(S_k(B))} + |m_k| \left| \int_{S_k(B)} w(x)\,\mathrm{d}x \right|^{-1} |S_k(B)|^{1/q} \\
&\lesssim \|b\|_{L^q(S_k(B))} + \|b\|_{L^q(S_k(B))} |S_k(B)|^{1/q'} |S_k(B)|^{-1} |S_k(B)|^{1/q} \\
&\lesssim \|b\|_{L^q(S_k(B))} \lesssim 2^{-k\varepsilon} |2^k B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}.
\end{aligned}$$

Thus, there exists a positive constant C_5 such that, for any $k \in \mathbb{Z}_+$,

$$\|b_{1,k}\|_{L^q(\mathbb{R}^n)} \leqslant C_5 2^{-k\varepsilon} |2^{k+1} B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}. \tag{3.17}$$

For each $k \in \mathbb{Z}_+$, let $a_{1,k} := 2^{k\varepsilon} b_{1,k} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)} / C_5 \|\chi_{2^{k+1}B}\|_{L^\varphi(\mathbb{R}^n)}$ and $\lambda_{1,k} := C_5 \|\chi_{2^{k+1}B}\|_{L^\varphi(\mathbb{R}^n)} / 2^{k\varepsilon} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}$. Then by (3.16) and (3.17) we conclude that for any $k \in \mathbb{Z}_+$, $b_{1,k} = \lambda_{1,k} a_{1,k}$ and $a_{1,k}$ is a (φ, q, w)-atom associated with the ball $2^{k+1}B$.

Now we deal with $b_{2,k}$ with $k \in \mathbb{Z}_+$. From Hölder's inequality, $0 < C_1 \leqslant w \leqslant C_2$ and $\varepsilon > nq(\varphi)/i(\varphi) \geqslant n$, we deduce that

$$\begin{aligned}
\|b_{2,k}\|_{L^q(\mathbb{R}^n)} &\lesssim |N_{k+1}| |S_k(B)|^{-1} |S_k(B)|^{1/q} \lesssim \sum_{j=k+1}^{\infty} |m_j| |S_k(B)|^{-1/q'} \\
&\lesssim \sum_{j=k+1}^{\infty} \|b\|_{L^q(S_j(B))} |S_j(B)|^{1/q'} |S_k(B)|^{-1/q'} \\
&\lesssim \sum_{j=k+1}^{\infty} 2^{-j\varepsilon} |2^j B|^{1/q} |S_j(B)|^{1/q'} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} |S_k(B)|^{-1/q'} \\
&\lesssim 2^{-k(\varepsilon - n)} |B| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} |S_k(B)|^{-1/q'} \sim 2^{-k\varepsilon} |2^{k+2} B| \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1},
\end{aligned}$$

which further implies that there exists a positive constant C_6 such that, for any $k\in\mathbb{Z}_+$,

$$\|b_{2,k}\|_{L^q(\mathbb{R}^n)}\leqslant C_6 2^{-k\varepsilon}|2^{k+2}B|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \tag{3.18}$$

For each $k\in\mathbb{Z}_+$, let $a_{2,k}:=2^{k\varepsilon}b_{2,k}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}/C_6\|\chi_{2^{k+2}B}\|_{L^\varphi(\mathbb{R}^n)}$ and $\lambda_{2,k}:=C_6\|\chi_{2^{k+2}B}\|_{L^\varphi(\mathbb{R}^n)}/2^{k\varepsilon}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}$. Moreover, it is easy to see that for each $k\in\mathbb{Z}_+$,

$$\operatorname{supp}(b_{2,k})\subset 2^{k+2}B\quad\text{and}\quad\int_{\mathbb{R}^n}b_{2,k}(x)w(x)\,\mathrm{d}x=0,$$

which, together with (3.18) and the definition of $a_{2,k}$, implies that for each $k\in\mathbb{Z}_+$, $b_{2,k}=\lambda_{2,k}a_{2,k}$ and $a_{2,k}$ is a (φ,q,w)-atom associated with the ball $2^{k+2}B$.

By the assumption $\varepsilon>nq(\varphi)/i(\varphi)$ we conclude that there exist $\widetilde{q}\in(q(\varphi),\infty)$ and $p_0\in(0,i(\varphi))$ such that $\varepsilon>n\widetilde{q}/p_0$, $\varphi\in\mathbb{A}_{\widetilde{q}}(\mathbb{R}^n)$ and φ is of uniformly lower type p_0, which, together with the definitions of $\lambda_{1,k}$ and $\lambda_{2,k}$ and Lemma 2.5 (v), further implies that for all $\lambda\in(0,\infty)$,

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{i=1}^{2}&\varphi\Big(B_{i,k},\frac{|\lambda_{i,k}|}{\lambda\|\chi_{B_{i,k}}\|_{L^\varphi(\mathbb{R}^n)}}\Big)\\&\lesssim\sum_{k=0}^{\infty}\varphi\Big(2^{k+1}B,\frac{C_5 2^{-k\varepsilon}}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big)+\sum_{k=0}^{\infty}\varphi\Big(2^{k+2}B,\frac{C_6 2^{-k\varepsilon}}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big)\\&\lesssim\sum_{k=0}^{\infty}2^{-k\varepsilon p_0}2^{kn\widetilde{q}}\varphi\Big(B,\frac{1}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big)\\&\sim\sum_{k=0}^{\infty}2^{-k(\varepsilon-n\widetilde{q}/p_0)p_0}\varphi\Big(B,\frac{1}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big)\sim\varphi\Big(B,\frac{1}{\lambda\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}}\Big),\end{aligned}$$

where, for each $i\in\{1,2\}$ and $k\in\mathbb{Z}_+$, $B_{i,k}$ denotes the ball associated with the atom $a_{i,k}$. Thus, (3.14) holds true. This completes the proof of the inclusion $H^{q,w,\varepsilon}_{\varphi,\mathrm{mol}}(\mathbb{R}^n)\subset H^{q,w}_{\varphi,\mathrm{at}}(\mathbb{R}^n)$ and hence the proof of Theorem 1.6. $\square$

4. Proof of Theorem 1.7

In this section we give the proof of Theorem 1.7. We begin with an auxiliary conclusion, which is just [6], Lemmas 2.11 and 2.13.

