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INFINITELY MANY SOLUTIONS FOR BOUNDARY VALUE PROBLEMS ARISING FROM THE FRACTIONAL ADVECTION DISPERSION EQUATION

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Abstract. We consider the existence of infinitely many solutions to the boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\Big)+\nabla F(t,u(t))=0 \quad \text{a.e. } t\in[0,T],$$
$$u(0)=u(T)=0.$$

Under more general assumptions on the nonlinearity, we obtain new criteria to guarantee that this boundary value problem has infinitely many solutions in the superquadratic, subquadratic and asymptotically quadratic cases by using the critical point theory.

Keywords: fractional boundary value problem; critical point theory; variational methods

MSC 2010: 26A33, 35G60

1. Introduction

In this paper, we consider the fractional Dirichlet boundary value problem

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\,{}_0D_t^{-\beta}(u'(t))+\frac{1}{2}\,{}_tD_T^{-\beta}(u'(t))\Big)+\nabla F(t,u(t))=0 \quad \text{a.e. } t\in[0,T],\\ u(0)=u(T)=0,\end{gathered} \tag{1.1}$$

where ${}_0D_t^{-\beta}$ and ${}_tD_T^{-\beta}$ are respectively the left and right Riemann-Liouville fractional integrals of order $0\leqslant\beta<1$, $F\colon\ [0,T]\times\mathbb{R}^N\to\mathbb{R}$, $F(t,x)$ is measurable in t for every

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$x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$. In particular, if $\beta = 0$, (1.1) reduces to the standard second-order boundary value problem, which has been extensively studied, for more detail one can see [21], [25], [19].

Fractional calculus and fractional differential equations have attracted wide concern recently, see [17], [24], [22], [26]. The interest in the physical models containing fractional differential operators is mainly due to the fact that they can describe the diffusion phenomenon more accurately in complex dynamic systems involving anomalous diffusion [5], [11], [10]. In [11], Fix and Roop set up a model for contaminant transport of ground-water flow, and prove the existence and uniqueness of the least squares approximation for a steady-state fractional advection-dispersion equation (FADE for short) with no advection terms by the least squares finite element analysis.

The equation in (1.1) is motivated by the steady-state FADE studied in [10],

$$\begin{gathered} -Da(p_0 D_t^{-\beta} + q_t D_T^{-\beta})Du + b(t)Du + c(t)u = f, \\ u(0) = u(T) = 0. \end{gathered} \tag{1.2}$$

In equation (1.2), let $D = \mathrm{d}/\mathrm{d}t$, $a = 1$, $p = q = 1/2$, $b(t) = c(t) = 0$, $f = f(t, u)$. Problem (1.2) is the scalar case of (1.1), in which $N = 1$ and $\nabla F(t, u) = f(t, u)$.

Recently, Jiao and Zhou [15] investigated (1.1) by the critical point theory. They established the variational structure for (1.1), and obtained two existence results of solutions for problem (1.1) by using the least action principle and the Mountain Pass Lemma. After that, many results on the existence and multiplicity for solutions of (1.1) or the related problems have been obtained, see, e.g., [16], [3], [2], [6], [27], [7], [8], [18], [23], [20], [13], [12]. Most of them, for example [15], [16], [6], [27], [8], [20], [12], treat the case where the nonlinearity $F(t, x)$ is superquadratic as $|x| \to \infty$ and the Ambrosetti-Rabinowitz superquadratic condition (see [1])

(AR) there are $\mu > 2$, $R > 0$ such that

$$0 < \mu F(t, x) \leqslant (\nabla F(t, x), x) \quad \forall\, |x| \geqslant R, \text{ a.e. } t \in [0, T],$$

is usually assumed except for [6], [12]. (AR) plays an important role in ensuring the boundedness of the Palais-Smale (PS) sequences of the energy functional, which is very crucial in applying the critical point theory. However, (AR) implies $F(t, x) \geqslant C_0|x|^\mu$ for large $|x|$ and some constant $C_0 > 0$, and there are many functions which are superquadratic at infinity, but do not satisfy (AR) for any $\mu > 2$. For example the function

$$F(t, x) = |x|^2 \ln(1 + |x|)$$

does not satisfy (AR).

For the superquadratic case, in a recent paper [7], the authors proved the existence of nontrivial solutions for problem (1.1) by the Mountain Pass Lemma based on some new superquadratic conditions, which are weaker than (AR). In papers [8] and [12], the results on the multiplicity of nontrivial solutions for problem (1.1) (in [12] for $\lambda = 1$) were obtained with the evenness assumption. The crucial difference between them is that Theorem 3.1 in [8] is established under (AR) but Theorem 3.2 in [12] is assumed without (AR). It is worth noting that some new superquadratic assumptions are made instead of (AR) in [12] namely

(A1) there exist $c > 0$ and $q > 2$ such that

$$|F(t,x)| \leqslant c(1 + |x|^q) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^N;$$

(A2) there exists $\sigma \geqslant 1$ such that

$$\sigma\mathcal{F}(t,x) \geqslant \mathcal{F}(t,sx) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^N \text{ and } s \in [0,1],$$

where

$$\mathcal{F}(t,x) = (\nabla F(t,x), x) - 2F(t,x);$$

(A3) $\lim\limits_{|x|\to\infty} F(t,x)/|x|^2 = \infty$ and $\lim\limits_{|x|\to\infty} (\nabla F(t,x), x)/|x|^2 = \infty$ uniformly for $t \in [0,T]$.

Let us recall that (A1) is usually assumed when (AR) is replaced by another superquadratic condition. (A2) and (A3) are superquadratic assumptions which just complement with (AR), which can be proved in a way similar to that in [28]. In this paper, motivated by [8], [12], [28], we will further study the existence of infinitely many nontrivial solutions of (1.1). Instead of (A2) and (A3), we will give more general superquadratic conditions near infinity.

Compared to the superquadratic case, as far as the authors are aware, there are few papers [6], [7], [8] concerning the case where $F(t,x)$ is subquadratic as $|x| \to \infty$. In paper [17], the authors proved the multiplicity of nontrivial solutions for problem (1.1) under the assumption

(A4) $F(t,x) = a(t)|x|^\gamma$, where $a(t) \in L^\infty([0,T], \mathbb{R}^+)$ and $1 < \gamma < 2$ is a constant.

There are many subquadratic functions in mathematical physics in problem like (1.1) except for $F(t,x) = a(t)|x|^\gamma$. In the present paper, we will establish some new existence criteria to guarantee that the problem (1.1) has infinitely many nontrivial solutions under more general assumptions by the genus property. However, not many multiplicity results for problem (1.1) exist in the asymptotically quadratic case. In the present paper, we will fill the gap.

In this paper, we study the existence of infinitely many nontrivial solutions of (1.1) under the assumption that $F(t,x)$ is even in x, i.e., $F(t,-x) = F(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^N$ separately for the above three cases. The structure of the paper is the following. In the next section, we present the necessary preliminary knowledge. After that, we prove our main results in Section 3. In Section 4, we give three examples as applications.

