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# More Multiplicity in the Rate of Interest; Poly-party and Poly-creditor Transactions. I

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omylu, avšak závažného. A to, když jsem napsal: „Nechci předpokládati zlou vůli . . .“ Odpověď pana Fitze mě přesvědčila o opaku. Nebudu proto napříště polemisovati, již také proto ne, že jde o anonyma, který své anonymity zneužívá k osobním a nekvalifikovatelným útokům.

Na jeho řečnickou otázku odpovím, až odloží svou masku, abych poznal, s kým mám tu čest. Zatím bych se na oplátku otázal, je-li si p. Fitz vědom toho, že nesprávným tvrzením o velmi špatné finanční situaci V. P. Ú. poškozuje těžce zájem pojištěnců. Kdyby bylo správné, co uvádí, muselo by nastati radikální zhoršení dávek a radikální zvýšení pojistného. Ptám se ho dále, jak je to dávno, co „nestranní odborníci“ propagovali zvýšení úrokové míry ze 4% na 4,5% a usilovali o snížení pojistného pro pojištění invalidní a starobní poukazem na zbytečně vysokou thesauraci peněz. Nechtěl by se p. Fitz raději poučiti o tom, jaké škody tyto snahy způsobily ústavům sociálního pojištění?

K závěru dovolím si připomenouti, že uveřejňováním kritik úrovně článku p. Fitze se zbytečně může vyvolati rozruch v kruzích pojištěnců, aniž se čímkoliv přispěje k odstranění domnělých závad. Volání po dosazení „nestranné komise odborníků“ příliš jasně prozrazuje, že hlavním motivem těchto kritik jest snaha vyvolávati pochybnosti z důvodů osobních. Každý odborník v republice, který se chce činně zúčastniti odborné práce při připravování novely pensijního zákona, má možnost své připomínky i případné námitky zaslati komisi ustanovené ministerstvem sociální péče a může býti jist, že jim bude věnována pozornost, jaké si zaslouží.

# More Multiplicity in the Rate of Interest; Poly-party and Poly-creditor Transactions.

By *D. P. Misra, M. Sc.* (Lucknow).

In my paper „Uniqueness Versus Multiplicity Of The Rate Of Interest In A Purely Financial Transaction“ published on page 71 and onward of the Journal of The Institute of Actuaries Vol. LXIV Part I, No. 308, (London, March 1933), the investigations have been carried out under the limitation that the total of repayments always exceeds that of capitals. This paper will briefly be referred to as „the first paper“.

In the present paper I propose to extend the limitation much more than in the first paper.

But, before the investigations are made in the extended domain, it would be quite worth while to examine the criticism of Dr. J. F. Steffensen (of Copenhogen) on page 169, ibid, on the first paper.

It is quite interesting to find that this paper could have aroused the interest of two eminent Actuaries Dr. Steffensen and Mr. G. J. Lidstone.

The proposition which Dr. Steffensen has tried to prove by elaborate and interesting Mathematics can very well be inferred from the second paragraph on page 81 of the first paper, viz.:

„One may again object that, no matter, which of the five rates of interest worked out in ex. 1 is uniformly applied throughout the term of the contract, at the end of the second year the accumulated amount of £. 1 lent to $B$ by $A$ at the end of the first year is in every case less than £. 5,2 to be paid back by $B$ to $A$ at the end of the second year, and therefore sometimes in this transaction $A$ is creditor and at others $B$ becomes creditor".

It is obvious from the last two clauses of the foregoing sentence that multiplicity in the rate of interest can arise, if at all, only in the transactions where both the parties take their turns, not necessarily alternate, as debtor and creditor. But, in a transaction where one and the same party remains creditor, the rate of interest cannot be more than one. If further the total of repayment is greater than that of capitals, in such a monocreditor transaction, the real positive rate of interest is, not only not more than one, but also unique, i. e. neither more, nor less, than one.

But, at the time of preparing the first paper, convinced as I was of the truth of this proposition empirically, I did not seriously attempt to prove it formally, and, therefore, did not include it in the first paper as a clear theorem.

I propose to show in the following lines that the proof advanced by Dr. Steffensen is not valid.

It should be noted that the hypothetical condition, i. e. the outstanding evaluated balance is always in favour of one and the same party, can be tone only for a rate of interest involved in the transaction and possibly for certain others but not for all the rates of interest in the universe. E. G. the value of all the outstanding payments positive or negative in the left side of

$$200 - 106v - 3v^2 + 297v^3 - 412v^4 = 0$$

remains negative for $v = 100/103$ which is a root of this equation and corresponds to $i = .03$ or $3\%$.

But for $v = \frac{1}{2}$ (taken for simplicity)

$$-106v - 3v^2 + 297v^3 - 412v^4 \quad \text{is negative, while} \quad 297v^3 - 412v^4$$

is positive.

Thus if there are certain values of $v$ in the interval (0, 1), for which $\sum_{n=r}^{n=\infty} v^n S_n > 0$ for $r > 0$ and therefore $\sum_{=}^{\infty} n\, v^n S_n > 0$, Dr. Steffensen has really proved that $f'(v) > 0$ only at these values of $v$ and not in the whole interval $(0, 1)$ for $v$.

This simply proves that $f(v)$ is an increasing function in some portion of the interval $(0, 1)$ of $v$ but by no means in the whole of it.

Obviously nothing in the hypothesis can prevent $f(v)$ from having more than one root in that portion of the interval $(0, 1)$ of $v$ where it is not monotonous.

Now it is the roots, in the interval $(\overset{\times}{0}, \overset{\times}{1})$, of $f(v) = 0$ that will give real positive rates of interest and there fore cannot be meaningless.

In case the series of payments positive or negative is actually interminable, Dr. Steffensens proof has a further defect; for in this case the series $\sum_{n=0}^{\infty} v^n S_n$ is actually infinite, and we are not at all justified to differentiate it term by term unless we prove that the series obtained after differentiation is not only convergent but also uniformly convergent. And, I think, there is nothing in the hypothesis which makes the uniform convergence of this series obvious.

Thus the proof of Dr. Steffensen about the absence of multiplicity in the rate of interest in a transaction in which one and the same party remains creditor throughout the currency of the transaction is not correct, true as the proposition is, as becomes obvious from Mr. Lidstone's proof (page 171 ibid.) which contains little Mathematics.

The only difficulty is that Mr. Lidstone's proof is applicable only to an actually terminable series of payments, and therefore, though quite sufficient for practical finance may not satisfy a pure Mathematician.

I have, therefore, given a rigorous mathematical proof of Dr. Steffensen's criterion in the appendix of this paper.

It is to be further noted that the criterion is only a sufficient condition for the abscence of multiplicity of the rate of interest but not at all a necessary one. I. e. we may get one and only one rate of interest although one and the same party may not remain creditor throughout the currency of the transaction. E. G.

Ex. 1. A advanced to B£ 100 at the beginning of the first year, got back from him £ 204 at the end of the first year, and advanced to B£ 103 finally at the end of the second year. The account was settled by this last payment. What was the rate of interest involved?

The equation of payment is

$100r^2 - 204r + 103 = 0$, where $r = 1 + i$, i. e. the accumulated amount of 1 at the end of one year. Thus $r = \frac{51 \pm \sqrt{26}}{50}$; $i = \frac{1 \pm \sqrt{26}}{50}$.