Lemma 4.1. *Let L be as in* (1.1) *and $f \in L^1(\mathbb{R}^n)$. Assume that L satisfies* (1.3) *and w is as in Theorem* 1.4. *Then*

$$\int_{\mathbb{R}^n} (-\Delta)^{1/2} L^{-1/2}(f)(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x)w(x)\,\mathrm{d}x \tag{4.1}$$

and

$$\int_{\mathbb{R}^n} L^{1/2}(-\Delta)^{-1/2}(f)(x)w(x)\,\mathrm{d}x = c_w \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x, \tag{4.2}$$

where c_w is a constant depending only on w.

Lemma 4.2. *Let L be as in* (1.1). *Then there exists a positive constant C such that for any $f \in L^2(\mathbb{R}^n)$, $\|(-\Delta)^{1/2}L^{-1/2}(f)\|_{L^2(\mathbb{R}^n)} \leqslant C\|f\|_{L^2(\mathbb{R}^n)}$.*

Proof. Let $f \in L^2(\mathbb{R}^n)$. It is known that $\nabla L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$ (see, for example, [11], (8.20)) and for any $u \in W^{1,2}(\mathbb{R}^n)$,

$$\|\nabla u\|_{L^2(\mathbb{R}^n)} \sim \|(-\Delta)^{1/2}u\|_{L^2(\mathbb{R}^n)}, \tag{4.3}$$

which implies that

$$\|(-\Delta)^{1/2}L^{-1/2}(f)\|_{L^2(\mathbb{R}^n)} \sim \|\nabla L^{-1/2}(f)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

This completes the proof of Lemma 4.2. $\square$

Now we prove Theorem 1.7 via Lemmas 3.2, 3.4, 4.1 and 4.2.

Proof of Theorem 1.7. We first prove (1.8). Let $f \in \widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ and let $M \in \mathbb{N}$ satisfy $M > nq(\varphi)/2i(\varphi) + 1/2$. From the assumption that $n \geqslant 3$ and

$$q(\varphi)[r(\varphi)]' < \frac{n}{nq(\varphi)/i(\varphi) - 1} \leqslant \frac{n}{n-1},$$

we deduce that $q(\varphi)[r(\varphi)]' < 2$. Then by Lemma 3.2 and Remark 3.5 (i) we see that there exist $\{\lambda_j\} \subset \mathbb{C}$ and a sequence $\{\alpha_j\}_j$ of $(\varphi, 2, M)_L$-atoms such that

$$f = \sum_j \lambda_j \alpha_j \quad \text{in } L^2(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{H_{\varphi,L}(\mathbb{R}^n)} \sim \Lambda(\{\lambda_j\alpha_j\}). \tag{4.4}$$

To prove $(-\Delta)^{1/2}L^{-1/2}(f) \in H_\varphi(\mathbb{R}^n)$, it suffices to show that for any $(\varphi, 2, M)_L$-atom α associated with the ball $B := B(x_0, r_0)$ and some $\varepsilon \in (nq(\varphi)/i(\varphi), \infty)$, $(-\Delta)^{1/2}L^{-1/2}(\alpha)$ is a $(\varphi, 2, 0, \varepsilon)$-molecule associated with B, up to a harmless constant multiple. If this claim holds true, this, (4.4) and Lemma 4.2 yield that $(-\Delta)^{1/2}L^{-1/2}(f) = \sum_j \lambda_j(-\Delta)^{1/2}L^{-1/2}(\alpha_j)$ is a molecular decomposition of $(-\Delta)^{1/2}L^{-1/2}(f)$, which, combined with Lemma 3.4 and (4.4) again, implies that $(-\Delta)^{1/2}L^{-1/2}(f) \in H_\varphi(\mathbb{R}^n)$ and $\|(-\Delta)^{1/2}L^{-1/2}(f)\|_{H_\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H_{\varphi,L}(\mathbb{R}^n)}$. By this, the arbitrariness of $f \in \widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ and the fact that $\widetilde{H}_{\varphi,L}(\mathbb{R}^n)$ is dense in $H_{\varphi,L}(\mathbb{R}^n)$, we conclude that for any $f \in H_{\varphi,L}(\mathbb{R}^n)$, $(-\Delta)^{1/2}L^{-1/2}(f) \in H_\varphi(\mathbb{R}^n)$ and

$$\|(-\Delta)^{1/2}L^{-1/2}(f)\|_{H_\varphi(\mathbb{R}^n)} \lesssim \|f\|_{H_{\varphi,L}(\mathbb{R}^n)}.$$

Now we prove that $(-\Delta)^{1/2}L^{-1/2}(\alpha)$ is a $(\varphi, 2, 0, \varepsilon)$-molecule up to a harmless constant multiple. Let $b \in D(L)$ be such that $\alpha = Lb$. By (4.1) and the fact that L is a self-adjoint operator on $L^2(\mathbb{R}^n)$, we see that

$$\int_{\mathbb{R}^n} (-\Delta)^{1/2}L^{-1/2}(\alpha)(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} Lb(x)w(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} b(x)Lw(x)\,\mathrm{d}x = 0. \tag{4.5}$$

Moreover, for $k \in \{0, 1, \ldots, 5\}$, it follows from Lemma 4.2 that

$$\|(-\Delta)^{1/2}L^{-1/2}(\alpha)\|_{L^2(S_k(B))} \lesssim \|\alpha\|_{L^2(\mathbb{R}^n)} \lesssim |B|^{1/2}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \tag{4.6}$$

