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SOLITON SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATION WITH CRITICAL EXPONENTIAL GROWTH IN $\mathbb{R}^N$

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Abstract. In this work, we study the existence of nonnegative and nontrivial solutions for the quasilinear Schrödinger equation

$$-\Delta_N u + b|u|^{N-2}u - \Delta_N(u^2)u = h(u), \quad x \in \mathbb{R}^N,$$

where Δ_N is the N-Laplacian operator, $h(u)$ is continuous and behaves as $\exp(\alpha|u|^{N/(N-1)})$ when $|u| \to \infty$. Using the Nehari manifold method and the Schwarz symmetrization with some special techniques, the existence of a nonnegative and nontrivial solution $u(x) \in W^{1,N}(\mathbb{R}^N)$ with $u(x) \to 0$ as $|x| \to \infty$ is established.

Keywords: N-Laplacian equation; critical exponential growth; Schwarz symmetrization; Nehari manifold

MSC 2010: 35D30, 35J20, 35J92

1. Introduction and main result

In this paper, we study the existence of solutions for a class of quasilinear Schrödinger equations of the form

$$-\Delta_N u + b|u|^{N-2}u - \Delta_N(u^2)u = h(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ $(N \geqslant 2)$ is the N-Laplacian operator, $b > 0$ is a constant and $h(u)$ is a continuous function having critical exponential growth.

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There has been recently a good amount of work on the quasilinear elliptic equations of the form

$$-\Delta u + V(x)u - \Delta(u^2)u = h(u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Such equations arise in various branches of mathematical physics and they have been the subject of extensive study in recent years. Part of the interest is due to the fact that solutions of (1.2) are related to the existence of solitary wave solutions for Schrödinger equations of the form

$$\mathrm{i}z_t = -\Delta z + W(x)z - h_1(|z|^2)z - k\Delta g(|z|^2)g'(|z|^2)z, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \tag{1.3}$$

where $z\colon \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W\colon \mathbb{R}^N \to \mathbb{R}$ is a given potential, k is a positive constant, g, h_1 are real functions.

Schrödinger equations of the form (1.3) appear naturally in mathematical physics and have been derived as models of several physical phenomena corresponding to various types of the nonlinear term g. The case $g(s) = s$ was used for the superfluid film equation in plasma physics by Kurihura in [18] (see also [19]). In the case $g(s) = (1+s)^{1/2}$, equation (1.3) models the self-channeling of a high-power ultra short laser in matter, see [7]. Equation (1.3) also appears in plasma physics and fluid mechanics, see [25], in mechanics, see [17], and in condensed matter theory, see [24].

Here we consider the case where $g(s) = s$, $k = 1$ and our special interest is in the existence of standing wave solutions, that is, solutions of type $z(t,x) = \exp(-\mathrm{i}\omega t)u(x)$, where $\omega \in \mathbb{R}$ and $u > 0$ is a real function. It is obvious that z satisfies (1.3) if and only if $u(x)$ solves the equation of elliptic type (1.2) with $h(u) = h_1(u^2)u$, where $V(x) = W(x) - \omega$ is the new potential.

Motivated by (1.2), Severo in [28] studied the equation

$$-\Delta_p u + V(x)|u|^{p-2}u - \Delta_p(u^2)u = h(u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where h has subcritical growth, that is, $|h(t)| \leqslant c(1+|t|^r)$, $(t \in \mathbb{R})$ with $2p-1 < r < 2p^* - 1$ if $1 < p < N$ and $r > 2p$ if $p = N$. Using Lion's compactness lemma (see [21]) and mountain pass lemma, the author proved the existence of a nontrivial solution in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ when $1 < p \leqslant N$. We also refer to the recent work of Wang et al. in [29], where the authors studied the problem (1.4) with $p = N$ and $h, V(x)$ satisfying the following conditions:

(H_1) $h(t) \in C(\mathbb{R})$ and $h(t) = o(|t|^{N-2}t)$ at the origin.

(H_2) There exists $\mu > 2N$ such that

$$0 < \mu H(t) \leqslant th(t) \quad \forall\, t > 0, \text{ where } H(t) = \int_0^t h(s)\,\mathrm{d}s.$$

(H$_3$) There exists $\alpha_0 > 0$ such that

$$\lim_{t\to\infty}(|h(t)|\exp(-\alpha|t|^{2N/(N-1)})) = \begin{cases} 0 & \text{if } \alpha > \alpha_0, \\ \infty & \text{if } \alpha < \alpha_0. \end{cases}$$

(H$_4$) Let α_0 be the constant in (H$_3$). Then there exists $\beta_0 > 0$ such that

$$\liminf_{t\to\infty}\big(th(t)\exp(-\alpha_0|t|^{2N/(N-1)})\big) \geqslant \beta_0 > 0.$$

(H$_5$) $V(x) \in C(\mathbb{R}^N)$ and there exists $V_0 > 0$ such that $V(x) \geqslant V_0$ in $\mathbb{R}^N$ and

$$\lim_{|x|\to\infty} V(x) = V_\infty < \infty, \quad \text{and} \quad V(x) \neq V_\infty \quad \forall\, x \in \mathbb{R}^N.$$

They used a similar argument as in [12], [21], [22] and obtained the following result.

Theorem 1.1. *Assume that* (H$_1$)–(H$_5$) *hold. Then problem* (1.4) *with* $p = N$ *possesses a nontrivial solution in* $W^{1,N}(\mathbb{R}^N)$.

Remark 1.2. It is easy to see that the function $h(t) = \lambda t|t|^{m-2}$ satisfies (H$_1$)–(H$_2$), but fails to verify (H$_3$)–(H$_4$), where $\lambda > 0$ and $m > 2N$.

The present paper is motivated by [28], [29], we are interested in the existence of nonnegative and nontrivial solutions to equation (1.4) with $V(x) = b > 0$ and $p = N$. The assumptions (H$_3$)–(H$_5$) are not necessary for our consideration. Our argument is new and different from that in [15], [23], [26], [28], [29]. To establish the existence of a nonnegative and nontrivial solution for (1.1), we will use the Nehari manifold method and Schwarz symmetrization as in [10], [11] with some special techniques.

Throughout this paper, let $E = W^{1,N}(\mathbb{R}^N)$. Since we are looking for a nonnegative and nontrivial solution, we suppose that $h(s) = 0$ in $(-\infty, 0]$ and satisfies the following hypotheses.

(A$_1$) The function $h(t) \in C^1(\mathbb{R}), h(t) > 0$ in $(0,\infty)$ and there exist the constants $b_1, \alpha_0 > 0$ and $q > 0$ such that

$$|h(t)| \leqslant b_1|t|^{q-1}R(\alpha_0, t) \quad \forall\, t \in \mathbb{R}, \tag{1.5}$$

where

$$R(\alpha_0, t) = \exp(\alpha_0|t|^{2N/(N-1)}) - S_{N-2}(\alpha_0, t), \tag{1.6}$$

$$S_{N-2}(\alpha_0, t) = \sum_{k=0}^{N-2}\frac{\alpha_0^k}{k!}|t|^{2kN/(N-1)}.$$

(A$_2$) The function $h(t)$ also satisfies

$$(q-1+2N)\frac{h(t)}{t} \leqslant h'(t), \quad (q+2N)H(t) \leqslant th(t) \quad \forall\, t \in \mathbb{R}\setminus\{0\}, \tag{1.7}$$

where $H(t)=\int_0^t h(s)\,\mathrm{d}s$.