Now since the only one repayment is greater than the total of two advances, the negative real rate of interest is certainly meaningless.

Thus we have only one rate of interest although both A and B are creditors in this transaction, showing the criterion to be only sufficient but not necessary.

As regards Dr. Steffensen's criticism of the first paper I have to say little, as his objection that the numerical examples in the first paper belong to what he calls unsound finance has been successfully met by Mr. W. J. Courcouf (page 552, J. I. A. Vol. LXIV, part III Nov. 1933) and by Mr. W. H. Carter, I. C. S. (page 94 et seq., Vol. LXV Part I No. 311, March 1934, J. I. A.).

The only difference is that these gentlemen have brought forward concrete examples of the multiplicity in the rate of interest from transactions involving both the interest and the mortality factors. But this, as Mr. Courcouf himself remarks, in no way differs from the principles of the first paper, for I excluded the consideration of problems involving mortality factor not because I thought them outside the scope of the first paper but because on account of their complexity they deserve a separate treatment.

In pure finance also, as Dr. Steffensen himself admits, transactions involving overdrawings can involve the multiplicity of the rare of interest. But his impression is that such transactions are rare. At any rate they are not rare at Lucknow.

Also, as ex. 2 will show, in order that the question of valuation may arise, it is not necessary, as Dr. Steffensen thinks, that payments should be fixed beforehand.

Turning to the main theme of the paper, the present writer notes that Mr. Lidstone's proof only tells us that in a monocreditor transaction there cannot be more than one (real positive) rate of interest. It does not tell us that in a monocreditor transaction there is always one and only one real positive rate of interest; for he says that if $i$ is a rate of interest then $i \pm h$ cannot be a rate of interest. But the existence if $i$ is not questioned. This shows that even among monocreditor transactions there may be such as may not have a real positive rate of interest.

From the first paper it is clear that when the total of repayments is greater than that of advances there is always at least one real positive rate of interest. But, since Mr. Lidstone has now proved that in a monocreditor transaction there cannot be more than one real positive rate of interest, it follows, that in a monocreditor transaction having the total of repayments greater than that of advances there is always one and only one real positive rate of interest.

The mystery in the meaning of ,,not more than one" should, therefore, be sought in such monocreditor transactions as have the total of repayments not greater than that of advances.

We shall, therefore, discuss the general cases of bicreditor transactions in which the total of repayments does not exceed the total of advances.

In the first paper the negative rate of interest was declared to be absurd. But that was done with the limitation of that paper. In the present paper it can be taken as obvious that both the zero and the real positive rates of interest are not meaningless in transactions where the total of repayments equals that of advances. Also all the real rates of interest (negative, zero, or positive) are not meaningless when the total of repayments is less than that of advances.

It should not be forgotten that these conventions about the zero and negative real rates of interest can be regarded as convenient only in theory. In practice their adoption may lead to untold confusion even there where the total of repayments does not exceed that of advances, as will be shown by numerical examples in this paper.

Sum of repayments equals that of advances.

(a) Lump sum capital. The equation (1) of the first paper i. e.

$$Ax^{l_r} - B_1x^{m_1} - B_2x^{m_2} - B_3x^{m_3} - \dots - B_r = 0 \qquad (1)$$

still remains the same, except that now

$$A = B_1 + B_2 + B_3 + \dots + B_r, \text{ where } x = (1 + i)^H.$$

Now by Decartes' rule of signs (1) can have only one real positive root, and, since $x = 1$ is obviously a root, $i = 0$ is the only rate of interest that has any meaning in this case.

(b) Capital invested by instalments. Now the only change in the condition is $A_1 + A_2 + A_3 + \dots + A_s = B_1 + B_2 + B_3 + \dots + B_r$.

(I) Last repayment after last investment. The equation (4) of the first paper is

$$A_1x^{l_r-p_1} + A_2x^{l_r-p_2} + \dots + A_sx^{l_r-p_s} - B_1x^{l_r-l_1} - B_2x^{l_r-l_2} - \dots - B_{r-1}x^{l_r-l_{r-1}} - B_r = 0. \qquad (4)$$

As before let $\psi(x)$ denote the left side of (4), so that $\psi(0) < 0$, $\psi(1) = 0$, and $\psi(\alpha) > 0$. Thus $x = 1$ is certainly a root of (4) giving $i = 0$ which is not meaningless. If further $\psi(1 - \varepsilon) > 0$, so that $\psi(1 + \varepsilon) < 0$, where $\mathrm{Lt}\ \varepsilon = 0$, (4) has an odd number of roots in each of the intervals $(\overset{\times}{0}, \overset{\times}{1})$ and $(\overset{\times}{1}, \overset{\times}{\alpha})$.

Now the roots in $(\overset{\times}{0}, \overset{\times}{1})$ give real negative values of $i$, which being meaningless in the present case do not contribute to the multiplicity, but roots in $(\overset{\times}{1}, \overset{\times}{\alpha})$ are each greater than 1, give real positive values of $i$, and do contribute to the multiplicity of the rate of interest.

If $\psi(1 - \varepsilon) < 0$, so that $\psi(1 + \varepsilon) > 0$, (4) has either no roots or an even number of them in each of the intervals $(\overset{\times}{0}, \overset{\times}{1})$ and $(1, \overset{\times}{\alpha})$.

Obviously the multiplicity of the rate of interest is quite probable in the present case.

($\alpha_1$) The non-overlapping of investments and epayments. Let the last instalment $A_s$ of the capital be invested before the first repayment $B_1$. In this case (4) having only one change of sign has only one real positive root viz. $x = 1$ giving $i = 0$ the only rate of interest not meaningless.

($\alpha_2$) A particular overlapping. If the investment of the last instalment $A_s$ of capital occurs simultaneously with the receipt or the due date of the first instalment $B_1$ of repayment, (4) becomes

$$A_1 x^{l_r - p_1} + A_2 x^{l_r - p_2} + \ldots + A_{s-1} x^{l_r - p_{s-1}} + (A_s - B_1) x^{l_r - l_1} - \\ - B_2 x^{l_r - l_2} - \ldots - B_{r-1} x^{l_r - l_{r-1}} - B_r = 0.$$

Obviously the conclusion is the same as in ($\alpha_1$), viz. zero is the only rate of interest not meaningless.

($\beta_1$) Overlapping. If restrictions ($\alpha_1$) and ($\alpha_2$) are removed we cannot say that there is one and only one change of sign. In fact (4) may have any number of changes of sign consistent with its degree.

All that we are certain of is that there may or may not be a number of real positive roots each $> 1$. We, therefore, cannot assert that there is one and only one rate of interest, viz., zero, since in this case all the real positive roots are not necessarily less than or equal to units.

It appears that in this case more than one rate of interest (including zero) can occur in one and the same transaction. The actual occurence of this multiplicity again depends on the absolute and relative magnitudes of the various constants in (4) besides the changes of sign.