When $k \in \mathbb{N}$ and $k \geqslant 6$, let $\widetilde{S}_k(B) := 2^{k+2}B \setminus 2^{k-2}B$. Take $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\psi \equiv 1$ on $S_k(B)$, $0 \leqslant \psi \leqslant 1$, $\operatorname{supp}(\psi) \subset \widetilde{S}_k(B)$ and $|\nabla\psi| \lesssim (2^k r_0)^{-1}$. Then by (4.3) we see that

$$\begin{aligned}\|(-\Delta)^{1/2}L^{-1/2}(\alpha)\|_{L^2(S_k(B))} &\\ \leqslant \|(-\Delta)^{1/2}(\psi L^{-1/2}\alpha)\|_{L^2(\widetilde{S}_k(B))} &\sim \|\nabla(\psi L^{-1/2}\alpha)\|_{L^2(\widetilde{S}_k(B))} \\ \lesssim \|\nabla L^{-1/2}(\alpha)\|_{L^2(\widetilde{S}_k(B))} &+ (2^k r_0)^{-1}\|L^{-1/2}(\alpha)\|_{L^2(\widetilde{S}_k(B))}.\end{aligned} \tag{4.7}$$

It follows from [30], (7.28) that for some $s \in (nq(\varphi)/i(\varphi), 2M)$,

$$\|\nabla L^{-1/2}(\alpha)\|_{L^2(\widetilde{S}_k(B))} \lesssim 2^{-sk}|B|^{1/2}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \tag{4.8}$$

For the sake of completeness, we give the proof of (4.8). By [11], Lemma 8.5, we see that there exist two positive constants C and c such that for all closed sets E and F in $\mathbb{R}^n$, $t \in (0, \infty)$ and $f \in L^2(E)$,

$$\|t\nabla \mathrm{e}^{-t^2L}f\|_{L^2(F)} \leqslant C\exp\Big\{-\frac{[\operatorname{dist}(E,F)]^2}{ct^2}\Big\}\|f\|_{L^2(E)}, \tag{4.9}$$

where $\operatorname{dist}(E,F) := \inf\{|x-y|\colon x \in E, y \in F\}$. Moreover, from the functional calculus of L, we deduce that for all $f \in L^2(\mathbb{R}^n)$,

$$\nabla L^{-1/2}f = \frac{1}{\pi^{1/2}}\int_0^\infty \nabla \mathrm{e}^{-tL}f\,\frac{\mathrm{d}t}{t^{1/2}}. \tag{4.10}$$

By the definition of a $(\varphi, 2, M)_L$-atom, we know that there exists $b \in \mathcal{D}(L^M)$ such that $\alpha = L^M b$, $\mathrm{supp}(b) \subset B$ and $\|b\|_{L^2(B)} \leqslant r_0^{2M}|B|^{1/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}$. Then from (4.10), the change of variables and Minkowski's inequality, it follows that for each $k \in \mathbb{N}$ with $k \geqslant 6$,

$$
\begin{aligned}
\text{(4.11)} \qquad & \|\nabla L^{-1/2}(\alpha)\|_{L^2(\widetilde{S}_k(B))} \\
&\lesssim \int_0^\infty \left\{\int_{\widetilde{S}_k(B)} |\nabla \mathrm{e}^{-t^2L}\alpha(x)|^2\,\mathrm{d}x\right\}^{1/2} \mathrm{d}t \\
&\sim \int_0^{r_0} \left\{\int_{\widetilde{S}_k(B)} |t\nabla \mathrm{e}^{-t^2L}\alpha(x)|^2\,\mathrm{d}x\right\}^{1/2} \frac{\mathrm{d}t}{t} \\
&\quad + \int_{r_0}^\infty \left\{\int_{\widetilde{S}_k(B)} |t\nabla (t^2L)^M \mathrm{e}^{-t^2L}b(x)|^2\,\mathrm{d}x\right\}^{1/2} \frac{\mathrm{d}t}{t^{2M+1}} \\
&=: \mathrm{H}_{k,1} + \mathrm{H}_{k,2}.
\end{aligned}
$$

For $\mathrm{H}_{k,1}$, by (4.9) we conclude that

$$
\begin{aligned}
\text{(4.12)} \qquad \mathrm{H}_{k,1} &\lesssim \int_0^{r_0} \exp\Big\{-\frac{(2^k r_0)^2}{ct^2}\Big\}\|\alpha\|_{L^2(B)}\frac{\mathrm{d}t}{t} \\
&\lesssim \left\{\int_0^{r_0} \frac{t^{2M-1}}{(2^k r_0)^{2M-1}}\frac{\mathrm{d}t}{t}\right\}\|\alpha\|_{L^2(B)} \sim 2^{-(2M-1)k}\|\alpha\|_{L^2(B)} \\
&\lesssim 2^{-(2M-1)k}|B|^{1/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}.
\end{aligned}
$$

Furthermore, similarly to (4.12), we see that

$$
\begin{aligned}
\mathrm{H}_{k,2} &\lesssim \int_{r_0}^\infty \exp\Big\{-\frac{(2^k r_0)^2}{ct^2}\Big\}\|b\|_{L^2(B)}\frac{\mathrm{d}t}{t^{2M+1}} \\
&\lesssim \int_{r_0}^\infty \frac{t^{(2M-1)}}{(2^k r_0)^{(2M-1)}}\frac{\mathrm{d}t}{t^{2M+1}}\|b\|_{L^2(B)} \\
&\lesssim 2^{-(2M-1)k}|B|^{1/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1},
\end{aligned}
$$

which, together with (4.11) and (4.12), implies that for all $k \in \mathbb{N}$ with $k \geqslant 6$,

$$
\|\nabla L^{-1/2}(\alpha)\|_{L^2(\widetilde{S}_k(B))} \lesssim 2^{-(2M-1)k}|B|^{1/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}.
$$

From this and the assumption that $M > nq(\varphi)/2i(\varphi) + 1/2$ we deduce that (4.8) holds true.