Remark 1.3. Obviously, the function

$$h(t)=\lambda t|t|^{m-2}\exp(\alpha_0|t|^{2N/(N-1)}), \quad t\geqslant 0,\ h(t)=0,\ t\leqslant 0, \tag{1.8}$$

satisfies (A$_1$)–(A$_2$), where the constants $\lambda>0$, $\alpha_0\geqslant 0$, and $m>2N$.

The main result in this paper is as follows.

Theorem 1.4. *Assume* (A$_1$)–(A$_2$). *Then, the problem* (1.1) *admits at least one nonnegative and nontrivial solution* $u_0(x)\in E=W^{1,N}(\mathbb{R}^N)$ *with* $u_0(x)\to 0$ *as* $|x|\to\infty$.

This paper is organized as follows. In Section 2, we set up the variational framework and prove some lemmas which will be used in the proof of Theorem 1.4. The proof of the main result is given in Section 3.

2. Preliminaries

Let Ω be a open subset of $\mathbb{R}^N$. We denote by $L^p(\Omega)$ $(p\geqslant 1)$ the usual Lebesgue spaces with the norm $\|u\|_p\equiv\|u\|_{L^p(\Omega)}=(\int_\Omega |u|^p\,\mathrm{d}x)^{1/p}$. Let $W^{1,p}(\Omega)(W_0^{1,p}(\Omega))$ be the usual Sobolev spaces with the norm

$$\|u\|_{W^{1,p}(\Omega)}=\left(\int_\Omega(|\nabla u|^p+|u|^p)\,\mathrm{d}x\right)^{1/p}. \tag{2.1}$$

In this paper, we will use the following lemmas.

Lemma 2.1 ([1], [5], [16]). *Let* Ω *be a open subset of* $\mathbb{R}^N$ *with a Lipschitz continuous boundary* $\partial\Omega$. *Then,*

(i) *the embedding* $W^{1,N}(\Omega)(W_0^{1,N}(\Omega))\hookrightarrow L^q(\Omega)$ *is continuous for* $q\in[N,\infty)$,

(ii) *if* Ω *is bounded, the embedding* $W^{1,N}(\Omega)(W_0^{1,N}(\Omega))\hookrightarrow L^q(\Omega)$ *is compact for* $q\in[N,\infty)$.

In particular, for $u\in W_0^{1,N}(\Omega)$,

$$\|u\|_N\leqslant(\omega_N^{-1}|\Omega|)^{1/N}\|\nabla u\|_N, \tag{2.2}$$

where $|\Omega|$ is the N-dimensional volume of Ω and ω_N is the volume of the unit sphere $B_1 \subset \mathbb{R}^N$, that is,

$$\omega_N = \frac{\pi^{N/2}}{\Gamma(1+N/2)}. \tag{2.3}$$

R e m a r k 2.2. Clearly, $N\omega_N$ is the surface area of the unit sphere ∂B_1 in $\mathbb{R}^N$.

Lemma 2.3 ([14]). *Let $u \in W_0^{1,N}(\Omega) \cap L^r(\Omega)$, where $r \in [1, \infty)$ and $\Omega \subseteq \mathbb{R}^N$ is an arbitrary domain. Then for $q \in [r, \infty)$,*

$$\|u\|_q \leqslant c(N, r) q^{1-1/N} \|\nabla u\|_N^{1-r/q} \|u\|_r^{r/q}. \tag{2.4}$$

The exponent $1 - 1/N$ of q is the best possible. In particular,

$$c(N, N) = \frac{1}{\sqrt{\pi}} \Big(\frac{\Gamma(N/2)\Gamma(2N)}{2\Gamma^2(N)}\Big)^{1/N} \equiv d_N. \tag{2.5}$$

R e m a r k 2.4. By Lemma 2.3, the embedding $W_0^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for every $q \in [N, \infty)$ and

$$\|u\|_{L^q(\Omega)} \leqslant d_N q^{1-1/N} \|u\|_{W_0^{1,N}(\Omega)}. \tag{2.6}$$

Lemma 2.5 (Trudinger-Moser inequality [10], [16]). *Let $N \geqslant 2$ and $\alpha > 0$.*

(i) *If $\Omega \subset \mathbb{R}^N$ is a bounded domain and $u \in W_0^{1,N}(\Omega)$, then*

$$\int_\Omega \exp(\alpha |u|^{N/(N-1)}) \, \mathrm{d}x < \infty. \tag{2.7}$$

(ii) *If $u \in E$, then*

$$\int_{\mathbb{R}^N} \big(\exp(\alpha |u|^{N/(N-1)}) - S_{N-2}(\alpha, u)\big) \, \mathrm{d}x < \infty. \tag{2.8}$$

We now establish the variational setting for problem (1.1). We observe that the natural energy functional associated to (1.1) is

$$I(u) = \frac{1}{N} \int_{\mathbb{R}^N} (1 + 2^{N-1}|u|^N)|\nabla u|^N \, \mathrm{d}x + \frac{b}{N} \int_{\mathbb{R}^N} |u|^N \, \mathrm{d}x - \int_{\mathbb{R}^N} H(u) \, \mathrm{d}x, \tag{2.9}$$

where $H(u) = \int_0^u h(s) \, \mathrm{d}s$. It should be pointed out that the functional I is not well defined in general in E. To overcome this difficulty, we employ an argument developed by Colin and Jeanjean in [6] (see also [28]). We make the change of variables $u = f(v)$ or $v = f^{-1}(u)$, where f is defined by

$$f'(t) = (1 + 2^{N-1}|f(t)|^N)^{-1/N}, \quad t \geqslant 0, \ f(0) = 0, \tag{2.10}$$

and by $f(t) = -f(-t)$ on $(-\infty, 0]$.

Lemma 2.6 ([6], [13], [28]).

(f_1) *The function f is uniquely defined, C^2 and invertible in $\mathbb{R}$,*
(f_2) $0 < f'(t) \leqslant 1$ *for all* $t \in \mathbb{R}$,
(f_3) $|f(t)| \leqslant |t|$ *for all* $t \in \mathbb{R}$,
(f_4) $f(t)/t \to 1$ *as* $t \to 0$,
(f_5) $|f(t)| \leqslant 2^{1/2N}|t|^{1/2}$ *for all* $t \in \mathbb{R}$,
(f_6) $\frac{1}{2}f(t) \leqslant tf'(t) \leqslant f(t)$ *for all* $t \in \mathbb{R}^+ = [0, \infty)$ *and* $f(t) \leqslant tf'(t) \leqslant \frac{1}{2}f(t)$ *for all* $t \in \mathbb{R}^- = (-\infty, 0]$,
(f_7) *there exists* $a \in (0, 2^{1/2N}]$ *such that* $f(t)/\sqrt{t} \to a$ *as* $t \to \infty$,
(f_8) *there exists* $b_0 > 0$ *such that*

$$|f(t)| \geqslant \begin{cases} b_0|t| & \text{if } |t| \leqslant 1, \\ b_0|t|^{1/2} & \text{if } |t| \geqslant 1. \end{cases} \tag{2.11}$$

So after the change of variables, we can write $I(u)$ as

$$J(v) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^N \,\mathrm{d}x + \frac{b}{N}\int_{\mathbb{R}^N} |f(v)|^N \,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(v)) \,\mathrm{d}x, \tag{2.12}$$

which is well defined on the space E under the assumptions (A_1)–(A_2).