(II) Let the investment of the last instalment $A_s$ of capital occur simultaneously with the receipt or the due date of the last instalment $B_r$ of repayment; so that (4) now becomes

$$A_1 x^{l_r - p_1} + A_2 x^{l_r - p_2} + \ldots + A_{s-1} x^{l_r - p_{s-1}} - B_1 x^{l_r - l_1} - B_2 x^{l_r - l_2} - \\ - \ldots - B_{r-1} x^{l_r - l_{r-1}} + A_s - B_r = 0. \qquad (4a)$$

Obviously the restrictions ($\alpha_1$) and ($\alpha_2$) cannot be imposed here.

Special cases ($\alpha_3$). Let the last but one instalment $A_{s-1}$ of capital become invested before the first instalment $B_1$ of repayment is received or becomes due, so that under the present condition (4a) becomes arranged in descending powers of $x$.

In case $A_s - B_r \leq 0$, there is only one change of sign, and by exactly the same reasoning as in ($\alpha_1$) and ($\alpha_2$) [except that, in case $A_s - B_r = 0$, the removal of zero roots is necessary] the equation (4a) has one and only one real positive root $x = 1$ giving $i = 0$ not meaningless.

In case $A_s - B_r > 0$, there are two changes of sign. Therefore by Decartes' rule of signs there are two real positive roots, since $x = 1$ is a real positive root.

Let the left side of (4a) be denoted by $\psi(x)$, so that $\psi(0) > 0$, $\psi(1) = 0$ [since $\Sigma A = \Sigma B$], and $\psi(\alpha) > 0$.

If further $\psi(1-\varepsilon) < 0$, so that $\psi(1+\varepsilon) > 0$, where $\mathrm{Lt}\ \varepsilon = 0$, then an odd number of roots lies in $(\overset{\times}{0}, \overset{\times}{1})$ and either none or an even number in $(\overset{\times}{1}, \overset{\times}{\alpha})$. But, since (4a) can have only two real positive roots, there can be no roots in $(\overset{\times}{1}, \overset{\times}{\alpha})$. Thus there is one root in $(\overset{\times}{0}, \overset{\times}{1})$ and another $x = 1$. Hence under the present case if $A_s - B_r > 0$ and $\psi(1-\varepsilon) < 0$, we have only zero rate of interest, negative rate of interest being inadmissible.

If $\psi(1-\varepsilon) > 0$, so that $\psi(1+\varepsilon) < 0$, 4(a) has no root in $(\overset{\times}{0}, \overset{\times}{1})$ and has one root $> 1$, and one $x = 1$, giving two rates of interest one zero and the other real and positive.

($\alpha_4$) Let us now suppose that the last but one instalment $A_{s-1}$ of the capital is invested simultaneously with the receipt or the due date of the first instalment $B_1$ of the repayment, so that (4a) reduces to

$$A_1 x^{l_r - p_1} + A_2 x^{l_r - p_2} + \ldots + A_{s-2} x^{l_r - p_{s-2}} + (A_{s-1} - B_1)\, x^{l_r - l_1} - \\ - B_2 x^{l_r - l_2} - \ldots - B_{r-1} x^{l_r - l_{r-1}} + A_s - B_r = 0.$$

This does not alter the number of changes of sign and the conclusion is the same as under ($\alpha_3$).

($\beta_2$) Greater overlapping. In absence of the conditions imposed in ($\alpha_3$) and ($\alpha_4$) all that we are certain of is that (4a) may or may not have more than one roots each $> 1$. Hence there may or may not be multiple rates of interest.

(III) Let the last instalment $A_s$ of capital be invested after the last instalment $B_r$ of repayment has been received or has become due, so that we have

$$A_1 x^{p_s - p_1} + A_2 x^{p_s - p_2} + \ldots + A_{s-1} x^{p_s - p_{s-1}} - B_1 x^{p_s - l_1} - B_2 x^{p_s - l_2} - \\ - \ldots - B_{r-1} x^{p_s - l_{r-1}} - B_r x^{p_s - l_r} + A_s = 0. \qquad (4b)$$

($\alpha_5$) Let us suppose that the last but one instalment $A_{s-1}$ of capital has been invested before the first instalment $B_1$ of repayment is received or becomes due, so that (4b) becomes arranged in descending powers of $x$ and has only two changes of sign, indicating there are only two real positive roots including $x = 1$. As before, it is easy to see that there are at most two non-negative rates of interest.

($\alpha_6$) If the last but one instalment $A_{s-1}$ of capital is invested simultaneously with the receipt or due date of the first instalment $B_1$ of repayment, (4b) reduces to

$$A_1 x^{p_s - p_1} + A_2 x^{p_s - p_2} + \ldots + A_{s-2} x^{p_s - p_{s-2}} + (A_{s-1} - B_1)\, x^{p_s - l_1} - \\ - B_2 x^{p_s - l_2} - \ldots - B_{r-1} x^{p_s - l_{r-1}} - B_r x^{p_s - l_r} + A_s = 0;$$

which does not alter the number of changes of sign, and therefore, the conclusion is the same as under $(\alpha_5)$.

$(\beta_3)$ If the conditions imposed in $(\alpha_5)$ and $(\alpha_6)$ are removed, all that we are certain of is that (4b) may or may not give us more than one real non-negative rates of interest.

Summary of the foregoing investigations.

We have thus arrived at the following generalisation:

If in a purely financial transaction the total of repayments is equal to the total capital, zero is always at least one of the rates of interest and in order that zero may be the only rate of interest involved in the transaction, it is sufficient (though by no means necessary) that

(1) the last instalment of capital has been invested before, or is invested simultaneously with, the receipt or due date of the first instalment of repayment; or

(2) the last but one instalment of capital has been invested before, or is invested simultaneously with, the receipt or due date of the first instalment of repayment, provided [in (2)] the last instalment of repayment is received or becomes due simultaneously with the investment of, and is greater than; the last instalment of capital.

But if either the last repayment has been received before the investment of last capital, or else if it is less than, and is received or becomes due with the investment of, the last instalment of capital, then besides zero rate of interest we may possibly have at most one real positive rate of interest.

Notwithstanding anything in the above investigation which apply both to the monocreditor and bicreditor transactions, if the transaction is a monocreditor one, zero is the only rate of interest (as is obvious from Lidstone's proof), no matter what the order of advances and repayments may be.

Sum of repayments less than that of capitals.

(a) Lump sum capital. The equation (1) of the first paper is

$$Ax^{l_r} - B_1x^{m_1} - B_2x^{m_2} - B_3x^{m_3} - \dots - B_r = 0 \qquad (1)$$

but now $A > B_1 + B_2 + B_3 + \dots + B_r$.

By Decartes' rule of signs (1) can have only one real positive root which is easily found to lie in (0, 1), making $1 + i < 1$ i. e. $i < 0$. If (1) has any negative roots, they also give rise to negative rates of interest, tremendous as they are in magnitude.

(b) Capital invested by instalments. The symbols have the same meaning as in the first paper, except that now

$$A_1 + A_2 + \dots + A_s > B_1 + B_2 + \dots + B_r.$$

(I) Last repayment after last investment. Equation (4) of the first paper is

$$A_1 x^{l_r - p_1} + A_2 x^{l_r - p_2} + \ldots + A_s x^{l_r - p_s} - B_1 x^{l_r - l_1} - B_2 x^{l_r - l_2} - \\ - \ldots - B_{r-1} x^{l_r - l_{r-1}} - B_r = 0. \qquad (4)$$

Denoting the left side by $\psi(x)$, we have $\varphi(0) < 0$, $\psi(1) > 0$, $\psi(\alpha) > 0$. There is, therefore, an odd number of roots in (0, 1) and either none or an even number of them each $> 1$. Also (4) may have negative roots.

