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Susil Kumar Jena

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Abstract. The two related Diophantine equations: $X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8$ and $Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4$, have infinitely many nontrivial, primitive integral solutions. We give two parametric solutions, one for each of these equations.

1 Introduction

In this note, we study the two related Diophantine equations

$$X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8 \tag{1}$$

and

$$Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4. \tag{2}$$

It seems that no parametric solutions are known for (1). Choudhry [1] has found parametric solutions of a similar equation

$$A^4 + 4B^4 = C^4 + 4D^4$$

involving only fourth powers. Though, the parametric solution of (2) is already known which is based on the identity

$$(p^4 + 2q^4)^4 = (p^4 - 2q^4)^4 + (2p^3q)^4 + 4(2pq^3)^4,$$

we give a new parametric solution of (2). For a historical background and references of these equations, and similar Diophantine problems on fourth powers, we refer to Guy ([3], pp. 215–218) and Dickson ([2], pp. 647–648). The parameterisations of (1) and (2) are based on a result from our paper [4] in which we proved the following theorem:

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Key words: Diophantine equation $A^4 + nB^4 = C^2$, Diophantine equation $A^4 - nB^4 = C^2$, Diophantine equation $X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8$, Diophantine equation $Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4$

Theorem 1. *(Jena, [4]) For any integer n, if (A_t, B_t, C_t) is a solution of the Diophantine equation*

$$A^4 + nB^4 = C^2 \tag{3}$$

with A, B, C as integers, then $(A_{t+1}, B_{t+1}, C_{t+1})$ is also the solution of the same equation such that

$$(A_{t+1}, B_{t+1}, C_{t+1}) = \{(A_t^4 - nB_t^4), (2A_tB_tC_t), (A_t^8 + 6nA_t^4B_t^4 + n^2B_t^8)\} \tag{4}$$

and if A_t, nB_t, C_t are pairwise coprime and A_t, nB_t are of opposite parity, then $A_{t+1}, nB_{t+1}, C_{t+1}$ will also be pairwise coprime and A_{t+1}, nB_{t+1} will be of opposite parity with $A_{t+1}, B_{t+1}, C_{t+1}$ as an odd, even, odd integer respectively.

Changing n to $-n$ at appropriate places in Theorem 1, we get its equivalent theorem:

Theorem 2. *For any integer n, if (A_t, B_t, C_t) is a solution of the Diophantine equation*

$$A^4 - nB^4 = C^2 \tag{5}$$

with A, B, C as integers, then $(A_{t+1}, B_{t+1}, C_{t+1})$ is also the solution of the same equation such that

$$(A_{t+1}, B_{t+1}, C_{t+1}) = \{(A_t^4 + nB_t^4), (2A_tB_tC_t), (A_t^8 - 6nA_t^4B_t^4 + n^2B_t^8)\} \tag{6}$$

and if A_t, nB_t, C_t are pairwise coprime and A_t, nB_t are of opposite parity, then $A_{t+1}, nB_{t+1}, C_{t+1}$ will also be pairwise coprime and A_{t+1}, nB_{t+1} will be of opposite parity with $A_{t+1}, B_{t+1}, C_{t+1}$ as an odd, even, odd integer respectively.

Theorem 1 and Theorem 2 are based on two equivalent polynomial identities

$$(a-b)^4 + 16ab(a+b)^2 = (a^2 + 6ab + b^2)^2;$$

and

$$(a+b)^4 - 16ab(a-b)^2 = (a^2 - 6ab + b^2)^2,$$

which can be used to parameterise (3) and (5) respectively.

2 Core Results

The following lemma will be used for obtaining the main results of this paper.

Lemma 1. *The Diophantine equation*

$$c^4 - 2d^4 = t^2 \tag{7}$$

has infinitely many non-zero, coprime integral solutions for (c, d, t).

Proof. The integral solutions of (7) are generated by using Theorem 2. If we take the initial solution of (7) as $(c_1, d_1, t_1) = (3, 2, 7)$, then from (6) we get the next solution

$$(c_2, d_2, t_2) = \left\{(c_1^4 + 2d_1^4), (2c_1d_1t_1), (c_1^8 - 6 \times 2c_1^4d_1^4 + 2^2d_1^8)\right\} = (113, 84, -7967).$$

We take $(c_2, d_2, t_2) = (113, 84, 7967)$ as c, d and t are raised to even powers in (7). Note that $(c_1, 2d_1, t_1) = (3, 4, 7)$ are pairwise coprime, $c_1 = 3$ is odd, and $2d_1 = 4$ is even. So, according to Theorem 2 we expect $(c_2, 2d_2, t_2)$ to be pairwise coprime, and c_2 and $2d_2$ to be of opposite parity. In fact, our expectation is true as $(c_2, 2d_2, t_2) = (113, 168, 7967)$ are pairwise coprime, $c_2 = 113$ is odd, and $2d_2 = 168$ is even. Thus, (7) has infinitely many non-zero and coprime integral solutions. □

It is easy to verify the two polynomial identities

$$(a + b)^4 - (a - b)^4 = 8ab(a^2 + b^2); \tag{8}$$

and

$$(c^4 - 2d^4)^2 + 4(cd)^4 = c^8 + 4d^8 \tag{9}$$

by direct computation.