2. Preliminaries and variational setting

In this section, we recall some related preliminaries and display the variational setting which has been established for our problem.

Definition 2.1 ([17]). Let $f(t)$ be a function defined on $[a,b]$ and $\gamma > 0$. The left and right Riemann-Liouville fractional integrals of order γ for the function $f(t)$ denoted by ${}_aD_t^{-\gamma} f(t)$ and ${}_tD_b^{-\gamma} f(t)$, respectively, are defined by

$$ {}_aD_t^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)\,\mathrm{d}s, \quad t \in [a,b], $$

and

$$ {}_tD_b^{-\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s)\,\mathrm{d}s, \quad t \in [a,b], $$

provided the right-hand sides are pointwise defined on $[a,b]$, where Γ is the gamma function.

Definition 2.2 ([17]). Let $f(t)$ be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order γ for function $f(t)$ denoted by ${}_aD_t^{\gamma} f(t)$ and ${}_tD_b^{\gamma} f(t)$, respectively, are defined by

$$ {}_aD_t^{\gamma} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} {}_aD_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_a^t (t-s)^{n-\gamma-1} f(s)\,\mathrm{d}s \right) $$

and

$$ {}_tD_b^{\gamma} f(t) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} {}_tD_b^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left(\int_t^b (s-t)^{n-\gamma-1} f(s)\,\mathrm{d}s \right), $$

where $t \in [a,b]$, $n-1 \leqslant \gamma < n$ and $n \in \mathbb{N}$.

Let $AC([a,b],\mathbb{R}^N)$ denote the space of absolutely continuous functions. For $k \in \mathbb{N}$,

$$ AC^k([a,b],\mathbb{R}^N) = \{ f \in C^{k-1}([a,b],\mathbb{R}^N) \colon\ f^{(k-1)} \in AC([a,b],\mathbb{R}^N) \}. $$

Definition 2.3 ([17]). Let $\gamma \geqslant 0$ and $n \in \mathbb{N}$. If $\gamma \in [n-1, n)$ and $f(t) \in AC^n([a,b], \mathbb{R}^N)$, then the left and right Caputo fractional derivatives of order γ for function $f(t)$, denoted by ${}_a^c D_t^\gamma f(t)$ and ${}_t^c D_b^\gamma f(t)$, respectively, are defined by

$$ {}_a^c D_t^\gamma f(t) = {}_a D_t^{\gamma-n} f^{(n)}(t) = \frac{1}{\Gamma(n-\gamma)} \biggl(\int_a^t (t-s)^{n-\gamma-1} f^{(n)}(s)\, \mathrm{d}s \biggr) $$

and

$$ {}_t^c D_b^\gamma f(t) = (-1)^n {}_t D_b^{\gamma-n} f^{(n)}(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \biggl(\int_t^b (s-t)^{n-\gamma-1} f^{(n)}(s)\, \mathrm{d}s \biggr), $$

for a.e. $t \in [a,b]$.

By the property of a semigroup of the Riemann-Liouville fractional integral operator (see [17]), we have

$$ \begin{aligned} {}_0 D_t^{-\beta}(u'(t)) &= {}_0 D_t^{-\beta/2}({}_0 D_t^{-\beta/2} u'(t)) = {}_0 D_t^{-\beta/2}({}_0 D_t^{1-\beta/2} u(t)), \\ {}_t D_T^{-\beta}(u'(t)) &= {}_t D_T^{-\beta/2}({}_t D_T^{-\beta/2} u'(t)) = -{}_t D_T^{-\beta/2}({}_t D_T^{1-\beta/2} u(t)). \end{aligned} $$

For any $u \in AC([0,T], \mathbb{R}^N)$ with $u(0) = u(T) = 0$, we have

$$ {}_0 D_t^{1-\beta/2} u(t) = {}_0^c D_t^{1-\beta/2} u(t), \quad {}_t D_T^{1-\beta/2} u(t) = {}_t^c D_T^{1-\beta/2} u(t); $$

then (1.1) is equivalent to the problem

$$ \begin{gathered} \frac{\mathrm{d}}{\mathrm{d}t} \Bigl(\frac{1}{2} {}_0 D_t^{\alpha-1} ({}_0^c D_t^\alpha u(t)) - \frac{1}{2} {}_t D_T^{\alpha-1} ({}_t^c D_T^\alpha u(t)) \Bigr) + \nabla F(t, u(t)) = 0 \ \text{ a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{gathered} \tag{2.1} $$

where $\alpha = 1 - \beta/2 \in (1/2, 1]$.

Remark 2.1. A function $u \in AC([0,T], \mathbb{R}^N)$ is a solution of problem (1.1) if and only if u is a solution of (2.1).

For $1 \leqslant p < \infty$ we denote

$$ \|u\|_p = \Bigl(\int_0^T |u(t)|^p\, \mathrm{d}t \Bigr)^{1/p}, \quad u \in L^p([0,T], \mathbb{R}^N), $$

and

$$ \|u\|_\infty = \max_{t \in [0,T]} |u(t)|, \quad u \in C([0,T], \mathbb{R}^N). $$

Set E^α as our workspace, it is the closure of $C_0^\infty([0,T],\mathbb{R}^N)$ with respect to the norm

$$\|u\| = \left[\int_0^T (|u(t)|^2 + |{}_0^cD_t^\alpha u(t)|^2)\,\mathrm{d}t\right]^{1/2} \quad \forall\, u \in E^\alpha, \tag{2.2}$$

where $\frac{1}{2} < \alpha \leqslant 1$ and $C_0^\infty([0,T],\mathbb{R}^N)$ denotes the set of all functions $u \in C^\infty([0,T],\mathbb{R}^N)$ with $u(0) = u(T) = 0$.

Evidently, E^α is a Hilbert space of functions $u \in L^2([0,T],\mathbb{R}^N)$ having an α-order Caputo fractional derivative ${}_0^cD_t^\alpha u \in L^2([0,T],\mathbb{R}^N)$ and satisfying $u(0) = u(T) = 0$ with the inner product

$$\langle u, v\rangle = \int_0^T [(u(t), v(t)) + ({}_0^cD_t^\alpha u(t),\, {}_0^cD_t^\alpha v(t))]\,\mathrm{d}t.$$

Proposition 2.1 ([15]). *The space E^α is reflexive and separable.*

Proposition 2.2 ([15]). *For all $u \in E^\alpha$, we have*

$$\|u\|_2 \leqslant \tau_2 \|{}_0^cD_t^\alpha u\|_2, \tag{2.3}$$

where $\tau_2 := T^\alpha/\Gamma(\alpha+1)$, and

$$\|u\|_\infty \leqslant \tau_\infty \|{}_0^cD_t^\alpha u\|_2, \tag{2.4}$$

where $\tau_\infty := T^{\alpha-1/2}/\Gamma(\alpha)(2\alpha-1)^{1/2}$.