Moreover, by $L^{-1/2}=\pi^{-1/2}\int_0^\infty \mathrm{e}^{-tL}t^{-1/2}\,\mathrm{d}t$, Minkowski's inequality and Hölder's inequality, we conclude that for each $k\in\mathbb{N}$ with $k\geqslant 6$,

$$\begin{aligned}(4.13)\qquad \|L^{-1/2}(\alpha)\|_{L^2(\widetilde{S}_k(B))} &\lesssim \int_0^{r_0^2}\left\{\int_{\widetilde{S}_k(B)}|\mathrm{e}^{-tL}\alpha(x)|^2\,\mathrm{d}x\right\}^{1/2}\frac{\mathrm{d}t}{t^{1/2}}\\ &\quad+\int_{r_0^2}^\infty\left\{\int_{\widetilde{S}_k(B)}|(tL)^M\mathrm{e}^{-tL}b(x)|^2\,\mathrm{d}x\right\}^{1/2}\frac{\mathrm{d}t}{t^{M+1/2}}\\ &=:\mathrm{I}_{k,1}+\mathrm{I}_{k,2}.\end{aligned}$$

From Minkowski's inequality, (1.2), Hölder's inequality and the fact that for all $x\in\widetilde{S}_k(B)$ and $y\in B$, $|x-y|\geqslant 2^{k-2}r_0-r_0\geqslant 2^{k-3}r_0$, we deduce that for each $k\in\mathbb{N}$ with $k\geqslant 6$,

$$\begin{aligned}(4.14)\qquad \|\mathrm{e}^{-tL}\alpha\|_{L^2(\widetilde{S}_k(B))} &\leqslant \int_B|\alpha(y)|\left\{\int_{\widetilde{S}_k(B)}|K_t(x,y)|^2\,\mathrm{d}x\right\}^{1/2}\mathrm{d}y\\ &\lesssim \|\alpha\|_{L^1(B)}\frac{1}{t^{n/2}}\mathrm{e}^{-(2^{k-3}r_0)^2/4t}|\widetilde{S}_k(B)|^{1/2}\\ &\lesssim 2^{kn/2}\frac{1}{t^{n/2}}\mathrm{e}^{-(2^{k-3}r_0)^2/4t}|B|^{3/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1},\end{aligned}$$

which further implies that

$$\begin{aligned}(4.15)\qquad \mathrm{I}_{k,1} &\lesssim 2^{kn/2}|B|^{3/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\int_0^{r_0^2}\frac{1}{t^{(n+1)/2}}\mathrm{e}^{-(2^{k-3}r_0)^2/4t}\,\mathrm{d}t\\ &\lesssim 2^{kn/2}|B|^{3/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}\int_0^{r_0^2}\frac{1}{t^{(n+1)/2}}\left[\frac{t}{(2^kr_0)^2}\right]^{M+n/4}\mathrm{d}t\\ &\lesssim 2^{-2M}r_0|B|^{1/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}.\end{aligned}$$

To estimate $\mathrm{I}_{k,2}$, we recall that for any $m\in\mathbb{N}$ there exists a positive constant c_m such that for all $t\in(0,\infty)$ and $x,y\in\mathbb{R}^n$,

$$\left|\frac{\partial^m K_t(x,y)}{\partial t^m}\right|\lesssim\frac{1}{t^{n/2+m}}\mathrm{e}^{-|x-y|^2/c_mt}$$

(see, for example, [23], Theorem 6.17). By this, similarly to (4.14), we see that

$$\begin{aligned}\|(tL)^M\mathrm{e}^{-tL}b\|_{L^2(\widetilde{S}_k(B))} &\leqslant \int_B|b(y)|\left\{\int_{\widetilde{S}_k(B)}\left|\frac{t^M\partial^MK_t(x,y)}{\partial t^M}\right|^2\mathrm{d}x\right\}^{1/2}\mathrm{d}y\\ &\lesssim 2^{kn/2}\frac{1}{t^{n/2}}\mathrm{e}^{-(2^{k-3}r_0)^2/c_Mt}r_0^{2M}|B|^{3/2}\|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1},\end{aligned}$$

which implies that

$$
\begin{aligned}
\mathrm{I}_{k,2} &\lesssim 2^{kn/2} r_0^{2M} |B|^{3/2} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \int_{r_0^2}^\infty \frac{1}{t^{M+(n+1)/2}} \mathrm{e}^{-(2^{k-3}r_0)^2/c_M t}\,\mathrm{d}t \\
&\lesssim 2^{kn/2} r_0^{2M} |B|^{3/2} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \int_{r_0^2}^\infty \frac{1}{t^{M+(n+1)/2}} \left[\frac{t}{(2^k r_0)^2}\right]^{M+n/4} \mathrm{d}t \\
&\lesssim 2^{-2M} r_0 |B|^{1/2} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}.
\end{aligned}
$$

This, (4.13), (4.15) and $s < 2M$ yield that

$$
\begin{aligned}
(2^k r_0)^{-1} \|L^{-1/2}(\alpha)\|_{L^2(\widetilde{S}_k(B))} &\lesssim 2^{-(2M+1)k} |B|^{1/2} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \\
&\lesssim 2^{-sk} |B|^{1/2} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1},
\end{aligned}
$$

where s is as in (4.8), which, combined with (4.7) and (4.8), implies that

$$
\|(-\Delta)^{1/2} L^{-1/2}(\alpha)\|_{L^2(S_k(B))} \lesssim 2^{-sk} |B|^{1/2} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}.
$$

By this inequality, (4.5) and (4.6), we see that $(-\Delta)^{1/2} L^{-1/2}(\alpha)$ is a $(\varphi, 2, 0, s)$-molecule up to a harmless constant multiple. Thus, the claim holds true.

Now we prove (1.9). By the assumption that $q(\varphi)[r(\varphi)]' < n/(nq(\varphi)/i(\varphi) - 1)$ we know that there exist $q \in (1, \infty)$ and $\gamma \in (0, 1)$ such that

$$
q(\varphi)[r(\varphi)]' < q < \frac{n}{nq(\varphi)/i(\varphi) - \gamma},
$$

which further implies that $\gamma + n/q > nq(\varphi)/i(\varphi)$. Due to this, Lemma 3.4 and Theorem 1.6, similarly to the proof of (1.8), it suffices to prove that for any $(\varphi, \infty, 0)$-atom a supported in the ball $B := B(x_0, r_0)$, $L^{1/2}(-\Delta)^{-1/2}(a)$ is a $(\varphi, q, w, \gamma + n/q)$-molecule associated with B, up to a harmless constant multiple.