Thus there is at least one real negative rote of interest and there may be multiple rates of interest both positive and negative.

($\alpha_1$) The non-overlapping of investments and repayments. If the last instalment $A_s$ of capital has been invested before the first instalment $B_1$ of repayment is received or becomes due, (4) becomes arranged in descending powers of $x$ and has only one change of sign. Hence (4) has only one real positive root which is found to lie in (0, 1) giving a real negative rate of interest. If further (4) has negative real roots, they also give rise to the negative rates of interest of a tremendous magnitude. Thus in the present case there are no positive real rates of interest and there is at least one real negative rate of interest.

($\alpha_2$) A particular overlapping. If the investment of the last instalment $A_S$ of capital occurs simultaneously with the receipt or due date of the first instalment $B_1$ of repayment, (4) becomes again arranged in descending powers of $x$ with one change of sign and with the same conclusion as under ($\alpha_1$).

($\beta_1$) If restrictions ($\alpha_1$) and ($\alpha_2$) are removed, there may or may not be real positive rates of interest. All that we are certain of is that there is at least one real negative rate of interest numerically less than 100% and either there are no real positive rates of interest or there is an even number of them. In this case it appears that, in spite of the total repayment being less than the total investment, there may exist two or an even number of real positive rates of interest, consistent with the degree of (4).

(II) Let the investment of the last instalment $A_s$ of capital occur simultaneously with the receipt or due date of the last instalment $B_r$ of repayment, so that (4) now becomes

$$A_1 x^{l_r - p_1} + A_2 x^{l_r - p_2} + \ldots + A_{s-1} x^{l_r - p_{s-1}} - B_1 x^{l_r - l_1} - B_2 x^{l_r - l_2} - \\ - \ldots - B_{r-1} x^{l_r - l_{r-1}} + A_s - B_r = 0. \qquad (4a)$$

Obviously the restrictions ($\alpha_1$) and ($\alpha_2$) cannot be imposed now.

Special cases.

($\alpha_3$) Let the last instalment, but one, $A_{s-1}$ of capital become invested before the first instalment $B_1$ of repayment is received or becomes due, so that (4a) becomes arranged in descending powers of $x$. In case $A_s - B_r \leqq 0$, there is only one change of sign and it is therefore easy to see that (4a) has only one real positive root in (0, 1) and no other real positive root $> 1$. Thus if $A_s \leqq B_r$ there is at least one negative real rate of interest and there is no real positive rate of interest.

If $A_s - B_r > 0$, there can be two real positive roots at most due to two changes of sign. Let the left side of (4a) be denoted by $\psi(x)$, so that $\psi(x) > 0$, $\psi(1) > 0$, $(\Sigma A > \Sigma B)$, and $\psi(\alpha) > 0$. Therefore either there are no real positive roots or there are two real positive roots. In case there is no real positive root, there is no real rate of interest greater than $-1$ (i. e. $-100\%$). And if $l_r - p_1$ is an even integer, $\psi(-\alpha) > 0$. And in that case either there are none or an even number of real negative roots.

Obviously then in the present case we may have no real rate of interest at all (positive, zero, or negative) but only imaginary or complex rates of interest which are certainly meaningless.

$(\alpha_4)$ Let the last but one instalment $A_{s-1}$ of capital be invested simultaneously with the receipt or due date of the first instalment $B_1$ of the repayment. This does nothing except make the last positive and the first negative terms of (4a) coalesce into one with out any change in the number of changes of sign. The conclusion is, therefore, the same as under $(\alpha_3)$.

$(\beta_2)$ Greater overlapping. In absence of conditions $(\alpha_3)$ and $(\alpha_4)$ all that we are certain of is that, if $A_s - B_r \leqq 0$, there is at least one real negative rate of interest and there are either none or an even number of real positive rates of interests; that, if $A_s - B_r < 0$, there may or may not be any real positive or negative rate of interest in case (4a) happens to be of an even degree, and there is at least one real negative rate of interest in case (4a) happens to be of an odd degree.

(III) Let the last instalment $A_s$ of the capital be invested after the last instalment $B_r$ of repayment has been received or has become due, so that (2) of the first paper reduces to

$$A_1 x^{p_s - p_1} + A_2 x^{p_s - p_2} + \ldots + A_{s-1} x^{p_s - p_{s-1}} - B_1 x^{p_s - l_1} - B_2 x^{p_s - l_2} - \\ - \ldots - B_{r-1} x^{p_s - l_{r-1}} - B_r x^{p_s - l_r} + A_s = 0. \qquad (4b)$$

$(\alpha_5)$ Let the last but one instalment $A_{s-1}$ of capital be invested before the first instalment $B_1$ of repayment is received or becomes due, so that (4b) becomes arranged in descending powers of $x$ and has only two changes of sign, indicating that (4b) can have either no, or two, real positive roots. Denoting the left side of (4b) by $\psi(x)$ we have $\psi(0) > 0$, $\varphi(1) > 0$, $\psi(+\alpha) > 0$ and $\psi(-\infty) \gtrless 0$ according as $p_S - p_1$ is even or odd.

Thus (4b) can have either two real positive roots in (0, 1) and none in $(1, \infty)$, or none in (0, 1) and two in $(1, \infty)$, or none in $(0, \infty)$. Also if $p_S - p_1$ is odd, (4b) has at least one real negative root and, if $p_S - p_1$, is even, (4b) may have no negative roots or an even number of them consistent with the degree of (4b).

Thus in this case there may be either two real positive rates of interest or none. Also if $p_S - p_1$ is even, there is either an even number of negative rates of interest or none, and if $p_S - p_1$ is odd there is at least one real negative rate of interest. So in the present case also we may have no real rate of interest at all positive, zero, or negative; all the rates obtainable from the equation of value may be imaginary or complex.

($\alpha_6$) If $A_{s-1}$ is invested simultaneously with the receipt or due date of $B_1$, the last positive and the first negative terms of (4b) coalesce into one giving rise to no additional change of sign, and therefore the conclusion remains the same as under ($\alpha_5$).

($\beta_3$) The removal of the conditions ($\alpha_5$) and ($\alpha_6$) can lead us to any number of real positive and for negative rates of interest consistent with the degree of the equation of value (4b).

It is to be noted that under ($\alpha_1$) and ($\alpha_2$) of the present section one and the same party remains creditor, still there may be more than one negative rates of interest. So the criterion proved by Mr. Lidstone is not applicable to the transactions where the total repayment is less than the total capital. The reason for this violation is made clear in the appendix.

Interminable series of advances and repayments.

Before supporting the theoretical conclusions by numerical examples we propose to investigate briefly the cases when one or both of the series of advances and repayments become interminable.

Only repayments interminable. Equation (8) of the first paper is

$$A_1 + A_2 z^{p_2-p_1} + A_3 z^{p_3-p_1} + \dots + A_s z^{p_s-p_1} - B_1 z^{l_1-p_1} - B_2 z^{l_2-p_1} - B_3 z^{l_3-p_1} - \dots - B_r z^{l_r-p_1} - \dots \text{ ad infinitum} = 0 \tag{8}$$

where $z = (1 + i)^{-h}$.