As in [28], we observe that if $v \in E \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ is a critical point of the functional J, that is, $J'(v)\varphi = 0$ for all $\varphi \in E$, where

$$J'(v)\varphi = \int_{\mathbb{R}^N} |\nabla v|^{N-2}\nabla v \nabla\varphi \,\mathrm{d}x + \int_{\mathbb{R}^N} [b|f(v)|^{N-2}f(v) - h(f(v))]f'(v)\varphi \,\mathrm{d}x, \tag{2.13}$$

then v is a weak solution of the equation

$$-\Delta_N v = g(v), \quad x \in \mathbb{R}^N, \tag{2.14}$$

where

$$g(s) = -b|f(s)|^{N-2}f(s)f'(s) + h(f(s))f'(s), \quad s \in \mathbb{R}, \tag{2.15}$$

and then $u = f(v)$ is a weak solution of (1.1). By Theorem 1 in [27], we can conclude that v is locally bounded in $\mathbb{R}^N$. So, we consider the existence of weak solutions to (2.14) in E.

We first construct a subspace $E_r \subset E$. The function $u \in L^p(\mathbb{R}^N)$ is called radially nonincreasing if $u(x) \leqslant u(y)$ when $|x| \geqslant |y|$ and $x, y \in \mathbb{R}^N$.

Lemma 2.7 ([3]). *If $u \in L^p(\mathbb{R}^N)$ $(p \geqslant 1)$ is a nonnegative and radially nonincreasing function, then one has*

$$|u(x)| \leqslant |x|^{-N/p} \omega_N^{-1/p} \|u\|_p \quad \forall\, x \in \mathbb{R}^N \setminus \{0\}. \tag{2.16}$$

In particular,

$$|u(x)| \leqslant |x|^{-1} \omega_N^{-1/N} \|u\|_N \quad \forall\, x \in \mathbb{R}^N \setminus \{0\}. \tag{2.17}$$

Denote

$$E_r = \{u \in E\colon\ u \text{ is nonnegative and radially nonincreasing in } \mathbb{R}^N\}. \tag{2.18}$$

Remark 2.8. By Lemma 2.7, we have $u(x) \to 0$ as $|x| \to \infty$ if $u(x) \in E_r$.

Lemma 2.9. *Let $s > N$. Then the embedding $E_r \hookrightarrow L^s(\mathbb{R}^N)$ is compact.*

Proof. Let $\{u_n\} \subset E_r$ be a bounded sequence in E. Without loss of a generality, we assume that $u_n \rightharpoonup 0$ in E and $\|u_n\|_E \leqslant M$ for all $n \in \mathbb{N}$ with some $M > 0$. For our purpose, it is sufficient to show that $u_n \to 0$ in $L^s(\mathbb{R}^N)$. By Lemma 2.1, we can assume $u_n \to 0$ in $L^s_{\rm loc}(\mathbb{R}^N)$ and $u_n(x) \to 0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Let B_1 be a unit sphere in $\mathbb{R}^N$ and $B_1^{\rm c} = \mathbb{R}^N \setminus \overline{B_1}$. Then, as $n \to \infty$, we obtain

$$\int_{\mathbb{R}^N} |u_n(x)|^s \,{\rm d}x = \int_{B_1} |u_n(x)|^s \,{\rm d}x + \int_{B_1^{\rm c}} |u_n(x)|^s \,{\rm d}x = \int_{B_1^{\rm c}} |u_n(x)|^s \,{\rm d}x + o(1). \tag{2.19}$$

Furthermore, from (2.17), it follows that

$$|u_n(x)|^s \leqslant C_1 |x|^{-s}, \quad x \in \mathbb{R}^N \setminus \{0\} \text{ and } n \in \mathbb{N}, \tag{2.20}$$

where $C_1 > 0$, independent of n. Using the assumption $s > N$ and the Lebesgue dominated convergence theorem, we derive that the function $|x|^{-s} \in L^1(B_1^{\rm c})$ and

$$\int_{B_1^{\rm c}} |u_n(x)|^s \,{\rm d}x \to 0 \quad \text{as } n \to \infty. \tag{2.21}$$

Then the limits (2.19) and (2.21) show that

$$\int_{\mathbb{R}^N} |u_n|^s \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{2.22}$$

Thus, the embedding $E_r \hookrightarrow L^s(\mathbb{R}^N)$ is compact. This completes the proof of Lemma 2.9. □

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^N$ be a domain (maybe unbounded) with the smooth boundary $\partial\Omega$. If $\{u_n\}$ is a sequence with $u_n \to u$ in $X = L^q(\Omega)$ $(q \geqslant 1)$, then there exists a subsequence $\{u_{kn}\} \subset \{u_n\}$ and $v \in X$ such that $|u_{kn}(x)| \leqslant v(x)$, a.e. in Ω and for any $k \geqslant 1$.*

Proof. Since $u_n \to u$ in X, we have $u_n(x) \to u(x)$ a.e. in Ω. Also we can extract a subsequence $\{u_{kn}\}$ of $\{u_n\}$ which we denote by $\{u_k\}$ such that

$$\|u_{k+1} - u_k\|_X \leqslant 2^{-k}, \quad k \geqslant 1. \tag{2.23}$$

Setting

$$g_n(x) = \sum_{k=1}^{n} |u_{k+1}(x) - u_k(x)|, \quad x \in \Omega,\ n = 1, 2, \ldots, \tag{2.24}$$

we get $g_n \in X$ and $\|g_n\|_X \leqslant 1$ for any $n \geqslant 1$. Obviously, $g_n(x)$ is monotonic increasing. By Levi's lemma and Brezis-Lieb lemma (see [4]), there exists a nonnegative function $g \in X$ such that $\|g_n - g\|_X \to 0$ and $g_n(x) \to g(x)$ a.e. in Ω as $n \to \infty$. Now, for $m > k \geqslant 2$, it follows that

$$|u_m(x) - u_k(x)| \leqslant |u_m(x) - u_{m-1}(x)| + \ldots + |u_{k+1}(x) - u_k(x)| \leqslant g(x) - g_{k-1}(x) \leqslant g(x)$$

for a.e. $x \in \Omega$. Letting $m \to \infty$, we obtain

$$|u(x) - u_k(x)| \leqslant g(x), \quad \text{for a.e. } x \in \Omega,\ k \geqslant 2 \tag{2.25}$$

and for $k \geqslant 2$, $|u_k(x)| \leqslant v(x)$ a.e. in Ω with $v(x) = |u(x)| + g(x) \in X$ and the proof is complete. □

Having established the main functional properties of the space E_r, we need a way to pass from functions in E to functions in E_r. A way to do this, suitable for our purpose, is given by a procedure called Schwarz symmetrization. We just describe roughly the idea for nonnegative functions and state the result that we will need in this paper, for details, see [2], [10], [20] and the references therein.