By (4.2) and $\int_{\mathbb{R}^n} a(x)\,\mathrm{d}x = 0$, we see that

$$
\int_{\mathbb{R}^n} L^{1/2}(-\Delta)^{-1/2}(a)(x) w(x)\,\mathrm{d}x = c_w \int_{\mathbb{R}^n} a(x)\,\mathrm{d}x = 0. \tag{4.16}
$$

For any $x \in \mathbb{R}^n$, let

$$
\begin{aligned}
J(x) := \int_0^\infty \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} &P_{t-s}(x - z) V(z) \\
&\times K_s(z, y)((-\Delta)^{-1/2} a)(y)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s\,\frac{\mathrm{d}t}{t^{3/2}}.
\end{aligned} \tag{4.17}
$$

From the functional calculus associated with $A := L$ or $-\Delta$, we deduce that there exists a positive constant c_0 such that, for a suitable function f on $\mathbb{R}^n$,

$$A^{1/2} f = c_0 \int_0^\infty (H_t f - f) \frac{\mathrm{d}t}{t^{3/2}}, \tag{4.18}$$

where $\{H_t\}_{t>0}$ denote the heat kernels of A. Furthermore, by the Kato-Trotter formula, we know that for all $t \in (0, \infty)$ and $x \in \mathbb{R}^n$,

$$K_t(x, y) = P_t(x - y) - \int_0^t \int_{\mathbb{R}^n} P_{t-s}(x - y) V(z) K_s(z, y) \,\mathrm{d}z \,\mathrm{d}s$$

(see, for example [6], (2.2)), which, combined with (4.18), implies that

$$\begin{aligned} L^{1/2}(-\Delta)^{-1/2}(a)(x) &= c_0 \int_0^\infty (K_t - I)(-\Delta)^{-1/2}(a)(x) \frac{\mathrm{d}t}{t^{3/2}} \\ &= c_0 \int_0^\infty (K_t - P_t)(-\Delta)^{-1/2}(a)(x) \frac{\mathrm{d}t}{t^{3/2}} \\ &\quad + c_0 \int_0^\infty (P_t - I)(-\Delta)^{-1/2}(a)(x) \frac{\mathrm{d}t}{t^{3/2}} \\ &= - c_0 J(x) + a(x). \end{aligned} \tag{4.19}$$

Moreover, by the equality $(-\Delta)^{-1/2} = \pi^{-1/2} \int_0^\infty \mathrm{e}^{t\Delta} t^{-1/2} \,\mathrm{d}t$, we see that for all $x \in \mathbb{R}^n$,

$$(-\Delta)^{-1/2}(a)(x) = C \int_{\mathbb{R}^n} \frac{a(y)}{|x - y|^{n-1}} \,\mathrm{d}y \tag{4.20}$$

with C a constant independent of a, which, together with $\|a\|_{L^\infty(\mathbb{R}^n)} \leqslant \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}$, implies that for any $x \in 2B$,

$$|(-\Delta)^{-1/2}(a)(x)| \lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}. \tag{4.21}$$

For $x \in \mathbb{R}^n \setminus 2B$, from (4.20), $\int_{\mathbb{R}^n} a(y) \,\mathrm{d}y = 0$, $\|a\|_{L^\infty(\mathbb{R}^n)} \leqslant \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1}$ and the mean value theorem, we deduce that

$$\begin{aligned} |(-\Delta)^{-1/2}(a)(x)| &\lesssim \left| \int_B [|x - y|^{1-n} - |x - x_0|^{1-n}] a(y) \,\mathrm{d}y \right| \\ &\lesssim r_0^{n+1} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} |x - x_0|^{-n}, \end{aligned}$$

which, together with (4.21), further implies that for all $x \in \mathbb{R}^n$,

$$|(-\Delta)^{-1/2}(a)(x)| \lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \left[1 + \frac{|x - x_0|}{r_0}\right]^{-n}. \tag{4.22}$$

Due to (4.22), similarly to [6], (4.8), we conclude that

$$\left\|J(\cdot)\Big(1+\frac{|\cdot-x_0|}{r_0}\Big)^{\gamma}\right\|_{L^q(\mathbb{R}^n)} \lesssim |B|^{1/q}\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)}\|\Delta^{-1}V\|_{L^\infty(\mathbb{R}^n)}, \tag{4.23}$$

where $J(\cdot)$ is as in (4.17). Indeed,

$$\begin{aligned} J(x) &= \int_0^{r_0^2}\int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\cdots+\int_{r_0^2}^{\infty}\int_0^{t/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\cdots+\int_{r_0^2}^{\infty}\int_{t/2}^{t}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\cdots \\ &=: J_1(x)+J_2(x)+J_3(x). \end{aligned} \tag{4.24}$$

To obtain (4.23), we need the following basic estimates, which are established in [7], Section 4, the details being omitted here. For $x\in\mathbb{R}^n$, $d\in(2,\infty)$ and $\beta,t\in(0,\infty)$, let

$$g(x):=(1+|x|)^{-d-\beta}, \quad g_t(x):=\frac{1}{t^{d/2}}g\Big(\frac{x}{t^{1/2}}\Big).$$

Then for any $t\in(0,\infty)$ and $x\in\mathbb{R}^n$,

$$\int_0^t g_s(x)\,\mathrm{d}s \lesssim |x|^{2-d}\Big[1+\frac{|x|}{t^{1/2}}\Big]^{-2-\beta} \tag{4.25}$$

and

$$\int_{t^2}^{\infty} g_s(x)\,\mathrm{d}s \lesssim t^{2-d}\Big[1+\frac{|x|}{t}\Big]^{-d+2}. \tag{4.26}$$

Moreover, for any $q\in(1,\infty)$, $\gamma\in(n/q',n]$ and $\beta,t\in(0,\infty)$,

$$\left\||x|^{\gamma-n}\Big[1+\frac{|x|}{t^{1/2}}\Big]^{-n-\beta}\right\|_{L^q(\mathbb{R}^n)} = C_{\gamma,\beta}t^{(\gamma-n+n/q)/2}, \tag{4.27}$$

and for any $0<\gamma<\beta<2$, $x\in\mathbb{R}^n$ and $r\in(0,\infty)$,

$$\int_{\mathbb{R}^n}|x-y|^{2-n}\Big[1+\frac{|x-y|}{r}\Big]^{-\beta}\Big[1+\frac{|y|}{r}\Big]^{-n+\gamma}\mathrm{d}y \lesssim r^2\Big[1+\frac{|x|}{r}\Big]^{\gamma+2-n-\beta}. \tag{4.28}$$