Total repayment equals total capital. We have $\Sigma A = \Sigma B$. Denoting the left side of (8) by $f(z)$, we have, $f(0) > 0$, $f(1) = 0$, and $f(+\infty) < 0$. Further if $f(1 - \varepsilon) < 0$, so that $f(1 + \varepsilon) > 0$, where Lt $\varepsilon = 0$, we have besides $z = 1$, an odd number of real positive roots in $\overset{\times}{(}0, 1)$ and an odd number in $\overset{\times}{(}1, \infty)$. But the odd number of roots each $> 1$ gives rise to negative rates of interest, which, being inadmissible under the present case, do not contribute to the multiplicity of the rate of interest. But the odd number of roots in $(0, \overset{\times}{1})$ giving rise to the real positive rates of interest does contribute to the multiplicity of the rate.

If $f(1 - \varepsilon) > 0$, so that $f(1 + \varepsilon) < 0$, there are either no real positive roots or an even number of them in $(0, \overset{\times}{1})$ and the same happens in $(\overset{\times}{1}, \infty)$. But the latter set does not contribute to the multiplicity. Thus zero is always a rate of interest in this case and in general we may have several real positive rates of interest.

If, however, the last instalment $A_S$ of capital has become invested before, or is invested simultaneously with, the first instalment $B_1$ of repayment, then (8) has only one change of sign and has, therefore, only one real positive root viz. $z = 1$ whence $i = 0$. Thus with this limitation zero is the only rate of interest.

*Total repayment less than total capital.* If $\Sigma A > \Sigma B$, we have $f(0) > 0$, $f(1) > 0$, and $f(\infty) < 0$; so that (8) has an odd number of real roots each $> 1$ and either none or an even number of them in $(0, 1)$. Non each of the roots $> 1$ giving rise to the negative rates of interest not meaningless in the present case, contributes to the multiplicity of the rate of interest. Also the roots in $(0, 1)$, if existant, give rise to an even number of real positive rates of interest. Notwithstanding anything in the real positive roots of (8), if it has negative real roots, they will also give rise to the negative real rates of interest.

In case $A_s$ has been invested before the receipt or due date of, or is invested simultaneously with, $B_1$, there is only one change of sign in (8) and therefore it has only one real positive root, which is easily found to be $> 1$. This means that there is only one real negative rate arising from positive roots and no real positive rate. But nothing in the hypothesis can prevent (8) from having negative real roots giving rise to negative real rates and therefore causing multiplicity in the rate of interest even in the absence of the overlapping of advances and repayments and therefore even when one and the same party remains creditor. *Thus with the present extension in the meaning of the rate of interest Dr. Steffensen's criterion proved by Mr. Lidstone is not applicable.*

*Series of Capitals alone interminable.* Now (8) is replaced by

$$A_1 + A_2 z^{p_2 - p_1} + \ldots + A_s z^{p_s - p_1} + \ldots \text{ ad infinitum}$$
$$- B_1 z^{l_1 - p_1} - B_2 z^{l_2 - p_1} - \ldots - B_r z^{l_r - p_1} = 0. \tag{8a}$$

*Total repayment equals total capital.* We have $f(0) > 0$, $f(1) = 0$, and $f(+\infty) > 0$. If, further, $f(1 - \varepsilon) > 0$, then $f(1 + \varepsilon) < 0$, where $\mathrm{Lt}\ \varepsilon = 0$. This means that (8a) has either no roots or an even number of them in $(0, \overset{\times}{1})$ and an odd number of real positive roots each $> 1$. Also $z = 1$ is always a root. Thus in this case besides the zero rate of interest (8a) may give an even number of real positive rates. If $f(1 - \varepsilon) < 0$, then $f(1 + \varepsilon) > 0$. In this case there is an odd number of real positive rates of interest.

If all the repayments were to be received or to become due between two successive advances, we should have only two changes of sign. And since $z = 1$ is a root, (8a) could have only one more real positive root $\lesseqgtr 1$. In this case besides zero rate of interest there may or may not be one positive real rate.

Total repayment less than total capital. Now we have $f(0) > 0$, $f(1) > 0$, and $f(+\infty) > 0$. Thus there may be none or an even number of real positive roots in (0, 1) and the same thing is true for roots each $> 1$. There may or may not be negative roots. In this case we may not have any real rates of interest positive, zero, or negative. On the other hand all the rates of interest may be imaginary or complex.

If all the repayments were to be received or to become due between two successive advances, (8a) would have either no or two real positive roots which may give two real positive rates or two real negative rates. Thus the general conclusion about the existence of real rates positive or negative is the same as before.

Both the series interminable. When the series of both the capitals and repayments become interminable, (8) becomes

$$\begin{aligned} &A_1 + A_2 z^{p_2 - p_1} + \ldots + A_s z^{p_s - p_1} + \ldots \text{ ad infinitum} \\ &- B_1 z^{l_1 - p_1} - B_2 z^{l_2 - p_1} - \ldots - B_r z^{l_r - p_1} - \ldots \text{ ad infinitum} = 0. \quad \text{(8b)} \end{aligned}$$

The equation has obviously an infinite number of changes of sign and therefore in this case there can be no maximum limit to the number of real positive roots, and also, in general, no limit to the number of real negative roots.

Total repayment equals total capital. We have $f(0) > 0$, $f(1) = 0$, $f(+\infty) \gtreqless 0$. Thus besides zero rate of interest, there may or may not be real positive rates.

Total repayment less than total capital. We have $f(0) > 0$, $f(1) > 0$, and $f(+\infty) \gtreqless 0$. In this case we may or may not have any real rates of interest positive or negative. All the rates may be imaginary or complex.

Numerical examples.

The symbolical treatment carried on till now makes, in general, the existence of the multiplicity in the rate of interest only of a probable character, and, I think, it is unable to segregate all the cases where multiplicity of a given order must occur. Following, therefore, the advice of Mr. W. Palin Elderton who always likes us to plunge the theoretical deductions into the cold water of numerical arithmetic, we shall now support the theoretical conclusions by means of numerical examples.

Total repayment greater than total capital.

Ex. 2. $A$ shares the business of $B$ by investing a capital of £ 1000 000 at the end of the year 1916. At the end of the year 1917 $A$'s capital including the profits grows to £ 3 500 000, out of which he withdraws £ 3 100 000. At the end of 1918 $A$'s capital including profits grows to £ 1 203 100 and $A$ invests an additional sum of £ 3 203 100. During the year 1919 the business sustains a heavy loss and at the end of 1919 $A$'s capital is reduced to £ 1 103 130. $A$ wants to withdraw the whole of his capital and have no concern in $B$'s business. $B$ agrees to sign a bill of exchange drawn by $A$ for the accumulated amount of £ 1 103 130 payable on December the 31st, 1924 the rate of interest being that actually realised by $A$ during his share in $B$'s business. For how much should $A$ draw the bill?

The amount of the bill should be £ 1 103 130 $(1 + i)^5$.