Let $u \in L^p(\mathbb{R}^N)$ $(1 \leqslant p \leqslant \infty)$ be such that $u(x) \geqslant 0$ a.e. in $\mathbb{R}^N$. For $t > 0$, set

$$\Omega(t) = \{x \in \mathbb{R}^N \colon u(x) > t\} \quad \text{and} \quad \mu(t) = \operatorname{meas}(\Omega(t)). \tag{2.26}$$

Here and in the sequel, $\operatorname{meas}(\Omega)$ is Lebesgue measure of Ω. Since $u \in L^p(\mathbb{R}^N)$, we have $\mu(t) < \infty$ for all $t > 0$. The Schwarz symmetrization constructs a radial function $u^* \colon \mathbb{R}^N \to \mathbb{R}$ such that

$$\{x \in \mathbb{R}^N \colon u^*(x) > t\} = B_{\varrho(t)} \quad \text{with meas } (B_{\varrho(t)}) = \mu(t), \tag{2.27}$$

where $B_{\varrho(t)}$ is the sphere with the radius $\varrho(t) > 0$ and the center at the origin. Thus, the sets where u and u^* are greater than t have the same Lebesgue measure. Obviously, the function u^* is radially nonincreasing. The most important properties of u^* are stated in the following results.

Lemma 2.11 ([2], [10], [20]).

(i) *Let $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and increasing function with $f(0) = 0$. Then,*

$$\int_{\mathbb{R}^N} f(u^*)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(u)\,\mathrm{d}x. \tag{2.28}$$

(ii) *Let $1 \leqslant p \leqslant \infty$. If $u \in E$ is a nonnegative function, then $u^* \in E$ and*

$$\int_{\mathbb{R}^N} |\nabla u^*|^p\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla u|^p\,\mathrm{d}x. \tag{2.29}$$

Remark 2.12. Lemma 2.11 shows that if $u \in E$ and $u(x) \geqslant 0$ in $\mathbb{R}^N$, then $u^* \in E_r$.

To prove the existence of nontrivial solutions for problem (2.14), we introduce the Nehari manifold

$$\begin{aligned} \mathcal{N} &= \{v \in E \setminus \{0\} \colon J'(v)v = 0\} \\ &= \Big\{v \in E \setminus \{0\} \colon \|\nabla v\|_N^N + \int_{\mathbb{R}^N} \big[b|f(v)|^{N-2}f(v) - h(f(v))\big]f'(v)v\,\mathrm{d}x = 0\Big\} \end{aligned} \tag{2.30}$$

and the fibering maps $\phi_v(t) = J(tv)$ for $t > 0$. Clearly, we have that $v \in \mathcal{N}$ if and only if $\phi_v'(1) = 0$ and, more generally, $tv \in \mathcal{N}$ if and only if $\phi_v'(t) = 0$. By the definition, one sees

$$\begin{aligned} \phi_v(t) &= J(tv) = \frac{1}{N}\|t\nabla v\|_N^N + \frac{b}{N}\int_{\mathbb{R}^N} |f(tv)|^N\,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(tv))\,\mathrm{d}x, \\ \phi_v'(t) &= t^{N-1}\|\nabla v\|_N^N + \int_{\mathbb{R}^N} [b|f(tv)|^{N-2}f(tv) - h(f(tv))]f'(tv)v\,\mathrm{d}x. \end{aligned} \tag{2.31}$$

Notice that, if $v \in \mathcal{N}$, then

$$J(v) = \frac{1}{N} \int_{\mathbb{R}^N} \big[h(f(v))f'(v)v - NH(f(v)) + b|f(v)|^{N-2}(f^2(v) - f(v)f'(v)v)\big] \, \mathrm{d}x. \tag{2.32}$$

In the following, under the assumptions (A$_1$)–(A$_2$), we derive some properties for $\mathcal{N}$.

Lemma 2.13. *The Nehari manifold $\mathcal{N} \neq \emptyset$.*

Proof. Choose the nonnegative function $v_1 \in C_0^\infty(\mathbb{R}^N) \subset E$ such that

$$\int_{\mathbb{R}^N} h(f(v_1))f'(v_1)v_1 \, \mathrm{d}x > 0 \tag{2.33}$$

and $\|v_1\|_E \leqslant \varrho$ for small $\varrho > 0$. For $t \geqslant 0$, let

$$\gamma(t) = J'(tv_1)tv_1 = t^N\|\nabla v_1\|_N^N + \int_{\mathbb{R}^N} (b(f(tv_1))^{N-1} - h(f(tv_1)))f'(tv_1)tv_1 \, \mathrm{d}x. \tag{2.34}$$

By Lemma 2.6, one sees

$$\int_{\mathbb{R}^N} (f(tv_1))^{N-1} f'(tv_1)tv_1 \, \mathrm{d}x \geqslant \frac{1}{2} b_0^N t^N \|v_1\|_N^N, \tag{2.35}$$

provided that $0 \leqslant tv_1(x) \leqslant 1$ in $\mathbb{R}^N$. Set $\Omega = \{x \in \mathbb{R}^N \colon v_1(x) \neq 0\}$. Then Ω is bounded in $\mathbb{R}^N$ and $\|v_1\|_{L^q(\Omega)} = \|v_1\|_{L^q(\mathbb{R}^N)} = \|v_1\|_q, \|v_1\|_{W_0^{1,N}(\Omega)} \leqslant \|v_1\|_E$. Furthermore, it follows from (2.6) and Lemma 2.6 that

$$\begin{aligned} 0 &\leqslant \int_{\mathbb{R}^N} h(f(tv_1))f'(tv_1)tv_1 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} h(f(tv_1))f(tv_1) \, \mathrm{d}x \\ &\leqslant b_1 \int_{\mathbb{R}^N} |f(tv_1)|^q [\exp(\alpha_0 |f(tv_1)|^{2N/(N-1)}) - S_{N-2}(\alpha_0, f(tv_1))] \, \mathrm{d}x \\ &\leqslant \sum_{k=N-1}^\infty \frac{b_1\alpha_0^k}{k!} t^{q+N} \int_{\mathbb{R}^N} |v_1|^{s_k} \, \mathrm{d}x \leqslant \sum_{k=N-1}^\infty \frac{b_1\alpha_0^k}{k!} t^{q+N} d_N^{s_k} s_k^{(1-1/N)s_k} \|v_1\|_E^{s_k} \\ &\leqslant b_1 t^{q+N} d_N^q \|v_1\|_E^q \sum_{k=N-1}^\infty a_k, \end{aligned} \tag{2.36}$$

where d_N is given in (2.5) and

$$\begin{aligned} s_k &= q_k + q, \quad \beta = \frac{N+q}{N-1}, \quad q_k = \frac{kN}{N-1}, \\ a_k &= \frac{\alpha_0^k}{k!} d_N^{kN/(N-1)} \|v_1\|_E^{kN/(N-1)} (\beta k)^{k+q(1-1/N)} \end{aligned} \tag{2.37}$$

with $k \geqslant N-1$. Since $\varrho > 0$ is so small that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \mathrm{e}\beta\alpha_0 \|v_1\|_E^{N/(N-1)} d_N^{N/(N-1)} \leqslant \mathrm{e}\beta\alpha_0 \varrho^{N/(N-1)} d_N^{N/(N-1)} < 1, \tag{2.38}$$

the positive series $\sum\limits_{k=N-1}^{\infty} a_k$ is convergent. Therefore, from (2.36), there is $C_2 > 0$ such that

$$0 \leqslant \int_{\mathbb{R}^N} h(f(tv_1))f'(tv_1)tv_1 \,\mathrm{d}x \leqslant C_2 t^{q+N} \|v_1\|_E^q. \tag{2.39}$$

Then it follows from (2.34)–(2.36) and (2.39) that $\gamma(t) > 0$ for small $t > 0$.