According to (2.3), *the norm $\|\cdot\|$ in E^α is equivalent to the norm*

$$\|u\|_\alpha = \|{}_0^cD_t^\alpha u\|_2 = \left(\int_0^T |{}_0^cD_t^\alpha u(t)|^2\,\mathrm{d}t\right)^{1/2}. \tag{2.5}$$

Lemma 2.1 ([15]). *Assume that the sequence $\{u_k\}$ converges weakly to u in E^α, i.e. $u_k \rightharpoonup u$. Then $u_k \to u$ in $C([0,T],\mathbb{R}^N)$, i.e. $\|u - u_k\|_\infty \to 0$ as $k \to \infty$.*

Corollary 2.1. *The embedding from E^α into $L^2([0,T],\mathbb{R}^N)$ is compact.*

Consider the energy functional $\varphi\colon E^\alpha \to \mathbb{R}$ given by

$$\varphi(u) = \int_0^T \Big[-\frac{1}{2}({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t)) - F(t,u(t))\Big]\,\mathrm{d}t. \tag{2.6}$$

It is known that φ is continuously differentiable on E^α by Theorem 4.1 in [15]. Moreover, for any $u, v \in E^\alpha$, the derivative of φ at u is

$$\begin{aligned}\langle \varphi'(u), v\rangle = & -\int_0^T \frac{1}{2}[({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha v(t)) + ({}_t^cD_T^\alpha u(t), {}_0^cD_t^\alpha v(t))]\,\mathrm{d}t \\ & - \int_0^T (\nabla F(t,u(t)), v(t))\,\mathrm{d}t.\end{aligned} \tag{2.7}$$

Lemma 2.2 ([15]). *Let $1/2 < \alpha \leqslant 1$ and let φ be defined by* (2.6). *If $u \in E^\alpha$ is a solution of the corresponding Euler equation $\varphi'(u) = 0$, then u is a solution of problem* (2.1).

Proposition 2.3 ([15]). *For any $u \in E^\alpha$ we have*

$$|\cos(\pi\alpha)|\|u\|_\alpha^2 \leqslant -\int_0^T ({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,\mathrm{d}t \leqslant \frac{1}{|\cos(\pi\alpha)|}\|u\|_\alpha^2. \tag{2.8}$$

Definition 2.4 ([21]). Let X be a real Banach space and let $\Phi\colon X \to \mathbb{R}$ be differentiable. We say that Φ satisfies the (PS) condition if any sequence $\{u_k\}$ in X such that $\{\Phi(u_k)\}$ is bounded and $\Phi'(u_k) \to 0$ as $k \to \infty$ contains a convergent subsequence.

3. Main results and proofs

3.1. Superquadratic case. Before presenting our theorems, we introduce the following assumptions.

(F1) $F(t,0) \equiv 0$ on $[0,T]$, and there exist $c_1 > 0, R_0 > 0$ such that

$$|\nabla F(t,x)| \leqslant c_1|x| \quad \forall\,|x| \leqslant R_0, \quad \text{a.e. } t \in [0,T];$$

(F2) $\lim\limits_{|x|\to\infty} |F(t,x)|/|x|^2 = \infty$, a.e. $t \in [0,T]$;

(F3) $\widetilde{F}(t,x) := \frac{1}{2}(\nabla F(t,x),x) - F(t,x) \geqslant 0\ \forall\,x \in \mathbb{R}^N$, a.e. $t \in [0,T]$;

(F4) there exists $c_2 > 0$ such that

$$0 \leqslant F(t,x) \leqslant c_2|x|^2\widetilde{F}(t,x) \quad \forall\,|x| \geqslant R_0,\ \text{a.e. } t \in [0,T].$$

Theorem 3.1. *Assume that* $F(t,x)$ *is even in* x *and satisfies* (F1), (F2), (F3), (F4), *then problem* (1.1) *has infinitely many large energy solutions.*

Remark 3.1. In our theorem, $F(t,x)$ is allowed to be sign-changing. Furthermore, even if $F(t,x) \geqslant 0$, assumptions (F2), (F3), and (F4) seem to be weaker than the superquadratic conditions required in the aforementioned reference. Two illustrating examples are given in Section 4.

We will use the Symmetric Mountain Pass Theorem to prove Theorem 3.1. Before proving Theorem 3.1, we give a sequence of lemmata.

Lemma 3.1. *Under assumptions* (F1), (F2), (F3), *and* (F4), *any sequence* $\{u_n\} \subset E^\alpha$ *satisfying*

$$\varphi(u_n) \to c > 0, \quad \langle \varphi'(u_n), u_n \rangle \to 0, \tag{3.1}$$

is bounded in E^α.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\|_\alpha \to \infty$. Let $v_n = u_n/\|u_n\|_\alpha$. Then $\|v_n\|_\alpha = 1$ and $\|v_n\|_2 \leqslant \tau_2 \|v_n\|_\alpha = \tau_2$. Observe that for n large,

$$c+1 \geqslant \varphi(u_n) - \frac{1}{2}\langle \varphi'(u_n), u_n \rangle = \int_0^T \Big[\frac{1}{2}(\nabla F(t,u_n(t)), u_n(t)) - F(t,u_n(t))\Big]\,\mathrm{d}t. \tag{3.2}$$

It follows from (2.6) and (3.1) that

$$\limsup_{n\to\infty} \int_0^T \frac{|F(t,u_n(t))|}{\|u_n\|_\alpha^2}\,\mathrm{d}t \geqslant \frac{|\cos(\pi\alpha)|}{2}. \tag{3.3}$$

For $0 \leqslant a < b$, let

$$\Omega_n(a,b) = \{t \in [0,T],\ a \leqslant |u_n(t)| < b\}.$$

Passing to a subsequence, we may assume $v_n \rightharpoonup v$ in E^α. Then by Lemma 2.1 and Corollary 2.1, $v_n \to v$ in $L^2([0,T])$, and $v_n \to v$ in $C([0,T])$.