To determine $i$, the equation of value is

$$1\,000\,000r^3 - 3\,100\,000r^2 + 3\,203\,100r - 1\,103\,130 = 0 \qquad (5)$$

where $r = 1 + i$. The affairs at the end of 1917 may be interpreted as with drawing £ 3 500 000 and investing £ 400 000. Thus — $-3\,500\,000r^2 + 400\,000r^2 = -3\,100\,000r^2$, and the equation of value remains the same. Similarly at the end of 1918.

Solving (5), we get, $r = 1{,}02$, $1{,}03$, or $1{,}05$; giving $i = 0{,}02$, $0{,}03$, or $0{,}05$.

A must certainly have been perplexed while fixing the amount of the bill, and, if he had learnt that in a monocreditor transaction there could not have been more than one rate of interest, he would certainly have quarrelled with Mathematics; for in quite a different sense he remained creditor throughout the currency of the transaction. Whether one and the same party remained creditor throughout the currency of $A$'s share in $B$'s business in the sense in which we have used a monocreditor transaction can be decided only at the end of the term, as the advances and repayments were not fixed beforehand. But the knowledge of the bicreditor nature of the transaction does not solve the difficulty involved in the multiplicity of the rate of interest.

Total repayment equals total capital.

Ex. 3. $A$ advanced to $B$ £ 1, £ 2, £ 3 at the beginnings of the years 1, 2, and 3, and $B$ paid him back £ 6 at the end of the 3rd year. What is the rate of interest involved in the transaction?

The equation of value is $r^3 + 2r^2 + 3r - 6 = 0$, where $r = 1 + i$, solving, we have, $r = 1$, or $\frac{-3 \pm \sqrt{-15}}{2}$, or $i = 0$, or $\frac{-5 \pm \sqrt{-15}}{2}$, of these only zero is admissible.

Ex. 4. $A$ paid to $B$ £ 100, and £ 103 at the beginnings of the years 1 and 3 and $B$ paid him back £ 203 at the beginning of the second year. What is the rate of interest.

The equation of value is: $100r^2 - 203r + 103 = 0$, whence $r = 1$, or $1{,}03$ i. e. $i = 0$ or $0{,}03$.

A positive rate of interest has been obtained because repayment was made in excess in anticipation of the advance still to be made. The only difficulty is that we have twe rates not one.

Ex. 5. $A$ and $B$ settle a contract: $A$ has to advance to $B$ £ 10 000, and £ 31 006 at the beginnings of the years 1 and 3 respectively and $B$ has to return $A$ £ 30 500, and £ 10 506 at the beginnings of the years 2 and 4 respectively.

(a) $B$ could not pay back £ 10 506 at the beginning of the 4th year but is willing to clear the account at the end of the 4th year. How much should $A$ fairly accept from $B$ at the end of the 4th year?

(b) $A$ cannot pay $B$ £ 31 006 at the beginning of the 3rd year. He is willing to for go £ 10 506 to be paid by $B$ at the beginning of the 4th year. He wants to settle the account immediately. What fair advice will you give to both the parties?

The answers are: (a) £ 10 506 $(1 + i)$, (b) $A$ must pay $B$ £ $[30\,500 - 10\,000\,(1 + i)]\,(1 + i)$, where $i$ is the rate of interest involved in the contract. To determine, $i$ the equation of value is

$$10\,000r^3 - 30\,500r^2 + 31\,006r - 10\,506 = 0,$$

where $r = 1 + i$, solving, we get, $r = 1$, 1,02, or 1,03, giving $i = 0$, 0,02, or 0,03.

The mean of the three values of $i$ is 0,016. Thus according to the convention put down at the end of the first paper $i$ may be taken to be 0,02. But obviously this convention would not be applicable to Ex. 4, where both the rates must be equidistant from the mean, if a dispute should arise about the rate of interest in case of a default of payment by either parts.

Total Repayment less than total capital.

Ex. 6. $A$ advanced to $B$ £ 100 and £ 103, 0 S., 3 d. at the ends of the years 1 and 3 respectively and withdrew from $B$ £ 203 at the end of the 2nd year. What was the rate of interest involved?

Equation of value is: $100r^2 - 203r + 103{,}0125 = 0$, giving

$$r = \frac{205}{200} \text{ or } \frac{201}{200} \text{ i. e. } i = 2\tfrac{1}{2}\%, \text{ or } \tfrac{1}{2}\%.$$

Ex. 7. $A$ advanced to a business £ 1 000 000 at the end of the first year, withdrew £ 30 000 and £ 95 200 at the ends of the years 2 and 3 respectively. At the end of the 4th year he advanced £ 5 172 012. At the end of the 5th year the business was dissolved and $A$ received £ 2 059 176 only as his share of the then capital. What rate of interest did $A$ realise by this?

The equation of value is

$$1\,000\,000r^4 + 30\,000r^3 - 959\,200r^2 + 5\,172\,012r - 2\,059\,176 = 0,$$

giving, $r = -2$, 0,98, 1,02, or 1,03; i. e. $i = -3$, $-0{,}02$, 0,02, or 0,03; four rates of interest out of which two are real and positive though $A$ gets back for less than he advances.

On this occasion it is fit to call properly negative rates those which arise out of the positive real roots of the equation of value, and improperly negative rates those which arise out of the negative roots of the equation of value. Obviously a transaction cannot have only improperly negative rates and no positive and or properly negative ones, unless it consists of advances only and of no repayments.

Ex. 8. $A$ and $B$ settled a contract. $A$ was to pay $B$ £ 100 and £ 110 at the beginning of the first year and at the end of the 2nd year respectively and in return $B$ was to pay back $A$ £ 203 at the end of the first year. (a) $A$ did not pay £ 110 at the end of the 2nd year. He wants to pay $B$ the accumulated amount of £ 110 at the end of the 6th year after the payment has been due. What sum should $A$ fairly pay $B$? (b) $B$ did not pay $A$ £ 203 at the end of the first year. He is prepared to forgo £ 110 due from $A$ at the end of the 2nd year and wants to settle the account at the end of the 2nd year. What sum should $A$ fairly accept from $B$?

The answers are: (a) $110\,(1+i)^6$, (b) $203\,(1+i)-110$.

To determine $i$, the equation of value is $100r^2-203r+110=0$ giving, $r=\dfrac{203\pm\sqrt{-2791}}{200}$, i. e. $i=\dfrac{3\pm\sqrt{-2791}}{200}$, i. e. both the rates of interest are complex and therefore absurd. Now both the theoretical investigations and numerical examples in this paper have shown us that, in case the total repayment is less than the total capital, besides negative real rates of interest (which may theoretically be taken to be admissible since the repayment is less than the advance) we have positive real rates of interest as in Ex. 7, and sometimes only positive real rates of interest as in Ex. 6, and at others only imaginary or complex rates of interest as in Ex. 8.

Also it is obvious that unless the rates of interest be real and positive we cannot handle the evaluations such as (a) and (b) of Ex. 8.

It may certainly be questioned why $A$ and $B$ should be erratic enough to make contract in such a way as may give rise to no real positive rates of interest.

The answer is: — because neither of them could foresee a default of payment in time, because their knowledge of algebra could not go beyond the solution of linear equations at the time of making contract, and above all because none could prevent them. To call such an event, therefore, erratic is really to evade the difficulty and not to solve it.