On the other hand, since $q > 0$, we choose $p > N$ such that $q > 2(p-N)$. We now prove $\gamma(t) \to -\infty$ as $t \to \infty$. Set

$$G(t) = t^{-p+1} h(f(tv_1))f'(tv_1)v_1 - h(f(v_1))f'(v_1)v_1, \quad t \geqslant 1. \tag{2.40}$$

We claim that $G'(t) \geqslant 0$ for $t \geqslant 1$. In fact, it follows from (1.7) and Lemma 2.6 that

$$\begin{aligned} G'(t) &= t^{-p-1}\big(h'(f)(f')^2t^2v_1^2 - (p-1)h(f)f'tv_1 + h(f)f''t^2v_1^2\big) \\ &\geqslant t^{-p-1}\frac{h(f)}{f}\big((q-1+2N)(f')^2t^2v_1^2 + f''ft^2v_1^2 - (p-1)f'ftv_1\big) \\ &\geqslant t^{-p-1}\frac{h(f)}{f}\big((q-1+2N)(f')^2t^2v_1^2 + f''ft^2v_1^2 - 2(p-1)(f')^2t^2v_1^2\big) \\ &= \frac{t^{-p+1}v_1^2h(f)}{f(1+2^{N-1}|f|^N)^{2/N}}\Big(q+1-2(p-N) - \frac{2^{N-1}|f|^N}{1+2^{N-1}|f|^N}\Big) \geqslant 0, \end{aligned} \tag{2.41}$$

provided that $q > 2(p-N)$, where $f = f(tv_1)$, $f' = f'(tv_1)$, $f'' = f''(tv_1)$. Then,

$$h(f(tv_1))f'(tv_1)tv_1 \geqslant t^p h(f(v_1))f'(v_1)v_1, \quad t \geqslant 1. \tag{2.42}$$

Moreover, the application of (2.34) yields

$$\gamma(t) \leqslant t^N \|\nabla v_1\|_N^N + bt^N \int_{\mathbb{R}^N} |v_1|^N \,\mathrm{d}x - t^p \int_{\mathbb{R}^N} h(f(v_1))f'(v_1)v_1 \,\mathrm{d}x \to -\infty \tag{2.43}$$

as $t \to \infty$. Then there exists $t_1 > 1$ such that $\gamma(t_1) = 0$. Obviously, $t_1v_1 \not\equiv 0$ in $\mathbb{R}^N$. In conclusion, $t_1v_1 \in \mathcal{N}$ and so $\mathcal{N} \neq \emptyset$. $\square$

Lemma 2.14. *The following properties hold:*

$$d = \inf_{v\in\mathcal{N}} J(v) \geqslant 0, \quad d_0 = \inf_{v\in\mathcal{N}} \{\|v\|_E\} > 0. \tag{2.44}$$

P r o o f. Let $v \in \mathcal{N}$. Then,

$$\|\nabla v\|_N^N + b\int_{\mathbb{R}^N} |f(v)|^{N-2} f(v) f'(v) v \,\mathrm{d}x = \int_{\mathbb{R}^N} h(f(v)) f'(v) v \,\mathrm{d}x. \tag{2.45}$$

By (1.7) and (2.32), we derive from Lemma 2.6 that

$$\begin{aligned} J(v) &= \frac{1}{N}\int_{\mathbb{R}^N} [h(f)f'(v)v - NH(f(v)) + b(|f(v)|^N - |f(v)|^{N-2}f(v)f'(v)v))]\,\mathrm{d}x \\ &\geqslant \frac{1}{N}\int_{\mathbb{R}^N} h(f(v))f(v)\Big(\frac{f'(v)v}{f(v)} - \frac{N}{q+2N}\Big)\,\mathrm{d}x + \frac{b}{2N}\int_{\mathbb{R}^N} |f(v)|^N\,\mathrm{d}x \\ &\geqslant \Big(\frac{1}{2} - \frac{N}{q+2N}\Big)\int_{\mathbb{R}^N} h(f(v))f(v)\,\mathrm{d}x + \frac{b}{2N}\int_{\mathbb{R}^N} |f(v)|^N\,\mathrm{d}x > 0. \end{aligned} \tag{2.46}$$

This shows that $J > 0$ on $\mathcal{N}$ and $d \geqslant 0$. Now let us show that $d_0 > 0$.

Assume, by contradiction, that there is $\{v_n\} \subset \mathcal{N}$ such that $0 < \|v_n\|_E \leqslant \varepsilon_n \to 0$ as $n \to \infty$. As in (2.36), we obtain

$$\begin{aligned} \|\nabla v_n\|_N^N + b\int_{\mathbb{R}^N} |f(v_n)|^{N-2} f(v_n) f'(v_n) v_n \,\mathrm{d}x &= \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n \,\mathrm{d}x \\ \leqslant \sum_{k=N-1}^{\infty} \frac{b_1\alpha_0^k}{k!}\|v_n\|_{q_k+q}^{q_k+q} &\leqslant \sum_{k=N-1}^{\infty} \frac{b_1\alpha_0^k}{k!} d_N^{q_k+q}(q_k+q)^{(1-1/N)(q_k+q)}\|v_n\|_E^{q_k+q} \\ &\leqslant b_1\|v_n\|_E^{q+N} d_N^q \beta^{q(N-1)/N} \sum_{k=N-1}^{\infty} b_k, \end{aligned} \tag{2.47}$$

where d_N is given in (2.5), β, q_k are in (2.37) and

$$b_k = \frac{\alpha_0^k}{k!}\beta^k (d_N^{N/(N-1)})^k \|v_n\|_E^{N(k-N+1)/(N-1)} k^{k+q(N-1)/N}, \quad k \geqslant N-1. \tag{2.48}$$

Since

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \mathrm{e}\alpha_0\beta\|v_n\|_E^{N/(N-1)} \leqslant \mathrm{e}\alpha_0\beta\varepsilon_n^{N/(N-1)} < 1, \tag{2.49}$$

the positive series $\sum\limits_{k=N-1}^{\infty} b_k$ is convergent. Denote $B_0 = \sum\limits_{k=N-1}^{\infty} b_k$.

On the other hand, we have from (f_6) and (f_8) (see Lemma 2.6) that

$$\begin{aligned}(2.50)\qquad \|\nabla v_n\|_N^N + b\int_{\mathbb{R}^N} |f(v_n)|^{N-2} f(v_n) f'(v_n) v_n \,\mathrm{d}x \\ \geqslant \|\nabla v_n\|_N^N + \frac{b}{2}\int_{\mathbb{R}^N} |f(v_n)|^N \,\mathrm{d}x \\ \geqslant \|\nabla v_n\|_N^N + \frac{1}{2} b b_0^N \|v_n\|_N^N \geqslant C_3 \|v_n\|_E^N,\end{aligned}$$

with $C_3 = \min\{1, bb_0^N/2\} > 0$. Then (2.47) and (2.50) show that

$$(2.51)\qquad 0 < C_3 \leqslant b_1 \|v_n\|_E^q d_N^q \beta^{q(N-1)/N} B_0 \leqslant b_1 \varepsilon_n^q d_N^q \beta^{q(N-1)/N} B_0,$$

which is impossible if ε_n is small enough. Thus, $d_0 = \inf\limits_{v\in\mathcal{N}} \{\|u\|_E\} > 0$. The proof is finished. □

Lemma 2.15. *There exists a nonnegative and nontrivial function $v_0 = v_0(x) \in \mathcal{N}$ such that $J(v_0) = d > 0$ and $v_0(x) \to 0$ as $|x| \to \infty$.*