Put $c^4 - 2d^4 = t^2$ from (7) in (9) to get

$$t^4 + 4(cd)^4 = c^8 + 4d^8. \tag{10}$$

Putting $a = c^4$ and $b = 2d^4$ in (8) we get

$$\begin{aligned}
&(c^4 + 2d^4)^4 - (c^4 - 2d^4)^4 = 16c^4d^4(c^8 + 4d^8);\\
\Rightarrow &(c^4 + 2d^4)^4 = (c^4 - 2d^4)^4 + (2cd)^4\{t^4 + 4(cd)^4\}; \quad \text{[from (10)]}\\
\Rightarrow &(c^4 + 2d^4)^4 = (c^4 - 2d^4)^4 + (2cdt)^4 + 4(2c^2d^2)^4. \qquad (11)
\end{aligned}$$

2.1 Diophantine equation $X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8$

Theorem 3. *The Diophantine equation*

$$X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8 \tag{12}$$

has infinitely many nontrivial, primitive integral solutions for (X_1, X_2, X_3, X_4). *To get the primitive solutions of* (12), *we assume that* $\gcd(X_1, X_2, X_3, X_4) = 1$.

Proof. In accordance with Lemma 1, we have infinitely many integral values of c, d, t with $c^4 - 2d^4 = t^2$ and $\gcd(c, 2d) = 1$ for which (10) has solutions. Comparing (12) with (10) we get $(X_1, X_2, X_3, X_4) = (t, cd, c, d)$. Since $X_3 = c$, $X_4 = d$ and $\gcd(c, 2d) = 1$, we get $\gcd(X_3, X_4) = 1$, and hence, $\gcd(X_1, X_2, X_3, X_4) = 1$. So, (12) has infinitely many nontrivial, primitive integral solutions for (X_1, X_2, X_3, X_4). □

Example 1.

$$\begin{aligned}
(c_1, d_1, t_1) &= (3, 2, 7) : (X_{1_1}, X_{2_1}, X_{3_1}, X_{4_1}) = (t_1, c_1 d_1, c_1, d_1) = (7, 6, 3, 2);\\
&\Rightarrow 7^4 + 4 \times 6^4 = 3^8 + 4 \times 2^8.\\
(c_2, d_2, t_2) &= (113, 84, 7967) : (X_{1_2}, X_{2_2}, X_{3_2}, X_{4_2})\\
&= (t_2, c_2 d_2, c_2, d_2) = (7967, 9492, 113, 84);\\
&\Rightarrow 7967^4 + 4 \times 9492^4 = 113^8 + 4 \times 84^8.
\end{aligned}$$

2.2 Diophantine equation $Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4$

Theorem 4. *The Diophantine equation*

$$Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4 \tag{13}$$

has infinitely many nontrivial, primitive integral solutions for (Y_1, Y_2, Y_3, Y_4). *To get the primitive solutions of* (13), *we assume that* $(Y_1, Y_2, Y_3, Y_4) = 1$.

Proof. Using Lemma 1, we get infinitely many integral values of c, d, t such that $c^4 - 2d^4 = t^2$ and $\gcd(c, 2d) = 1$ for which (11) is satisfied. Comparing (13) with (11) we get

$$(Y_1, Y_2, Y_3, Y_4) = \big\{(c^4 + 2d^4), (c^4 - 2d^4), 2cdt, 2c^2d^2\big\}.$$

Since $\gcd(c, 2d) = 1$, we have

$$\gcd\big((c^4 + 2d^4), (c^4 - 2d^4)\big) = 1.$$

Thus, $\gcd(Y_1, Y_2) = 1$; or, $\gcd(Y_1, Y_2, Y_3, Y_4) = 1$. So, (13) has infinitely many nontrivial, primitive integral solutions for (Y_1, Y_2, Y_3, Y_4). □

Example 2.

$$\begin{aligned}
(c_1, d_1, t_1) &= (3, 2, 7) :\\
(Y_{1_1}, Y_{2_1}, Y_{3_1}, Y_{4_1}) &= \big\{(c_1^4 + 2d_1^4), (c_1^4 - 2d_1^4), 2c_1 d_1 t_1, 2c_1^2 d_1^2\big\} = (113, 49, 84, 72);\\
&= \big\{(c_1^4 + 2d_1^4), t_1^2, 2c_1 d_1 t_1, 2c_1^2 d_1^2\big\} = (113, 7^2, 84, 72).\\
\Rightarrow 113^4 &= 49^4 + 84^4 + 4 \times 72^4 = 7^8 + 84^4 + 4 \times 72^4.
\end{aligned}$$

$$\begin{aligned}
(c_2, d_2, t_2) &= (113, 84, 7967) :\\
(Y_{1_2}, Y_{2_2}, Y_{3_2}, Y_{4_2}) &= \big\{(c_2^4 + 2d_2^4), (c_2^4 - 2d_2^4), 2c_2 d_2 t_2, 2c_2^2 d_2^2\big\}\\
&= (262621633, 63473089, 151245528, 180196128);\\
&= \big\{(c_2^4 + 2d_2^4), t_2^2, 2c_2 d_2 t_2, 2c_2^2 d_2^2\big\}\\
&= (262621633, 7967^2, 151245528, 180196128).\\
\Rightarrow 262621633^4 &= 63473089^4 + 151245528^4 + 4 \times 180196128^4;\\
&= 7967^8 + 151245528^4 + 4 \times 180196128^4.
\end{aligned}$$

3 Conclusion

We make no attempt of giving the complete parametric solutions to the two Diophantine equations of the title. There might exist some singular solutions. It is expected that the prospective scholars will continue further exploration to find the complete solutions of these two equations.

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Author's address:
(Former Professor of Electronics and Telecommunication Engineering)
Plot No-170, Sector-5, Niladri Vihar, Chandrasekharpur, Bhubaneswar-751021, Odisha, India

E-mail: susil_kumar@yahoo.co.uk

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