If $v \equiv 0$ on $[0,T]$, then $v_n \to 0$ in $L^2([0,T])$ and $v_n \to 0$ in $C([0,T])$. Hence, it follows from (F1) that there exists $c_1 > 0$ such that

$$\begin{aligned}\int_{\Omega_n(0,R_0)} \frac{|F(t,u_n(t))|}{|u_n(t)|^2}|v_n(t)|^2\,\mathrm{d}t &\leqslant c_1 \int_{\Omega_n(0,R_0)} |v_n(t)|^2\,\mathrm{d}t \\ &\leqslant c_1 \int_{[0,T]} |v_n(t)|^2\,\mathrm{d}t \to 0.\end{aligned} \tag{3.4}$$

In view of (F3) and (F4) and (3.2), we have

$$\begin{aligned}
(3.5)\qquad & \int_{\Omega_n(R_0,\infty)} \frac{|F(t,u_n(t))|}{|u_n(t)|^2}|v_n(t)|^2\,\mathrm{d}t \\
&\leqslant \|v_n\|_\infty^2 \int_{\Omega_n(R_0,\infty)} \frac{|F(t,u_n(t))|}{|u_n(t)|^2}\,\mathrm{d}t \\
&\leqslant c_2\|v_n\|_\infty^2 \int_{\Omega_n(R_0,\infty)} \widetilde{F}(t,u_n(t))\,\mathrm{d}t \\
&\leqslant c_2\|v_n\|_\infty^2 \Big(c+1-\int_{\Omega(0,R_0)} \widetilde{F}(t,u_n)\,\mathrm{d}t\Big) \\
&\leqslant c_2\|v_n\|_\infty^2 \Big(c+1+\int_{\Omega(0,R_0)} |\widetilde{F}(t,u_n)|\,\mathrm{d}t\Big) \to 0.
\end{aligned}$$

Hence, from (3.4) and (3.5) we get

$$\begin{aligned}
\int_0^T \frac{|F(t,u_n(t))|}{\|u_n\|_\alpha^2}\,\mathrm{d}t = {} & \int_{\Omega_n(0,R_0)} \frac{|F(t,u_n(t))|}{|u_n(t)|^2}|v_n(t)|^2\,\mathrm{d}t \\
& + \int_{\Omega_n(R_0,\infty)} \frac{|F(t,u_n(t))|}{|u_n(t)|^2}|v_n(t)|^2\,\mathrm{d}t \to 0,
\end{aligned}$$

which contradicts (3.3). Set $A := \{t \in [0,T], |v(t)| \neq 0\}$. If $v \not\equiv 0$ on $[0,T]$, then $\operatorname{meas}(A) > 0$. For a.e. $t \in A$, we have $\lim\limits_{n\to\infty} |u_n(t)| = \infty$. Hence, $A \subset \Omega_n(R_0,\infty)$ for large $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, let $\chi\colon [0,T] \to \mathbb{R}$ be the indicator of Ω_n, that is

$$\chi(t) = \begin{cases} 1, & t \in \Omega_n, \\ 0, & t \notin \Omega_n. \end{cases}$$

It follows from (F1), (F2) and Fatou's Lemma that

$$\begin{aligned}
(3.6)\ 0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|_\alpha^2} = \lim_{n\to\infty} \frac{\varphi(u_n)}{\|u_n\|_\alpha^2} \\
&\leqslant \lim_{n\to\infty} \Big[\frac{1}{2|\cos(\pi\alpha)|} - \int_0^T \frac{F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2\,\mathrm{d}t\Big] \\
&= \lim_{n\to\infty} \Big[\frac{1}{2|\cos(\pi\alpha)|} - \int_{\Omega_n(0,R_0)} \frac{F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2\,\mathrm{d}t \\
&\qquad - \int_{\Omega_n(R_0,\infty)} \frac{F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2\,\mathrm{d}t\Big] \\
&\leqslant \limsup_{n\to\infty} \Big[\frac{1}{2|\cos(\pi\alpha)|} + \frac{c_1}{2}\int_0^T |v_n(t)|^2\,\mathrm{d}t - \int_{\Omega_n(R_0,\infty)} \frac{F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2\,\mathrm{d}t\Big]
\end{aligned}$$

$$\leqslant \frac{1}{2|\cos(\pi\alpha)|} + \frac{c_1\tau_2^2}{2} - \liminf_{n\to\infty} \int_{\Omega_n(R_0,\infty)} \frac{F(t,u_n(t))}{|u_n(t)|^2} |v_n(t)|^2 \,\mathrm{d}t$$
$$= \frac{1}{2|\cos(\pi\alpha)|} + \frac{c_1\tau_2^2}{2} - \liminf_{n\to\infty} \int_0^T \frac{F(t,u_n(t))}{|u_n(t)|^2} [\chi_{\Omega_n(R_0,\infty)}(t)] |v_n(t)|^2 \,\mathrm{d}t$$
$$\leqslant \frac{1}{2|\cos(\pi\alpha)|} + \frac{c_1\tau_2^2}{2} - \int_0^T \liminf_{n\to\infty} \frac{F(t,u_n(t))}{|u_n(t)|^2} [\chi_{\Omega_n(R_0,\infty)}(t)] |v_n(t)|^2 \,\mathrm{d}t$$
$$= -\infty,$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E^α. □

Lemma 3.2. *Under assumptions* (F1), (F2), (F3), *and* (F4), *any sequence satisfying* (3.1) *has a convergent subsequence in* E^α.

P r o o f. Lemma 3.1 implies that $\{u_n\}$ is bounded in E^α. Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u \in E^\alpha$, it is clear that

$$\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle = \langle \varphi'(u_n), u_n - u\rangle - \langle \varphi'(u), u_n - u\rangle \to 0. \tag{3.7}$$

For $u_n \to u$ uniformly in $C([0,T])$, we have

$$\int_0^T |\nabla F(t,u_n(t)) - \nabla F(t,u(t))| |u_n(t) - u(t)| \,\mathrm{d}t \to 0. \tag{3.8}$$

Observe that

$$\begin{aligned} &\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle \\ &\quad = -\int_0^T ({}_0^cD_t^\alpha(u_n(t) - u(t)), {}_t^cD_T^\alpha(u_n(t) - u(t)))\,\mathrm{d}t \\ &\quad\quad - \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u(t)), u_n(t) - u(t))\,\mathrm{d}t. \end{aligned} \tag{3.9}$$

Combining it with (2.8), we get

$$\begin{aligned} |\cos(\pi\alpha)| \|u_n - u\|_\alpha^2 \leqslant &\langle \varphi'(u_n) - \varphi'(u), u_n - u\rangle \\ &+ \int_0^T (\nabla F(t,u_n(t)) - \nabla F(t,u(t)), u_n(t) - u(t))\,\mathrm{d}t. \end{aligned} \tag{3.10}$$

So we have $\|u_n - u\|_\alpha \to 0$ for $n \to \infty$. □

Lemma 3.3. *Under assumptions* (F1) *and* (F2), *for any finite dimensional subspace* $\widetilde{E} \subset E^{\alpha}$ *we have*

$$\varphi(u) \to -\infty \quad \textit{if } \|u\|_{\alpha} \to \infty \textit{ and } u \in \widetilde{E}.$$

P r o o f. Arguing by contradiction, assume that for a sequence $\{u_n\} \subset \widetilde{E}$ with $\|u_n\|_{\alpha} \to \infty$, there is $M_2 > 0$ such that

$$\varphi(u_n) \geqslant -M_2 \quad \forall\, n \in \mathbb{N}.$$

Set $v_n = u_n/\|u_n\|_{\alpha}$. Then $\|v_n\|_{\alpha} = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E^{α}. Since $\widetilde{E}$ is finite dimensional, we have $v_n \to v \in \widetilde{E}$ and $v_n \to v$ in $C([0,T])$. Obviously, one has $\|v\|_{\alpha} = 1$. The contradiction will be achieved in a way similar to that for (3.6). □