Proof. We first show that there exists a minimizing sequence for d in $\mathcal{N} \cap E_r$. To this aim, let $\{z_n\} \subset \mathcal{N}$ be a minimizing sequence for d. The fact $J(z_n) = J(|z_n|)$ implies that $\{|z_n|\}$ is also a minimizing sequence, so we can assume from the beginning that $z_n \geqslant 0$ a.e. in $\mathbb{R}^N$, that is, $J(z_n) \to d$, $J'(z_n) \to 0$ in E^* as $n \to \infty$. Similarly as in the proof of Lemma 3.1 in [29], we can prove that the sequence $\{z_n\}$ is bounded in E. Let $w_n = z_n^* \in E_r$ be the Schwarz symmetrization of z_n. Then, noticing that the functions $g_1(t) = h(f(t))f'(t)t$ and $g_2(t) = |f(t)|^{N-2} f(t) f'(t) t$ are nonnegative and increasing in $\mathbb{R}^+$, we have from Lemma 2.11 that

$$\begin{aligned}(2.52)\qquad &\int_{\mathbb{R}^N} |\nabla w_n|^N \,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla z_n|^N \,\mathrm{d}x, \quad \int_{\mathbb{R}^N} g_1(w_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} g_1(z_n)\,\mathrm{d}x, \\ &\int_{\mathbb{R}^N} g_2(w_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} g_2(z_n)\,\mathrm{d}x, \quad \int_{\mathbb{R}^N} |f(w_n)|^N \,\mathrm{d}x = \int_{\mathbb{R}^N} |f(z_n)|^N \,\mathrm{d}x, \\ &\int_{\mathbb{R}^N} H(f(w_n))\,\mathrm{d}x = \int_{\mathbb{R}^N} H(f(z_n))\,\mathrm{d}x.\end{aligned}$$

Since $z_n \in \mathcal{N}$, then

$$\begin{aligned}\|\nabla w_n\|_N^N + \int_{\mathbb{R}^N} b g_2(w_n)\,\mathrm{d}x &\leqslant \|\nabla z_n\|_N^N + \int_{\mathbb{R}^N} b g_2(z_n)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} g_1(z_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} g_1(w_n)\,\mathrm{d}x.\end{aligned}$$

Hence, setting

$$\gamma_n(t) = J'(tw_n)tw_n = \|\nabla tw_n\|_N^N + b\int_{\mathbb{R}^N} g_2(tw_n)\,\mathrm{d}x - \int_{\mathbb{R}^N} g_1(tw_n)\,\mathrm{d}x, \tag{2.53}$$

we have $\gamma_n(1) \leqslant 0$ and $\gamma_n(t) > 0$ for small $t > 0$. This shows that there exists $t_n \in (0,1]$ such that $\gamma_n(t_n) = 0$ and $t_n w_n \in \mathcal{N}$. On the other hand, it is noted that the functions $G_1(t) = h(f(t))f'(t)t - NH(f(t))$ and $G_2(t) = |f(t)|^N - |f(t)|^{N-2}f(t)f'(t)t$ are increasing in $\mathbb{R}^+$.

Then, it follows from (2.32) and (2.52) that

$$\begin{aligned} d \leqslant J(t_n w_n) &= \frac{1}{N}\int_{\mathbb{R}^N} (G_1(t_n w_n) + bG_2(t_n w_n))\,\mathrm{d}x \\ &\leqslant \frac{1}{N}\int_{\mathbb{R}^N} (G_1(w_n) + bG_2(w_n))\,\mathrm{d}x \\ &= \frac{1}{N}\int_{\mathbb{R}^N} (G_1(z_n) + bG_2(z_n))\,\mathrm{d}x = J(z_n). \end{aligned} \tag{2.54}$$

This implies that $\{t_n w_n\}$ is also a minimizing sequence for d and $t_n w_n \in \mathcal{N} \cap E_r$. Let $v_n = t_n w_n \geqslant 0$. We can assume that, up to a subsequence, $v_n \rightharpoonup v_0$ in E. By Lemma 2.9, it follows that $v_n \to v_0$ in $L^s(\mathbb{R}^N)$ for all $s > N$, and, again up to a subsequence, $v_n(x) \to v_0(x)$ a.e. in $\mathbb{R}^N$. So $v_0(x) \geqslant 0$ a.e. in $\mathbb{R}^N$ and $v_0 \in E_r$. We now show that $v_0 \in \mathcal{N}$ and $J(v_0) = d$. We first claim that $v_0 \neq 0$ in E. Otherwise, $v_n \to 0$ in E as $n \to \infty$. Arguing as in the proof of (2.51), it is impossible. So, one has $\|v_0\|_E > 0$.

Noticing that $v_n \in \mathcal{N}$, one has

$$\|\nabla v_n\|_N^N + b\int_{\mathbb{R}^N} |f(v_n)|^{N-2}f(v_n)f'(v_n)v_n\,\mathrm{d}x = \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n\,\mathrm{d}x. \tag{2.55}$$

Then, the application of the weak lower semicontinuity of norms and Fatou's lemma yields

$$\begin{aligned} &\|\nabla v_0\|_N^N + b\int_{\mathbb{R}^N} |f(v_0)|^{N-2}f(v_0)f'(v_0)v_0\,\mathrm{d}x \\ &\qquad \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n\,\mathrm{d}x. \end{aligned} \tag{2.56}$$

In what follows we make use of the following results (to be proved later):

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n\,\mathrm{d}x &= \int_{\mathbb{R}^N} h(f(v_0))f'(v_0)v_0\,\mathrm{d}x, \\ \lim_{n\to\infty}\int_{\mathbb{R}^N} H(f(v_n))\,\mathrm{d}x &= \int_{\mathbb{R}^N} H(f(v_0))\,\mathrm{d}x. \end{aligned} \tag{2.57}$$

Clearly, if

$$\|\nabla v_0\|_N^N + b\int_{\mathbb{R}^N} |f(v_0)|^{N-2} f(v_0) f'(v_0) v_0 \,\mathrm{d}x = \int_{\mathbb{R}^N} h(f(v_0)) f'(v_0) v_0 \,\mathrm{d}x, \tag{2.58}$$

then $v_0 \in \mathcal{N}$. So, arguing by contradiction, we let

$$\|\nabla v_0\|_N^N + b\int_{\mathbb{R}^N} |f(v_0)|^{N-2} f(v_0) f'(v_0) v_0 \,\mathrm{d}x < \int_{\mathbb{R}^N} h(f(v_0)) f'(v_0) v_0 \,\mathrm{d}x. \tag{2.59}$$

Let $\gamma(t) = J'(tv_0)tv_0$. Clearly, $\gamma(t) > 0$ for small $t > 0$ and $\gamma(1) < 0$. So there exists $t \in (0,1)$ such that $\gamma(t) = 0$ and $tv_0 \in \mathcal{N}$. Then we have from (2.32) and (2.57) that

$$\begin{aligned} d \leqslant J(tv_0) &= \frac{1}{N}\int_{\mathbb{R}^N} (G_1(tv_0) + bG_2(tv_0)) \,\mathrm{d}x < \frac{1}{N}\int_{\mathbb{R}^N} (G_1(v_0) + bG_2(v_0)) \,\mathrm{d}x \\ &\leqslant \frac{1}{N}\liminf_{n\to\infty} \int_{\mathbb{R}^N} (G_1(v_n) + bG_2(v_n)) \,\mathrm{d}x = \liminf_{n\to\infty} J(v_n) = d. \end{aligned}$$

This contradiction proves that (2.58) holds and then $v_0 \in \mathcal{N}$. Again, applying the weak lower semicontinuity of norms, we get $J(v_0) \leqslant \liminf_{n\to\infty} J(v_n) = d$. On the other hand, for every $v \in \mathcal{N}$, $J(v) \geqslant d$. So, $J(v_0) = d$. Furthermore, from (2.46), it follows $d > 0$.