In all probability the multiplicity and complexity in the rate of interest are not at all due to some hidden defect in transactions but in the assumption of a uniform rate of interest. This assumption has already been objected to*) in view of the fantastic results, when applied to the accumulation of even a small principal for a very long time.

Whether the objection is a valid one from this point of view or not is difficult to decide. But certainly the assumption of a uniform rate of interest is objectionable, because it sometimes leads us to complex or imaginary rates of interest and to no others as in Ex. 8.

The best way to meet the difficulty seems to assume the rate of interest to be a function of time and then to seek the solution for this function.

Let $i_t$ denote the effective rate of interest per unit per year at the end of $t$ years and $f(t)$ the accumulated amount of 1 at the end of $t$ years. Thus valuing the contract in Ex. 8 at the end of $(t+2)$ years from the beginning of the contract, we have,

$$100\, f(t+2) - 203 \,.\, \frac{f(t+2)}{f(1)} + 110 \,.\, \frac{f(t+2)}{f(2)} = 0. \tag{6}$$

Obviously $f(1)$ and $f(2)$ must be constants, so that the equation reduces to $E^2 f(t) = 0$, where $E^x f(t) = f(t+x)$, of which the only solution is $f(t) = 0$, which is certainly inadmissible in the present case, as $f(0) = 1$.

We shall, therefore, assume that the law of variation of the rate of interest operates on each sum from the time the payment or receipt of each sum is made or becomes due and not before. It is due to the lack of realising this assumption that we got an inadmissible solution as we must have done. For at the end of the first year although the rate of interest for £ 100 is $i_1$, yet for £ 203 it is $i_0$. Similarly at the end of the second year from the beginning of the contract the rate of interest for £ 100 is $i_2$, for £ 203 it is $i_1$, and for £ 110 it is $i_0$ and so on for any time onward.

Under this form of the variation of rate of interest, (6) is replaced by

$$100\, f(t+2) - 203\, f(t+1) + 110\, f(t) = 0,$$

i. e.

$$(100E^2 - 203E + 110)\, f(t) = 0.$$

The subsidiary equation is $100m^2 - 203m + 110 = 0$, giving

$$m = \frac{203 \pm \sqrt{-1}\sqrt{2791}}{200},$$

$$\therefore\ f(t) \equiv A_1 \left(\frac{203 + \sqrt{-1}\sqrt{2791}}{200}\right)^t + B_1 \left(\frac{203 - \sqrt{-1}\sqrt{2791}}{200}\right)^t$$

where $A_1$ and $B_1$ are arbitrary constants.

*) By Todhunter and others.

Putting $\cos\Theta = \dfrac{203}{\sqrt{44\,000}}$, $\sin\Theta = \dfrac{\sqrt{2791}}{\sqrt{44\,000}}$, it is easy to see after the application of De Moivre's theorem in Trigonometry, that $f(t) \equiv (1{,}1)^{\frac{1}{2}t}(A\cos t\Theta + B\sin t\Theta)$, where $A$ and $B$ are real arbitrary constants

$$.\because\; i_t = \frac{1}{f(t)} \cdot \frac{df(t)}{dt} =$$

$$= \frac{[A\log_e\sqrt{1{,}1} + B\Theta]\cos t\Theta + [B\log_e\sqrt{1{,}1} - A\Theta]\sin t\Theta}{A\cos t\Theta + B\sin t\Theta}.$$

Putting $t = 0$, we have, $f(0) = A$, and $i_0 = \dfrac{A\log_e\sqrt{1{,}1} + B\Theta}{A}$,

$$\text{whence } B = \frac{f(0)}{\Theta}[i_0 - \log_e\sqrt{1{,}1}].$$

Obviously $f(0) = 1$, but the data in the problem are insufficient to determine $i_0$. In case $i_0$ be known some how or other, we have,

$$A = 1, \text{ and } B = \frac{i_0 - \log_e\sqrt{1{,}1}}{\Theta}.$$

$$.\because\; f(t) \equiv (1{,}1)^{\frac{1}{2}t}\left[\cos t\Theta + \frac{i_0 - \frac{1}{2}\log_e\sqrt{1{,}1}}{\Theta}\sin t\Theta\right],$$

where $\Theta$ is measured in radians, as in order to evaluate $B$ we had to use differentiation. Thus

$$f(1) = \sqrt{1{,}1}\left[\frac{203}{\sqrt{44\,000}} + \frac{i_0 - \frac{1}{2}\log_e 1{,}1}{\Theta} \cdot \sqrt{\frac{2791}{44\,000}}\right],$$

which is positive even if $i_0 \lesseqgtr 0$.

Also

$$f(6) = (1{,}1)^3\left[\cos 6\Theta + \frac{i_0 - \frac{1}{2}\log 1{,}1}{\Theta}\sin 6\Theta\right],$$

which seems to be negative if $i_0 = 0$.

Obviously the solution by finite difference equation is a remedy against the multiplicity of the rate of interest only to some extent but entirely against its complexity, as the difference equation, corresponding to the algebraic equation of value, being linear has one and only one general solution i. e. one and only one primitive. But the law of the variation of rate of interest found in this way has the following disadvantages: —

(1) Obviously the order of the finite difference equation would, in general, be the same as the degree of the corresponding algebraic equation of value, so that the greater the period of currency of a tran-

saction, the more numerous the arbitrary constants to be evaluated in primitive; and, in general, the conditions in the problem may be insufficient for this evaluation.

(2) The solution may lead to negative accumulated amounts or to those less than the corresponding principals.

The only way to meet (1) is to know a sufficient number of initial, boundary, or any other, conditions to determine the constants in the primitive, e. g., the values of $i_t$ and its successive differential coefficients for $t = 0$; and the way to meet (2) is to choose the given conditions in such a way as just not to make the accumulated amount negative or less than the corresponding principal as the case may be.

Hence the finite difference equation method is quite a good device to remove the complex or imaginary nature of the rate of interest but entirely breaks down when applied to the removal of multiple rates, as to determine the constants the initial conditions can be chosen in a variety of ways, giving rise to another multiplicity, though suppressing that of rates, quite unique as the primitive undoubtedly is. The method may, therefore, be used only when the equation of value leads either to now to two positive real rates of interest. In other cases the remedy suggested at the end of the first paper may be used.

The remedies suggested here are undoubtedly more than arbitrary, but there seems to be no help.

Application of Integral Equations may give a more satisfactory result, but, partly from want of leisure and partly from keeping the size of the paper in view, no attempt is made here by the auther to solve the indicial Integral Equations formed in the present case to draw the attention of Mathematicians, or even to investigate whether the solution exists at all.

The equation of value in Ex. 8 can be written as

$$100\,f(2) - 203\,f(1) + 110 = 0. \tag{7}$$

It is shown in works on mathematical theory of finance that

$$f(t) = e^{\int_0^t i_t dt}.$$

:. (7) becomes
$$100e^{\int_0^2 i_t dt} - 203e^{\int_0^1 i_t dt} + 110 = 0. \tag{9}$$

Or, if we prefer,

$$100e^{\int_0^2 i_t dt} - 203e^{\int_1^2 i_t dt} + 110 = 0, \tag{10}$$

or
$$100e^{\int_0^2 i_t dt} - 203 \cdot \frac{e^{\int_0^2 i_t dt}}{e^{\int_0^1 i_t dt}} + 110 = 0. \tag{11}$$

Of these (10) and (11) are identical as they are based on the assumption which gave $f(t) = 0$ as the solution of the finite difference equation, while (9) is distinct from them as it is based on the other assumption in the formation of the finite difference equation.