Corollary 3.1. *Under assumptions* (F1) *and* (F2), *for any finite dimensional subspace* $\widetilde{E} \subset E^{\alpha}$, *there is* $R = R(\widetilde{E}) > 0$ *such that*

$$\varphi(u) \leqslant 0 \quad \forall\, u \in \widetilde{E},\ \|u\| \geqslant R.$$

Let $\{e_j\}$ *be an orthonormal basis of* E^{α} *and define* $X_j = \mathbb{R}e_j$,

$$Y_k = \bigoplus_{j=1}^{k} X_j, \quad Z_k = \bigoplus_{j=k+1}^{\infty} X_j, \quad k \in \mathbb{N}. \tag{3.11}$$

Lemma 3.4. *Let*

$$\gamma_k := \sup_{\substack{u \in Z_k \\ \|u\|_{\alpha}=1}} \|u\|_2. \tag{3.12}$$

Then

$$\gamma_k \to 0 \quad \textit{for } k \to \infty. \tag{3.13}$$

P r o o f. The result can be proved in a way similar to that in ([29], Lemma 3.8) as the embedding from E^{α} into $L^2([0,T])$ is compact. □

It follows from (F1) that there exists $c_3 > 0$ such that

$$|F(t,u(t))| \leqslant c_3|u(t)|^2 \quad \forall\, u \in E^{\alpha},\ \|u\|_{\alpha} = R_0. \tag{3.14}$$

By Lemma 3.4, we can choose $m \geqslant 1$ such that

$$\|u\|_2^2 \leqslant \frac{|\cos(\pi\alpha)|}{4c_3}\|u\|_{\alpha}^2 \quad \forall\, u \in Z_m. \tag{3.15}$$

Lemma 3.5. *Under assumption* (F1), *there exist constants* $\varrho > 0$, $\alpha_1 > 0$ *such that* $\varphi|_{\partial B_\varrho \cap Z_m} \geqslant \alpha_1$.

Proof. By (2.6), (3.14), and (3.15), for all $u \in Z_m$, $\|u\|_\alpha = R_0 := \varrho$, we have

$$\begin{aligned}\varphi(u) &= -\int_0^T \frac{1}{2}({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,\mathrm{d}t - \int_0^T F(t,u(t))\,\mathrm{d}t \\ &\geqslant \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - c_3\int_0^T |u(t)|^2\,\mathrm{d}t \\ &= \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - c_3\|u\|_2^2 \\ &\geqslant \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \frac{|\cos(\pi\alpha)|}{4}\|u\|_\alpha^2 \\ &= \frac{|\cos(\pi\alpha)|}{4}\|u\|_\alpha^2 = \frac{|\cos(\pi\alpha)|}{4}R_0^2 := \alpha_1.\end{aligned}$$

□

We say that $\varphi \in C^1(X,\mathbb{R})$ satisfies $(C)_c$-condition if any sequence $\{u_n\}$ such that

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1+\|u_n\|) \to 0 \tag{3.16}$$

has a convergent subsequence. In [4], [14], the authors proved if (PS) condition is replaced by the weaker $(C)_c$-condition, the deformation lemmas still hold.

Lemma 3.6 ([25], Symmetric Mountain Pass Theorem). *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. Suppose that $\varphi \in C^1(E,\mathbb{R})$ satisfies $(C)_c$-condition for all $c > 0$ and*
- (I1) $\varphi(0) = 0$, $\varphi(-u) = \varphi(u)$ *for all* $u \in X$;
- (I2) *there exist constants* ϱ, $\alpha > 0$ *such that* $\varphi|_{\partial B_\varrho \cap Z} \geqslant \alpha$;
- (I3) *for any finite dimensional subspace* $\widetilde{X} \subset X$, *there is* $R = R(\widetilde{X}) > 0$ *such that*

$$\varphi(e) \leqslant 0 \quad \forall\, e \in \widetilde{X} \setminus B_R(0).$$

Then φ possesses an unbounded sequence of critical values.

Proof of Theorem 3.1. Let $X = E^\alpha$, $Y = Y_m$, $Z = Z_m$. By Lemma 3.1, Lemma 3.2, Lemma 3.5, and Corollary 3.1, all conditions of Lemma 3.6 are satisfied. Thus, problem (1.1) possesses infinitely many solutions with large energy. □

3.2. Subquadratic case. For the subquadratic case, we assume:
(F5) There are constants $1 < \gamma_1 < \gamma_2 < 2$, $a_1 \geqslant 0$, and $a_2 \geqslant 0$ such that

$$|F(t,x)| \leqslant a_1|x|^{\gamma_1} + a_2|x|^{\gamma_2} \quad \forall\, x \in \mathbb{R}^N,\ t \in [0,T].$$

(F6) There exist a nonempty open set $J \subset [0,T]$ and $\delta, \eta > 0$, $\gamma_3 \in (1,2)$ such that

$$F(t,x) \geqslant \eta |x|^{\gamma_3} \quad \forall (t,x) \in J \times \mathbb{R}^N,\ |x| \leqslant \delta.$$

Theorem 3.2. *Suppose that* (F5) *and* (F6) *are satisfied, and that* $F(t,x)$ *is even in* x*. Then problem* (1.1) *has infinitely many nontrivial solutions.*

Remark 3.2. If $F(t,x) = a(t)|x|^{\gamma}$, then assumption (A4) implies that (F5) and (F6) with $\gamma_1 = \gamma_3 = \gamma < \gamma_2 < 2$, $a_1 = \sup\limits_{[0,T]} |a(t)|$ and $a_2 = 0$.

In order to find nontrivial critical points of φ, we will use the "genus" properties, so we recall the following definitions and results (see [25] and [9]).

Let X be a Banach space, $\psi \in C^1(X,\mathbb{R})$, and $c \in \mathbb{R}$. We set

$$\Sigma = \{A \subset X - \{0\}\colon\ A \text{ is closed in } X \text{ and symmetric with respect to } 0\},$$
$$K_c = \{u \in X\colon\ \psi(u) = c,\ \psi'(u) = 0\}, \quad \psi^c = \{u \in X\colon\ \psi(u) \leqslant c\}.$$

Definition 3.1 ([25]). For $A \in \Sigma$, we say the genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\varphi \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property.