Furthermore, for any $x\in\mathbb{R}^n$, $\gamma\in(0,2]$ and $r\in(0,\infty)$,

$$\int_{\mathbb{R}^n}V(y)\Big[1+\frac{|x-y|}{r}\Big]^{-n+\gamma}\mathrm{d}y \lesssim r^{n-2}\|\Delta^{-1}V\|_{L^\infty(\mathbb{R}^n)}. \tag{4.29}$$

Now we prove (4.22) by using (4.25) through (4.29). From (4.22) and (1.2), it follows that for all $s \in (0, r_0^2)$ and $y, z \in \mathbb{R}^n$,

$$
\begin{aligned}
&K_s(y,z)|(-\Delta)^{-1/2}(a)(y)| \\
&\qquad \lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \frac{1}{s^{n/2}} \mathrm{e}^{-|y-z|^2/8s} \left[1 + \frac{|y-z|}{s^{1/2}}\right]^{-n} \left[1 + \frac{|y-x_0|}{r_0}\right]^{-n} \\
&\qquad \lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \frac{1}{s^{n/2}} \mathrm{e}^{-|y-z|^2/8s} \left[1 + \frac{|y-z| + |y-x_0|}{r_0}\right]^{-n} \\
&\qquad \lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \frac{1}{s^{n/2}} \mathrm{e}^{-|y-z|^2/8s} \left[1 + \frac{|z-x_0|}{r_0}\right]^{-n},
\end{aligned}
$$

which, together with the change of variables and (4.25) (taking $d = n+1$ in this case), implies that

$$
\begin{aligned}
|J_1(x)| &\lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \int_0^{r_0^2} \int_0^t \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} P_{t-s}(x-z) V(z) \\
&\qquad \times \frac{1}{s^{n/2}} \mathrm{e}^{-|y-z|^2/8s} \left[1 + \frac{|z-x_0|}{r_0}\right]^{-n} \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}s \, \frac{\mathrm{d}t}{t^{3/2}} \\
&\lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \int_0^{r_0^2} \int_{\mathbb{R}^n} P_s(x-z) V(z) \left[1 + \frac{|z-x_0|}{r_0}\right]^{-n} \mathrm{d}z \, \frac{\mathrm{d}s}{s^{1/2}} \\
&\lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \int_{\mathbb{R}^n} |x-z|^{1-n} \left[1 + \frac{|x-z|}{r_0}\right]^{-N} V(z) \left[1 + \frac{|z-x_0|}{r_0}\right]^{-n} \mathrm{d}z,
\end{aligned}
$$

where $N \in (n+\gamma, \infty)$ is a positive constant. Now we further see that

$$
\begin{aligned}
|J_1(x)| \left[1 + \frac{|x-x_0|}{r_0}\right]^{\gamma} &\lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \int_{\mathbb{R}^n} |x-z|^{1-n} \left[1 + \frac{|x-z|}{r_0}\right]^{-N+\gamma} \\
&\qquad \times V(z) \left[1 + \frac{|z-x_0|}{r_0}\right]^{-n+\gamma} \mathrm{d}z,
\end{aligned}
$$

which, together with Minkowski's inequality, (4.27) and (4.29), implies that

$$
\begin{aligned}
&\left\| J_1(\cdot) \left(1 + \frac{|\cdot - x_0|}{r_0}\right)^{\gamma} \right\|_{L^q(\mathbb{R}^n)} \\
&\qquad \lesssim r_0 \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \int_{\mathbb{R}^n} \left\| |x-z|^{1-n} \left[1 + \frac{|x-z|}{r_0}\right]^{-N+\gamma} \right\|_{L^q(\mathbb{R}^n)} \\
&\qquad\qquad \times V(z) \left[1 + \frac{|z-x_0|}{r_0}\right]^{-n+\gamma} \mathrm{d}z \\
&\qquad \lesssim r_0^{2-n+n/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \int_{\mathbb{R}^n} V(z) \left[1 + \frac{|z-x_0|}{r_0}\right]^{-n+\gamma} \mathrm{d}z \\
&\qquad \lesssim |B|^{1/q} \|\chi_B\|_{L^\varphi(\mathbb{R}^n)}^{-1} \|\Delta^{-1} V\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned} \tag{4.30}
$$

For J_2, from (4.22), (1.2), (4.25) (taking $d = n$ in this case) and the estimate

$$\left[1+\frac{|x-x_0|}{r_0}\right]^{\gamma} \lesssim \left[\frac{t^{1/2}}{r_0}\right]^{2\gamma}\left[1+\frac{|y-x_0|}{r_0}\right]^{\gamma}\left[1+\frac{|y-z|}{t^{1/2}}\right]^{\gamma}\left[1+\frac{|z-x|}{t^{1/2}}\right]^{\gamma} \tag{4.31}$$

for any $t \in (r_0^2, \infty)$, we deduce that

$$\begin{aligned}
&|J_2(x)|\left[1+\frac{|x-x_0|}{r_0}\right]^{\gamma} \\
&\quad\lesssim r_0 \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\left[1+\frac{|x-x_0|}{r_0}\right]^{\gamma}\int_{r_0^2}^{\infty}\int_0^{t/2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\mathrm{e}^{-|x-z|^2/2t} \\
&\qquad\qquad \times V(z)\frac{1}{s^{n/2}}\mathrm{e}^{-|y-z|^2/4s}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s\,\frac{\mathrm{d}t}{t^{3/2}} \\
&\quad\lesssim r_0 \|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\left[1+\frac{|x-x_0|}{r_0}\right]^{\gamma}\int_{r_0^2}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\mathrm{e}^{-|x-z|^2/2t} \\
&\qquad\qquad \times V(z)|z-y|^{2-n}\left[1+\frac{|y-z|}{t^{1/2}}\right]^{-N}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n}\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}t}{t^{3/2}} \\
&\quad\lesssim r_0^{1-2\gamma}\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\int_{r_0^2}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} t^{(2\gamma-n-3)/2}\mathrm{e}^{-|x-z|^2/4t}V(z) \\
&\qquad\qquad \times |z-y|^{2-n}\left[1+\frac{|z-y|}{t^{1/2}}\right]^{-N+\gamma}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n+\gamma}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t,
\end{aligned}$$