These should be solved for $i_t$ as a function of $t$ and we should then see which gives the most satisfactory results. The general problem may be tackled afterwards by means of Integral Equations.

In concluding the biparty and bicreditor portion of this paper I should like to emphasize again that the difficulty of the rate of interest as regards multiplicity and as regards absurdity in its being imaginary andfor complex (and also its being zero or negative considered from a practical point of view) should not be evaded by feeling that bicreditor transactions are rare.

These results may be quite uncomfortable, as Dr. Steffensen seems to think, since the supports of Messrs. Courcouf and Carter, already referred to, tell us that multiplicity of rate does actually arise in practice. But in the words of Sir Oliver Lodge, if we are wise, we shall not let any truth be suppressed, uncomfortable as it may seem at first.

The methods developed in this paper may lead, in suitable hands, to the evolution of some methods for forecasting the future rate of interest almost with Mathematical exactness in the long run.

The complete and satisfactory solution of new difficulties can be left to Pure Mathematics and its progress.

Polyparty and Polycreditor transactions.

Till now we have confined our attention to the transactions involving only two parties and there fore two creditors at most. We now propose to consider (only in outline) the transaction which involve more than two parties and may or may not be polycreditor.

Such transactions can very well be realised in practice. E. G· A person may have his accounts in more than one bank, or more than one account in the same bank, or both; and withdrawals, deposits, overdrawings, transfers from one account into another and from one bank into another may take place according to the financial position and necessities of the depositor.

What has till now been called the equation of value may now more fitly be called the equation of equilibrium.

Triparty and tricreditor transaction.

To simplify the investigation we shall deal with a three party transaction, and the conclusions arrived at may be taken to be general as the method is applicable to any number of parties.

Let $A$, $B$, and $C$ be the three parties, and let the payment made by $A$ to $B$ at the end of $t_i$ years be denoted by $A_{t_iB}$ and so on.

Thus the equation of euqilibrium between $A$ and $B$ is

$$A_{t_1B}v^{t_1} + A_{t_2B}v^{t_2} + \ldots + A_{t_sB}v^{t_s} - B_{n_1A}v^{n_1} - \ldots - B_{n_rA}v^{n_r} = 0. \quad (8c)$$

That between $B$ and $C$ is

$$B_{g_1C}v^{g_1} + \ldots + B_{g_kC}v^{g_k} - C_{h_1B}v^{h_1} - \ldots - C_{h_lB}v^{h_l} = 0. \quad (9a)$$

That between $C$ and $A$ is

$$C_{c_1A}v^{c_1} + \ldots + C_{c_mA}v^{c_m} - A_{d_1C}v^{d_1} - \ldots - A_{d_nC}v^{d_n} = 0. \quad (10a)$$

That between $A$ and others is

$$A_{t_1B}v^{t_1} + \ldots + A_{t_sB}v^{t_s} + A_{d_1C}v^{d_1} + \ldots + A_{d_nC}v^{d_n} - B_{n_1A}v^{n_1} - \\ - \ldots - B_{n_rA}v^{n_r} - C_{c_1A}v^{c_1} - \ldots - C_{c_mA}v^{c_m} = 0. \quad (11a)$$

That between $B$ and others is

$$B_{n_1A}v^{n_1} + B_{n_2A}v^{n_2} + \ldots + B_{n_rA}v^{n_r} + B_{g_1C}v^{g_1} + \ldots + B_{g_kC}v^{g_k} - \\ A_{t_1B}v^{t_1} - \ldots - A_{t_sB}v^{t_s} - C_{h_1B}v^{h_1} - \ldots - C_{h_lB}v^{h_l} = 0. \quad (12)$$

And finally that between $C$ and others is

$$C_{h_1B}v^{h_1} + \ldots + C_{h_lB}v^{h_l} + C_{c_1A}v^{c_1} + \ldots + C_{c_mA}v^{c_m} - A_{d_1C}v^{d_1} - \\ - \ldots - A_{d_nC}v^{d_n} - B_{g_1C}v^{g_1} - \ldots - B_{g_kC}v^{g_k} = 0. \quad (13)$$

Confining the attention to the rational measures of time, we have clearly no loss of generality in assuming the powers of to be positive integers; for a year is only an arbitrary unit of time which can be replaced by the Highest Common Factor of any number of rational measures of time.

The rate of interest between every two parties and that realised by each party on the whole cannot be the same, unless the equations (8c) . . . (12) have a common root which is not meaningless for the rate of interest.

Obviously these equations may not have a common root necessarily except $v = 0$, giving $i = \infty$. So it is not necessary that every party realises the same rate of interest. Nor can the application of Difference and Integral equations necessarily make each party realise the same rate of interest, for it is not necessary for either the difference or Integral equations both corresponding to (8c), . . ., (13) to have the same solution or have such solutions as may have at least one common factor not meaningless for the rate.

We have six algebraic equations and, in order that they may have at least one common root after removing the zero roots, their coefficients must satisfy five independent simultaneous eliminant equations which can be formed in any given case. And since the number of coefficients is obviously not less than five the eliminant equations are not inconsistent for some particular values of the coefficients in (8c), . . ., (13). If further the eliminant equations have positive real solutions (which in all probability are infinite in number in this case) the sums of money involved in (8c), . . ., (13) are quite possible. But even in this particular case there is no guarantee that the common root or roots will not be meaningless for the rate of interest in every case when the above conditions are satisfied.

Application of Difference and Integral equations may take us some step further, but we must admit that the problem of polyparty and polycreditor transactions, in general, in determinate as regards an all round rate of interest i. e. an all round rate may or may not exist.

We are aware that the problem of three or more bodies is yet in determinate in Astronomy, of course, in quite a different sense. Now if in animate bodies can well evade the revelation of their indeterminateness, much more can the transactions which greater or less depend on human free will guided by financial circumatances.

We may still await the progress of the various branches of Science for a complete and satisfactory solution.

## Appendix.

In view of the fact that the proof advanced by Dr. Steffensen for the rate of interest being not more than one in a monocreditor transaction is defective and that Mr. Lidstone's proof, quite correct as it is, is applicable only to a terminable series of payments, the author proposes to advance a proof free from flaws and to state the criterion in lucid language.

Since the author can prove this criterion only on the basis of two theorems in the theory of algebraic equations and since these two theorems do not seem alredy to exist to the best of author's enquiry and have been investigated by the present writer, it is quite proper to discuss them first.

Lemma I. If a rational integral algebraic equation of any degree finite or infinite has a positive real root $\alpha$ and if, when $\alpha$ is substituted for the unknown, the expressions obtained by omitting, from the left side of the equation, successively the first term, the first two terms, the first three terms, and so on, are of the same sign except that some, but not all, of these expressions may be zero, the equation can have no real positive root other than $\alpha$.

(To be continued.)