Now, we prove the first limit in (2.57). Since $\{v_n\}$ is bounded in E, we assume $\|v_n\|_E \leqslant M (n \geqslant 1)$ for some constant $M > 0$. Then, it follows from $J(v_n) \to d$ and (1.7) that there exists $M_1 > 0$ such that

$$0 \leqslant \int_{\mathbb{R}^N} h(f(v_n)) f'(v_n) v_n \,\mathrm{d}x \leqslant M_1, \quad 0 \leqslant \int_{\mathbb{R}^N} H(f(v_n)) \,\mathrm{d}x \leqslant M_1 \quad \forall\, n \geqslant 1 \tag{2.60}$$

with some constant $M_1 > 0$. For any $r > 0$, (2.60) gives

$$0 \leqslant \int_{B_r} h(f(v_n)) f'(v_n) v_n \,\mathrm{d}x \leqslant M_1, \quad 0 \leqslant \int_{B_r} H(f(v_n)) \,\mathrm{d}x \leqslant M_1 \quad \forall\, n \geqslant 1. \tag{2.61}$$

Arguing as in the proof of Lemma 4 in [9] and Lemma 2.1 in [8], we have $g_n(x) = h(f(v_n(x)) f'(v_n(x)) \to g_0(x) = h(f(v_0(x)) f'(v_0(x))$ in $L^1(B_r)$ as $n \to \infty$. Then, by Lemma 2.10, there exists $g(x) \in L^1(B_r)$ such that $|g_n(x)| \leqslant g(x)$ a.e. in B_r for all $n \geqslant 1$. Noticing that (2.17), we obtain $|v_n(x)| \leqslant M|x|^{-1}$ for $x \neq 0$ and $n \geqslant 1$. Then $|g_n(x) v_n(x)| \leqslant Mg(x)|x|^{-1} \leqslant M\tau^{-1} g(x)$ in $D_\tau = B_r \setminus B_\tau$ for any $\tau \in (0,r)$. Clearly, $B_r = D_\tau \cup B_\tau$. By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_{D_\tau} h(f(v_n)) f'(v_n) v_n \,\mathrm{d}x = \int_{D_\tau} h(f(v_0)) f'(v_0) v_0 \,\mathrm{d}x. \tag{2.62}$$

On the other hand, from Lemma 2.5, it follows that $|v_0|^q R(\alpha_0, v_0) \in L^1(B_r)$, where $R(\alpha_0, t)$ is defined by (1.6). Furthermore, using the integral absolute continuity, we derive that for any small $\varepsilon > 0$, there exists $\tau > 0$ such that

$$b_1 \int_{B_\tau} |v_0|^q R(\alpha_0, v_0)\,\mathrm{d}x \leqslant \varepsilon, \tag{2.63}$$

where the constant $b_1 > 0$ is given in (A_1). Clearly, if we can prove

$$\limsup_{n\to\infty} \int_{B_\tau} h(f(v_n))f'(v_n)v_n\,\mathrm{d}x \leqslant b_1 \int_{B_\tau} |v_0|^q R(\alpha_0, v_0)\,\mathrm{d}x, \tag{2.64}$$

then it follows from (2.62), (2.63), and (2.64) that

$$\lim_{n\to\infty} \int_{B_r} h(f(v_n))f'(v_n)v_n\,\mathrm{d}x = \int_{B_r} h(f(v_0))f'(v_0)v_0\,\mathrm{d}x. \tag{2.65}$$

We now prove (2.64) and let $k = N-1$. By Lemmas 2.9 and 2.10, there exists a subsequence $\{v_{(N-1)n}\} \subset \{v_n\}$ and $\mu_{N-1} \in L^{q+q_{N-1}}(B_\tau)$ such that $|v_{(N-1)n}(x)| \leqslant \mu_{N-1}(x)$, a.e. in B_τ for all $n \in \mathbb{N}$. Then, by Fatou's lemma, we get

$$\limsup_{n\to\infty} \int_{B_\tau} |v_{(N-1)n}(x)|^{q+q_{N-1}}\,\mathrm{d}x \leqslant \int_{B_\tau} |v_0(x)|^{q+q_{N-1}}\,\mathrm{d}x, \tag{2.66}$$

where $q_k = kN/(N-1)$, $k \geqslant N-1$.

Likewise, by Lemmas 2.9 and 2.10, we take a subsequence $\{v_{Nn}\} \subset \{v_{(N-1)n}\}$ and $\mu_N \in L^{q+q_N}(B_\tau)$ such that $|v_{Nn}(x)| \leqslant \mu_N(x)$, a.e. in B_τ for all $n \geqslant 1$. Furthermore, using Fatou's lemma, one sees

$$\begin{aligned}\limsup_{n\to\infty} &\int_{B_\tau} |v_{Nn}(x)|^{q+q_N}\,\mathrm{d}x \\ &\leqslant \int_{B_\tau} \limsup_{n\to\infty} |v_{Nn}(x)|^{q+q_N}\,\mathrm{d}x = \int_{B_\tau} |v_0(x)|^{q+q_N}\,\mathrm{d}x.\end{aligned} \tag{2.67}$$

Continuing this line of reasoning, we obtain a subsequence $\{v_{kn}\} \subset \{v_{(k-1)n}\}$ $(k \geqslant N)$ with the property

$$\limsup_{n\to\infty} \int_{B_\tau} |v_{kn}|^{q+q_k}\,\mathrm{d}x \leqslant \int_{B_\tau} |v_0|^{q+q_k}\,\mathrm{d}x \quad \forall\, k \geqslant N-1. \tag{2.68}$$

By a diagonal process, we take a subsequence $\{v_{mm}\} \subset \{v_n\}$. For convenience, we set $v_m = v_{mm}$ with $m \geqslant 1$. Thus, the application of assumption (A_1) and Lemma 2.6 gives that

$$\limsup_{n\to\infty} \int_{B_\tau} |v_n|^{q+q_k}\,\mathrm{d}x \leqslant \int_{B_\tau} |v_0|^{q+q_k}\,\mathrm{d}x \quad \forall\, k \geqslant N-1 \tag{2.69}$$

and

$$\text{(2.70)}\quad \limsup_{n\to\infty}\int_{B_\tau}|h(f(v_n))f'(v_n)v_n|\,\mathrm{d}x \leqslant \limsup_{n\to\infty}\sum_{k=N-1}^{\infty}\frac{b_1\alpha_0^k}{k!}\int_{B_\tau}|v_n|^{q+q_k}\,\mathrm{d}x$$

$$\leqslant \sum_{k=N-1}^{\infty}\frac{b_1\alpha_0^k}{k!}\limsup_{n\to\infty}\int_{B_\tau}|u_n|^{q+q_k}\,\mathrm{d}x \leqslant \sum_{k=N-1}^{\infty}\frac{b_1\alpha_0^k}{k!}\int_{B_\tau}|v_0|^{q+q_k}\,\mathrm{d}x$$

$$= b_1\int_{B_\tau}|v_0|^q R(\alpha_0, v_0)\,\mathrm{d}x.$$

This yields (2.64). Therefore, limit (2.65) holds. Similarly, we can derive that

$$\text{(2.71)}\qquad \lim_{n\to\infty}\int_{B_r}H(f(v_n))\,\mathrm{d}x = \int_{B_r}H(f(v_0))\,\mathrm{d}x.$$