where $N \in (2\gamma, \infty)$ is a positive constant. Letting $N = \beta + \gamma$ with $0 < \gamma < \beta < 1$ and applying Minkowski's inequality, (4.28) and (4.29), we see that

$$\begin{aligned}
&\left\|J_2(\cdot)\Big(1+\frac{|\cdot-x_0|}{r_0}\Big)^{\gamma}\right\|_{L^q(\mathbb{R}^n)} \\
&\quad\lesssim r_0^{1-2\gamma}\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\int_{r_0^2}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} t^{(2\gamma-n-3)/2}\|\mathrm{e}^{-|\cdot-z|^2/4t}\|_{L^q(\mathbb{R}^n)}V(z) \\
&\qquad \times |z-y|^{2-n}\left[1+\frac{|z-y|}{t^{1/2}}\right]^{-N+\gamma}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n+\gamma}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t \\
&\quad\lesssim r_0^{1-2\gamma}\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\int_{r_0^2}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} t^{(2\gamma+n/q-n-3)/2}V(z) \\
&\qquad \times \left[\frac{t^{1/2}}{r_0}\right]^{\beta}|z-y|^{2-n}\left[1+\frac{|z-y|}{r_0}\right]^{-\beta}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n+\gamma}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t. \\
&\quad\lesssim r_0^{1-2\gamma-\beta}\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1} \\
&\qquad \times \int_{\mathbb{R}^n} r_0^{2\gamma+\beta+n/q+1-n}V(z)\left[1+\frac{|z-x_0|}{r_0}\right]^{-n+2+\gamma-\beta}\mathrm{d}z \\
&\quad\lesssim |B|^{1/q}\|\chi_B\|_{L^{\varphi}(\mathbb{R}^n)}^{-1}\|\Delta^{-1}V\|_{L^{\infty}(\mathbb{R}^n)}.
\end{aligned} \tag{4.32}$$

For J_3, it follows from (4.22), (1.2), the change of variables and (4.25) with $d=n$ that

$$
\begin{aligned}
|J_3(x)| &\lesssim r_0\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\int_{r_0^2}^\infty\int_{t/2}^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} P_{t-s}(x-z)V(z)\\
&\qquad\times\frac{1}{t^{n/2}}\mathrm{e}^{-|y-z|^2/2t}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s\,\frac{\mathrm{d}t}{t^{3/2}}\\
&\lesssim r_0\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\int_{r_0^2}^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x-z|^{2-n}\left[1+\frac{|x-z|}{t^{1/2}}\right]^{-N}\\
&\qquad\times V(z)\frac{1}{t^{n/2}}\mathrm{e}^{-|y-z|^2/2t}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n}\mathrm{d}y\,\mathrm{d}z\,\frac{\mathrm{d}t}{t^{3/2}},
\end{aligned}
$$

where $N\in(n,\infty)$ is a positive constant, which, combined with (4.31), further implies that

$$
\begin{aligned}
&|J_3(x)|\left[1+\frac{|x-x_0|}{r_0}\right]^{\gamma}\\
&\quad\lesssim r_0^{1-2\gamma}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\int_{r_0^2}^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|x-z|^{2-n}\left[1+\frac{|x-z|}{t^{1/2}}\right]^{-N+\gamma}t^{(2\gamma-3)/2}\\
&\qquad\times V(z)\frac{1}{t^{n/2}}\mathrm{e}^{-|y-z|^2/4t}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n+\gamma}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t.
\end{aligned}
$$

By this, Minkowski's inequality, (4.27), (4.26) with $d=2n+1-2\gamma-n/q$ and (4.29), we conclude that

$$
\begin{aligned}
&\left\|J_3(\cdot)\left(1+\frac{|\cdot-x_0|}{r_0}\right)^{\gamma}\right\|_{L^q(\mathbb{R}^n)}\\
&\quad\lesssim r_0^{1-2\gamma}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\int_{r_0^2}^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}t^{(2\gamma+(n/q)-n-1)/2}V(z)\\
&\qquad\times\frac{1}{t^{n/2}}\mathrm{e}^{-|y-z|^2/2t}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n+\gamma}\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\\
&\quad\lesssim r_0^{2-2n+n/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}V(z)\\
&\qquad\times\left[1+\frac{|z-y|}{r_0}\right]^{2\gamma+1+(n/q)-2n}\left[1+\frac{|y-x_0|}{r_0}\right]^{-n+\gamma}\mathrm{d}y\,\mathrm{d}z\\
&\quad\lesssim r_0^{2-n+n/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\int_{\mathbb{R}^n}V(z)\left[1+\frac{|z-x_0|}{r_0}\right]^{2\gamma+1+(n/q)-2n}\mathrm{d}z\\
&\quad\lesssim |B|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}\|\Delta^{-1}V\|_{L^\infty(\mathbb{R}^n)},
\end{aligned}
$$

which, combined with (4.24), (4.30) and (4.32), implies that (4.23) holds true.

From (4.19), (4.23) and $\|a\|_{L^\infty(\mathbb{R}^n)}\leqslant\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}$, it follows that

$$
\|L^{1/2}(-\Delta)^{-1/2}(a)\|_{L^q(4B)}\lesssim\|J\|_{L^q(4B)}+\|a\|_{L^q(\mathbb{R}^n)}\lesssim|B|^{1/q}\|\chi_B\|^{-1}_{L^\varphi(\mathbb{R}^n)}. \tag{4.33}
$$