Lemma 3.7 ([25]). *Let* ψ *be an even* C^1 *functional on* X *satisfying the* (PS)-*condition. For any* $n \in \mathbb{N}$*, set*

$$\Sigma_n = \{A \in \Sigma\colon\ \gamma(A) \geqslant n\}, \quad c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \psi(u).$$

(i) *If* $\Sigma_n \neq \emptyset$ *and* $c_n \in \mathbb{R}$*, then* c_n *is a critical value of* ψ*.*

(ii) *If there exists* $r \in \mathbb{N}$ *such that*

$$c_n = c_{n+1} = \ldots = c_{n+r} = c \in \mathbb{R}$$

and $c \neq \psi(0)$*, then* $\gamma(K_c) \geqslant r + 1$*.*

Lemma 3.8. *Under assumption* (F5), φ *is bounded from below and satisfies the* (PS) *condition.*

P r o o f. We first prove that φ is bounded from below. By (2.6), (2.8), (F5), and the Hölder inequality we have

$$\begin{aligned}(3.17)\quad \varphi(u) &= -\int_0^T \frac{1}{2}({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,\mathrm{d}t - \int_0^T F(t,u(t))\,\mathrm{d}t \\ &\geqslant \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \int_0^T a_1|u(t)|^{\gamma_1}\,\mathrm{d}t - \int_0^T a_2|u(t)|^{\gamma_2}\,\mathrm{d}t \\ &\geqslant \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \sum_{i=1}^2\Big(\int_0^T |a_i|^{2/(2-\gamma_i)}\,\mathrm{d}t\Big)^{(2-\gamma_i)/2}\Big(\int_0^T |u(t)|^2\,\mathrm{d}t\Big)^{\gamma_i/2} \\ &\geqslant \frac{|\cos(\pi\alpha)|}{2}\|u\|_\alpha^2 - \sum_{i=1}^2 d_i\|u\|_\alpha^{\gamma_i},\end{aligned}$$

where $d_i = a_iT^{(2-\gamma_i)/2}\tau_2^{\gamma_i}$, $i=1,2$. Since $1<\gamma_1<\gamma_2<2$, (3.17) implies $\varphi(u)\to\infty$ as $\|u\|_\alpha\to\infty$. Consequently, φ is bounded from below. Next, we prove φ satisfies the (PS)-condition.

Assume that $\{u_k\}\subset E^\alpha$ is a sequence such that $\{\varphi(u_k)\}$ is bounded and $\varphi'(u_k)\to 0$ as $k\to\infty$. Then by (3.17) there exists $A>0$ such that

$$\|u_k\|_\alpha \leqslant A, \quad k\in\mathbb{N}.$$

Since the proof that any bounded (PS)-sequence converges in E^α is the same as Lemma 3.2, we omit it. □

P r o o f of Theorem 3.2. In view of the proof of Lemma 3.8, $\varphi\in C^1(E^\alpha,\mathbb{R})$ is bounded from below and satisfies the (PS)-condition. By the assumption (F5) and the fact that $F(t,x)$ is even in x, it is obvious that φ is even and $\varphi(0)=0$. Set

$$\varphi^c = \{u\in E^\alpha\colon\ \varphi(u)\leqslant c\},$$

where c is a constant. In order to apply Lemma 3.7, we prove now that

$$(3.18)\qquad \forall\, n\in\mathbb{N}\ \exists\ \varepsilon>0 \text{ such that } \gamma(\varphi^{-\varepsilon})\geqslant n.$$

For any $n\in\mathbb{N}$, we take n disjoint open sets J_i, $i=1,2,\dots,n$, such that

$$\bigcup_{i=1}^n J_i\subset J.$$

Let $u_i\in E^\alpha\setminus\{0\}$ with $\operatorname{supp}(u_i)\subset J_i$, $\|u_i\|_\alpha=1$, $i=1,2,\dots,n$, and

$$\begin{aligned}E_n &= \operatorname{span}\{u_1,u_2,\dots,u_n\},\\ S_n &= \{u\in E_n\colon\ \|u\|_\alpha=1\}.\end{aligned}$$

For any $u \in E_n$ there exists $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, such that

$$u(t) = \sum_{i=1}^{n} \lambda_i u_i(t), \quad t \in [0, T]. \tag{3.19}$$

Then

$$\|u\|_{\gamma_3} = \left(\int_0^T |u|^{\gamma_3}\, \mathrm{d}t\right)^{1/\gamma_3} = \left(\sum_{i=1}^{n} |\lambda_i|^{\gamma_3} \int_{J_i} |u_i|^{\gamma_3}\, \mathrm{d}t\right)^{1/\gamma_3}, \tag{3.20}$$

and

$$\begin{aligned} \|u\|_\alpha^2 &= \int_0^T |{}_0^cD_t^\alpha u(t)|^2\, \mathrm{d}t = \sum_{i=1}^{n} \lambda_i^2 \int_{J_i} |{}_0^cD_t^\alpha u_i(t)|^2\, \mathrm{d}t \\ &= \sum_{i=1}^{n} \lambda_i^2 \int_0^T |{}_0^cD_t^\alpha u_i(t)|^2\, \mathrm{d}t = \sum_{i=1}^{n} \lambda_i^2 \|u_i\|_\alpha^2 = \sum_{i=1}^{n} \lambda_i^2. \end{aligned} \tag{3.21}$$

Since all norms in a finite dimensional normed space are equivalent, there is a constant $c' > 0$ such that

$$c'\|u\|_\alpha \leqslant \|u\|_{\gamma_3}, \quad u \in E_n. \tag{3.22}$$

By (F6), (2.6), (3.19), (3.20), (3.21), and (3.22), we have

$$\begin{aligned} \varphi(su) &= -\int_0^T \frac{1}{2}({}_0^cD_t^\alpha su, {}_t^cD_T^\alpha su)\, \mathrm{d}t - \int_0^T F(t, su)\, \mathrm{d}t \\ &\leqslant \frac{s^2}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \sum_{i=1}^{n} \int_{J_i} F(t, s\lambda_i u_i)\, \mathrm{d}t \\ &\leqslant \frac{s^2}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \eta s^{\gamma_3} \sum_{i=1}^{n} |\lambda_i|^{\gamma_3} \int_{J_i} |u_i|^{\gamma_3}\, \mathrm{d}t \\ &= \frac{s^2}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \eta s^{\gamma_3} \|u\|_{\gamma_3}^{\gamma_3} \\ &\leqslant \frac{s^2}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \eta (c's)^{\gamma_3} \|u\|_\alpha^{\gamma_3} \\ &= \frac{s^2}{2|\cos(\pi\alpha)|} - \eta (c's)^{\gamma_3} \quad \forall\, u \in S_n,\ 0 < s \leqslant \delta\Big(\max_{1\leqslant i\leqslant n} \|u_i\|_\infty\Big)^{-1}. \end{aligned} \tag{3.23}$$

Inequality (3.23) implies that there exist $\varepsilon > 0$ and $\sigma > 0$ such that

$$\varphi(\sigma u) < -\varepsilon, \quad u \in S_n. \tag{3.24}$$

Let

$$S_n^\sigma = \{\sigma u\colon\ u \in S_n\},$$
$$\Omega = \left\{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n\colon\ \sum_{i=1}^{n} \lambda_i^2 < \sigma^2\right\}.$$

Then it follows from (3.24) that $u \in S_n^\sigma$,

$$\varphi(u) < -\varepsilon,$$

which, together with the fact that $\varphi \in C^1(E^\alpha, \mathbb{R})$ is even, implies that

$$S_n^\sigma \subset \varphi^{-\varepsilon} \in \Sigma. \tag{3.25}$$

On the other hand, it follows from (3.19) and (3.21) that there exists an odd homeomorphism mapping $\varphi \in C(S_n^\sigma, \partial\Omega)$. By the properties of the genus (see Proposition 7.5 (iii) and Proposition 7.7 in [25]), we have

$$\gamma(\varphi^{-\varepsilon}) \geqslant \gamma(S_n^\sigma) = n, \tag{3.26}$$

so the proof of (3.18) follows. Set

$$c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \varphi(u).$$

It follows from relation (3.26) and the fact that φ is bounded from below on E^α that $-\infty < c_n \leqslant -\varepsilon < 0$, that is, for any $n \in \mathbb{N}$, c_n is a real negative number. By Lemma 3.7 (i), φ has infinitely many nontrivial critical points, so Problem (1.1) possesses infinitely many nontrivial solutions. □

3.3. Asymptotically quadratic case. Let us introduce the following assumptions on F:

(F7) For all $(t, x) \in [0, T] \times \mathbb{R}^N$, $F(t, x) \geqslant 0$, there exist constants $\nu_1 > 0$ and $R_1 > 0$ such that
$$F(t, x) \leqslant \nu_1 |x|^2 \quad \forall\, t \in [0, T],\ |x| \geqslant R_1;$$

(F8) $\lim\limits_{|x|\to 0} F(t, x)/|x|^2 = \infty$ for $t \in [0, T]$ uniformly, and there exist constants $\nu_2 > 0$, $R_2 > 0$, and $\sigma \in [1, 2)$ such that
$$F(t, x) \leqslant \nu_2 |x|^\sigma \quad \forall\, t \in [0, T],\ |x| \leqslant R_2;$$

(F9) $\liminf\limits_{|x|\to\infty} F(t, x)/|x| \geqslant \lambda > 0$ for $t \in [0, T]$ uniformly.

Theorem 3.3. *Assume that* (F7), (F8), (F9) *hold and that* $F(t,x)$ *is even in* x. *Then problem* (1.1) *has infinitely many small energy solutions.*

Lemma 3.9 ([29], Dual Fountain Theorem)**.** *Assume that the functional* φ *satisfies*

(H1) X *is a Banach space,* $\varphi \in C^1(X,\mathbb{R})$ *is even, subspaces* X_k, Y_k, Z_k *are defined by* (3.11)*; and there is a constant* $k_0 > 0$ *such that for each* $k \geqslant k_0$ *there exists* $\varrho_k > r_k > 0$ *such that*

(H2) $d_k := \inf\limits_{\substack{u\in Z_k\\ \|u\|\leqslant \varrho_k}} \varphi(u) \to 0$ *as* $k \to \infty$*;*

(H3) $i_k := \max\limits_{\substack{u\in Y_k\\ \|u\|= r_k}} \varphi(u) < 0$*;*

(H4) $\xi_k := \inf\limits_{\substack{u\in Z_k\\ \|u\|= \varrho_k}} \varphi(u) \geqslant 0$*;*

(H5) φ *satisfies the* $(\mathrm{PS})_c^*$ *condition for every* $c \in [d_{k_0}, 0)$.

Then φ *has a sequence of negative critical values converging to* 0.

Remark 3.3. The function φ satisfies the $(\mathrm{PS})_c^*$ condition means: if $\{u_{n_j}\} \subset X$ is any sequence such that $n_j \to \infty$, $u_{n_j} \in Y_{n_j}$, $\varphi(u_{n_j}) \to c$, and $(\varphi|_{Y_{n_j}})'(u_{n_j}) \to 0$, then $\{u_{n_j}\}$ contains a subsequence converging to a critical point of φ. It is obvious that if φ satisfies the $(\mathrm{PS})_c^*$ condition, then φ satisfies the $(\mathrm{PS})_c$ condition.

Proof of Theorem 3.3. In the following, we prove that φ satisfies all conditions of Lemma 3.9.

Step 1. Prove φ satisfies the $(\mathrm{PS})_c^*$-condition. The proof is standard, and we can see the first step of proof in ([18], Theorem 3.2), so we omit it.

Step 2. We show that F satisfies Lemma 3.9 (H2)–(H4). By (2.4), for any $u \in E^\alpha$ with $\|u\|_\alpha \leqslant R_2/\tau_\infty$ we have

$$\|u\|_\infty \leqslant R_2,$$

where R_2 and τ_∞ are the constants in (F8) and (2.4) respectively. Then for $u \in Z_k$ and $\|u\|_\alpha \leqslant R_2/\tau_\infty$, by (F8) and the Hölder inequality, we have

$$\begin{aligned}\varphi(u) &= -\int_0^T \frac{1}{2}({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,\mathrm{d}t - \int_0^T F(t,u(t))\,\mathrm{d}t \\ &\geqslant \frac{1}{2}|\cos(\pi\alpha)|\|u\|_\alpha^2 - \int_0^T \nu_2|u(t)|^\sigma\,\mathrm{d}t \\ &\geqslant \frac{1}{2}|\cos(\pi\alpha)|\|u\|_\alpha^2 - \nu_2T^{(2-\sigma)/2}\|u\|_2^\sigma.\end{aligned} \tag{3.27}$$

Consequently, (3.27) implies

$$\varphi(u) \geqslant \frac{1}{2}|\cos(\pi\alpha)|\|u\|_\alpha^2 - \nu_2T^{(2-\sigma)/2}\gamma_k^\sigma\|u\|_\alpha^\sigma \tag{3.28}$$

for all $\|u\|_\alpha \leqslant R_2/\tau_\infty$, where γ_k is defined by (3.12) in Lemma 3.4. For any $k \in \mathbb{N}$, let

$$\varrho_k := \left(\frac{4\nu_2 T^{(2-\sigma)/2}\gamma_k^\sigma}{|\cos(\pi\alpha)|}\right)^{1/(2-\sigma)}. \tag{3.29}$$

Then we have

$$\varrho_k \to 0 \quad \text{for } k \to \infty. \tag{3.30}$$

Evidently, there exists a positive integer $k_1 > 0$ such that

$$\varrho_k < \frac{R_2}{\tau_\infty} \quad \forall\, k \geqslant k_1. \tag{3.31}$$

For any $k \geqslant k_1$, (3.28) together with (3.29) and (3.31) yields

$$\xi_k := \inf_{\substack{u\in Z_k\\ \|u\|_\alpha=\varrho_k}} \varphi(u) \geqslant \frac{|\cos(\pi\alpha)|}{4}\varrho_k^2 > 0.$$

This shows the condition of (H4) in Lemma 3.9 is satisfied.