When $j \in \mathbb{N}$ with $j \geqslant 2$ by (4.19), (4.23), $\operatorname{supp}(a) \subset B$ and the fact that for any $x \in S_j(B)$, $|x - x_0| \sim 2^j r_0$, we conclude that

$$\begin{aligned}
\|L^{1/2}(-\Delta)^{-1/2}(a)\|_{L^q(S_j(B))} &\sim \|J\|_{L^q(S_j(B))} \\
&\sim 2^{-\gamma j}\|J(\cdot)\Big(1 + \frac{|\cdot - x_0|}{r_0}\Big)^{\gamma}\|_{L^q(S_j(B))} \\
&\lesssim 2^{-\gamma j}|B|^{1/q}\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)} \\
&\sim 2^{-(\gamma+n/q)j}|2^{j+1}B|^{1/q}\|\chi_B\|^{-1}_{L^{\varphi}(\mathbb{R}^n)},
\end{aligned}$$

which, combined with (4.16) and (4.33), further implies that $L^{1/2}(-\Delta)^{-1/2}(a)$ is a $(\varphi, q, 0, \gamma + n/q)$-molecule up to a harmless constant multiple. This completes the proof of Theorem 1.7. □

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References

[1] *A. Bonami, S. Grellier, L. D. Ky*: Paraproducts and products of functions in $\mathrm{BMO}(\mathbb{R}^n)$ and $\mathcal{H}^1(\mathbb{R}^n)$ through wavelets. J. Math. Pures Appl. (9) *97* (2012), 230–241. French summary.

[2] *A. Bonami, T. Iwaniec, P. Jones, M. Zinsmeister*: On the product of functions in BMO and H^1. Ann. Inst. Fourier *57* (2007), 1405–1439.

[3] *T. A. Bui, J. Cao, L. D. Ky, D. Yang, S. Yang*: Musielak-Orlicz-Hardy spaces associated with operators satisfying reinforced off-diagonal estimates. Anal. Geom. Metr. Spaces (electronic only) *1* (2013), 69–129.

[4] *J. Cao, D.-C. Chang, D. Yang, S. Yang*: Boundedness of second order Riesz transforms associated to Schrödinger operators on Musielak-Orlicz-Hardy spaces. Commun. Pure Appl. Anal. *13* (2014), 1435–1463.

[5] *X. T. Duong, L. Yan*: Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. J. Am. Math. Soc. *18* (2005), 943–973.

[6] *J. Dziubański, J. Zienkiewicz*: A characterization of Hardy spaces associated with certain Schrödinger operators. Potential Anal. *41* (2014), 917–930.

[7] *J. Dziubański, J. Zienkiewicz*: On isomorphisms of Hardy spaces associated with Schrödinger operators. J. Fourier Anal. Appl. *19* (2013), 447–456.

[8] *C. L. Fefferman, E. M. Stein*: H^p spaces of several variables. Acta Math. *129* (1972), 137–193.

[9] *J. García-Cuerva, J. L. Rubio de Francia*: Weighted Norm Inequalities and Related Topics. North-Holland Mathematics Studies 116, North-Holland, Amsterdam, 1985.

[10] *L. Grafakos*: Modern Fourier Analysis. Graduate Texts in Mathematics 250, Springer, New York, 2009.

[11] *S. Hofmann, G. Lu, D. Mitrea, M. Mitrea, L. Yan*: Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates. Mem. Am. Math. Soc. *1007* (2011), 78 pages.

[12] *S. Hofmann, S. Mayboroda*: Hardy and BMO spaces associated to divergence form elliptic operators. Math. Ann. *344* (2009), 37–116.
[13] *S. Hofmann, S. Mayboroda, A. McIntosh*: Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces. Ann. Sci. Éc. Norm. Supér. (4) *44* (2011), 723–800. French summary.
[14] *S. Hou, D. Yang, S. Yang*: Lusin area function and molecular characterizations of Musielak-Orlicz Hardy spaces and their applications. Commun. Contemp. Math. *15* (2013), Article ID1350029, 37 pages.
[15] *S. Janson*: Generalizations of Lipschitz spaces and an application to Hardy spaces and bounded mean oscillation. Duke Math. J. *47* (1980), 959–982.
[16] *R. Jiang, D. Yang*: Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates. Commun. Contemp. Math. *13* (2011), 331–373.
[17] *R. Jiang, D. Yang*: New Orlicz-Hardy spaces associated with divergence form elliptic operators. J. Funct. Anal. *258* (2010), 1167–1224.
[18] *R. Jiang, D. Yang, D. Yang*: Maximal function characterizations of Hardy spaces associated with magnetic Schrödinger operators. Forum Math. *24* (2012), 471–494.
[19] *L. D. Ky*: Endpoint estimates for commutators of singular integrals related to Schrödinger operators. To appear in Rev. Mat. Iberoam.
[20] *L. D. Ky*: New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators. Integral Equations Oper. Theory *78* (2014), 115–150.
[21] *L. D. Ky*: Bilinear decompositions and commutators of singular integral operators. Trans. Am. Math. Soc. *365* (2013), 2931–2958.
[22] *J. Musielak*: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics 1034, Springer, Berlin, 1983.
[23] *E. M. Ouhabaz*: Analysis of Heat Equations on Domains. London Mathematical Society Monographs Series 31, Princeton University Press, Princeton, 2005.
[24] *M. M. Rao, Z. D. Ren*: Theory of Orlicz Spaces. Pure and Applied Mathematics 146, Marcel Dekker, New York, 1991.
[25] *Y. A. Semenov*: Stability of L^p-spectrum of generalized Schrödinger operators and equivalence of Green's functions. Int. Math. Res. Not. *12* (1997), 573–593.
[26] *B. Simon*: Functional Integration and Quantum Physics. AMS Chelsea Publishing, Providence, 2005.
[27] *J.-O. Strömberg*: Bounded mean oscillation with Orlicz norms and duality of Hardy spaces. Indiana Univ. Math. J. *28* (1979), 511–544.
[28] *J.-O. Strömberg, A. Torchinsky*: Weighted Hardy Spaces. Lecture Notes in Mathematics 1381, Springer, Berlin, 1989.
[29] *L. Yan*: Classes of Hardy spaces associated with operators, duality theorem and applications. Trans. Am. Math. Soc. *360* (2008), 4383–4408.
[30] *D. Yang, S. Yang*: Musielak-Orlicz Hardy spaces associated with operators and their applications. J. Geom. Anal. *24* (2014), 495–570.
[31] *D. Yang, S. Yang*: Local Hardy spaces of Musielak-Orlicz type and their applications. Sci. China Math. *55* (2012), 1677–1720.

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