By (3.28), for any $u \in Z_k$ with $\|u\|_\alpha \leqslant \varrho_k$, we have

$$\varphi(u) \geqslant -\nu_2 T^{(2-\sigma)/2}\gamma_k^\sigma \|u\|_\alpha^\sigma.$$

Observing that $\varphi(0) = 0$ by (F8), we obtain

$$0 \geqslant \inf_{\substack{u\in Z_k\\ \|u\|_\alpha\leqslant\varrho_k}} \varphi(u) \geqslant -\nu_2 T^{(2-\sigma)/2}\gamma_k^\sigma \varrho_k^\sigma.$$

This together with (3.13) and (3.30) implies

$$d_k = \inf_{\substack{u\in Z_k\\ \|u\|_\alpha\leqslant\varrho_k}} \varphi(u) \to 0, \quad k \to \infty.$$

So, the condition of (H2) in Lemma 3.9 holds.

For any $u \in Y_k$, there exists a constant $C_k > 0$ such that

$$\|u\|_2 \geqslant C_k\|u\|_\alpha. \tag{3.32}$$

By (F8), there exists a constant $\delta_k > 0$ such that

$$F(t,u) \geqslant \frac{|u|^2}{C_k^2|\cos(\pi\alpha)|} \quad \forall\, |u| \leqslant \delta_k. \tag{3.33}$$

By (2.4), for any $u \in E^\alpha$ with $\|u\|_\alpha \leqslant \delta_k/\tau_\infty$, we have

$$\|u\|_\infty \leqslant \delta_k.$$

Combining this with (3.32) and (3.33), for any $u \in Y_k \setminus \{0\}$ with $\|u\|_\alpha \leqslant \delta_k/\tau_\infty$, we obtain

$$
\begin{aligned}
\varphi(u) &= -\int_0^T \frac{1}{2}({}_0^cD_t^\alpha u(t), {}_t^cD_T^\alpha u(t))\,\mathrm{d}t - \int_0^T F(t,u(t))\,\mathrm{d}t,\\
&\leqslant \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \int_0^T \frac{|u|^2}{C_k^2|\cos(\pi\alpha)|}\,\mathrm{d}t\\
&= \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \frac{\|u\|_2^2}{C_k^2|\cos(\pi\alpha)|}\\
&\leqslant \frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2 - \frac{\|u\|_\alpha^2}{|\cos(\pi\alpha)|}\\
&= -\frac{1}{2|\cos(\pi\alpha)|}\|u\|_\alpha^2.
\end{aligned}
\tag{3.34}
$$

Choosing $0 < r_k < \min\{\varrho_k, \delta_k/\tau_\infty\}$, inequality (3.34) implies

$$i_k := \max_{\substack{u\in Y_k\\ \|u\|_\alpha = r_k}} \varphi(u) \leqslant -\frac{1}{2|\cos(\pi\alpha)|}r_k^2 < 0 \quad \forall\, k \in \mathbb{N}.$$

Hence, the condition (H3) in Lemma 3.9 is satisfied.

Thus all the conditions in Lemma 3.9 hold. Therefore, by Lemma 3.9, φ has a sequence of negative critical values $c_n = \varphi(u_n)$ converging to 0, that is φ has infinitely many solutions with small energy. $\square$

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. In Problem (1.1), let $F(t,x) = (1+\sin t)|x|^2 \ln(\frac{1}{2} + |x|)$, and

$$\widetilde{F}(t,x) = \frac{1}{2}(\nabla F(t,x), x) - F(t,x) = (1+\sin t)\frac{|x|^3}{1+2|x|},$$

so it is easy to verify that all the conditions (F1)–(F4) are satisfied. Then by Theorem 3.1, Problem (1.1) has infinitely many solutions $\{u_n\}$ on E^α. But F does not satisfy (AR). In fact, for $\kappa > 2$ we have

$$(\nabla F(t,x),x) - \kappa F(t,x) = (2-\kappa)(1+\sin t)|x|^2 \ln\Big(\frac{1}{2}+|x|\Big) + (1+\sin t)\frac{|x|}{\frac{1}{2}+|x|}|x|^2 \to -\infty, \quad |x| \to \infty.$$

Likewise, it is easy to check that the function

$$F(t,x) = a(t)\sum_{i=1}^{m} b_i |x|^{\beta_i},$$

where $b_1 > 0$, $b_i \in \mathbb{R}$, $i = 2,3,\ldots,m$, $\beta_1 > \beta_2 > \ldots > \beta_m \geqslant 2$, $a \in C([0,T],\mathbb{R})$ such that $0 < \min\limits_{[0,T]} a \leqslant \max\limits_{[0,T]} a < \infty$, does not satisfy (AR), but satisfies (F1)–(F4). Hence, exactly the same conclusions hold true by Theorem 3.1.

Example 4.2. In Problem (1.1), let $T = 1$ and

$$F(t,x) = \frac{\cos t}{1+|t|^{1/2}}|x|^{4/3} + \frac{\sin t}{1+|t|^{1/3}}|x|^{3/2}.$$

Then

$$\nabla F(t,x) = \frac{4\cos t}{3(1+|t|^{1/2})}|x|^{-2/3}x + \frac{3\sin t}{2(1+|t|^{1/3})}|x|^{-1/2}x.$$

It is easy to see that

$$|\nabla F(t,x)| \leqslant \frac{4}{3(1+|t|^{1/2})}|x|^{1/3} + \frac{3}{2(1+|t|^{1/3})}|x|^{1/2} \quad \forall (t,x) \in [0,1]\times\mathbb{R}^N,\ |x| \leqslant 1,$$

and

$$F(t,x) \geqslant \frac{1}{4}|x|^{4/3} \quad \forall (t,x) \in [0,1]\times\mathbb{R}^N,\ |x| \leqslant 1.$$

Then F satisfies all the assumptions in Theorem 3.2, that is

$$\frac{4}{3} = \gamma_1 = \gamma_3 < \gamma_2 = \frac{3}{2}, \quad a_1 = 1, \quad a_2 = 1,$$
$$\delta = 1, \quad \eta = \frac{1}{4}, \quad J = (0,1).$$

By Theorem 3.2, Problem (1.1) has infinitely many nontrivial solutions.

Example 4.3. In Problem (1.1), let

$$F(t,x) = \frac{(1+\sin^2 t)(|x|^2 + \ln(1+|x|^2))}{1+\arctan|x|}. \tag{4.1}$$

Hence, F is asymptotically quadratic at infinity. Since

$$\lim_{|x|\to 0}\frac{F(t,x)}{|x|^2}=\lim_{|x|\to 0}\frac{(1+\sin^2 t)(|x|^2+\ln(1+|x|^2))}{(1+\arctan|x|)|x|^2}=\infty$$

and

$$\lim_{|x|\to \infty}\frac{F(t,x)}{|x|}=\lim_{|x|\to \infty}\frac{(1+\sin^2 t)(|x|^2+\ln(1+|x|^2))}{(1+\arctan|x|)|x|}=\infty,$$

we conclude that (F7), (F9) are satisfied, and (F8) holds when $\sigma=1$. By Theorem 3.3, Problem (1.1) has infinitely many nontrivial solutions.

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