In the following, we prove that, for any small $\varepsilon > 0$, there exists a large $r_0 > 1$ such that $r \geqslant r_0$ and

$$\text{(2.72)}\qquad \int_{B_r^{\mathrm{c}}}|h(f(v_n))f'(v_n)v_n|\,\mathrm{d}x < \varepsilon,\quad \int_{B_r^{\mathrm{c}}}|H(f(v_n))|\,\mathrm{d}x < \varepsilon \quad \forall\, n \geqslant 1,$$

$$\int_{B_r^{\mathrm{c}}}|h(f(v_0))f'(v_0)v_0|\,\mathrm{d}x < \varepsilon,\quad \int_{B_r^{\mathrm{c}}}|H(f(v_0))|\,\mathrm{d}x < \varepsilon.$$

In fact, it follows from (2.17) that

$$\text{(2.73)}\qquad \int_{B_r^{\mathrm{c}}}|v_n|^{q+q_k}\,\mathrm{d}x \leqslant M^{q_k+q}\omega_N^{-(q_k+q)/N}\int_r^{\infty}\int_{|\omega|=1}\varrho^{N-1-q_k-q}\,\mathrm{d}\omega\,\mathrm{d}\varrho$$

$$= M^{q_k+q}N\omega_N^{1-(q_k+q)/N}\frac{r^{N-q_k-q}}{q_k+q-N}$$

$$\leqslant \frac{r^{-q}}{q}M^qN\omega_N^{(N-q)/N}\big(M^{N/(N-1)}\omega_N^{-1/(N-1)}\big)^k$$

$$\equiv A^k\frac{r^{-q}}{q}M^qN\omega_N^{(N-q)/N}$$

and

$$\text{(2.74)}\qquad \int_{B_r^{\mathrm{c}}}|h(f(v_n))f'(v_n)v_n|\,\mathrm{d}x \leqslant \sum_{k=N-1}^{\infty}\frac{b_1\alpha_0^k}{k!}\int_{B_r^{\mathrm{c}}}|v_n|^{q+q_k}$$

$$\leqslant b_1\frac{r^{-q}}{q}M^qN\omega_N^{(N-q)/N}\sum_{m=N-1}^{\infty}\frac{(A\alpha_0)^m}{m!}\to 0 \quad \text{as } r\to\infty.$$

This shows that for any $\varepsilon > 0$, there is a large $r_0 > 1$ such that $r > r_0$ and

$$\int_{B_r^c} |h(f(v_n))f'(v_n)v_n| \, dx < \varepsilon \quad \forall n \geqslant 1. \tag{2.75}$$

The other three inequalities in (2.72) can be proved similarly. Now, the application of (2.65) and (2.75) yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} h(f(v_n))f'(v_n)v_n \, dx = \int_{\mathbb{R}^N} h(f(v_0))f'(v_0)v_0 \, dx. \tag{2.76}$$

Similarly, we can show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} H(f(v_n)) \, dx = \int_{\mathbb{R}^N} H(f(v_0)) \, dx \tag{2.77}$$

and (2.57) holds. This completes the proof of Lemma 2.15. □

3. Proof of Theorem 1.4

We now can prove the main result in this paper by dint of lemmas in Section 2.

P r o o f of Theorem 1.4. Clearly, it is sufficient to prove that v_0 is a critical point for J in E, that is, $J'(v_0)v = 0$ for all $v \in E$ and thus $J'(v_0) = 0$ in E^*, where v_0 is in the position of Lemma 2.15.

For every $v \in E$, we choose $\varepsilon > 0$ such that $w_s = v_0 + sv \neq 0$ for all $s \in (-\varepsilon, \varepsilon)$. Clearly, $w_0 = v_0$. Define a function $\varphi\colon (-\varepsilon, \varepsilon) \times (0, \infty) \to \mathbb{R}$ by

$$\begin{aligned}\varphi(s,t) &= J'(t(w_s))t(w_s)\\ &= -\int_{\mathbb{R}^N} h(f(t(w_s)))f'(t(w_s))t(w_s) \, dx\\ &\quad + \|\nabla t(w_s)\|_N^N + b\int_{R^N} |f(t(w_s))|^{N-2} f(t(w_s))f'(t(w_s))t(w_s) \, dx.\end{aligned}$$

Then $J'(v_0)v_0 = 0$ and

$$\begin{aligned}\frac{\partial\varphi}{\partial t}(0,1) &= b(N-1)\int_{\mathbb{R}^N} (f(v_0))^{N-2}((f'(v_0))^2 v_0^2 - f(v_0)f'(v_0)v_0) \, dx\\ &\quad + \int_{\mathbb{R}^N} (b(f(v_0))^{N-1} f''(v_0)v_0^2 + F(v_0)) \, dx,\end{aligned} \tag{3.1}$$

where $f(v_0) \geqslant 0$, $f(v_0) - f'(v_0)v_0 \geqslant 0$, $f''(v_0) < 0$ and

$$
\begin{aligned}
(3.2)\quad F(v_0) &= (N-1)h(f)f'(v_0)v_0 - h(f)f''(v_0)v_0^2 - h'(f)(f'(v_0))^2v_0^2 \\
&\leqslant (N-1)h(f)f'(v_0)v_0 - h(f)f''(v_0)v_0^2 - \frac{(q-1+2N)h(f)(f'(v_0))^2v_0^2}{f} \\
&\leqslant \frac{h(f)}{f}[2(N-1)(f'(v_0))^2v_0^2 - (q-1+2N)(f'(v_0))^2v_0^2 - f''(v_0)fv_0^2] \\
&= \frac{v_0^2h(f)}{f(1+2^{N-1}|f|^N)^{2/N}}\Big(\frac{2^{N-1}|f|^N}{1+2^{N-1}|f|^N} - q - 1\Big) \leqslant 0,
\end{aligned}
$$

provided that $q > 0$, where $f = f(v_0)$. Therefore, $\partial\varphi/\partial t(0,1) < 0$.

So, by the implicit function theorem, there exists ε_0, $0 < \varepsilon_0 \leqslant \varepsilon$, and a C^1 function $t\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ such that $t(0) = 1$ and $\varphi(s, t(s)) = 0$ for all $s \in (-\varepsilon_0, \varepsilon_0)$. This also shows that $t(s) \neq 0$, at least for ε_0 very small. Therefore, $t(s)(u + sv) \in \mathcal{N}$. Denote $t = t(s), w_s = v_0 + sv$ and

$$
\begin{aligned}
(3.3)\qquad \phi(s) &= J(t(w_s)) \\
&= \frac{1}{N}\int_{\mathbb{R}^N}(|\nabla t(w_s)|^N + b|f(t(w_s))|^N)\,\mathrm{d}x - \int_{\mathbb{R}^N} H(f(t(w_s)))\,\mathrm{d}x.
\end{aligned}
$$

We see that the function $\phi(s)$ is differentiable and has a minimum point at $s = 0$. Thus,

$$
(3.4)\qquad 0 = \phi'(0) = t'(0)J'(v_0)v_0 + J'(v_0)v.
$$

Since $v_0 \in \mathcal{N}$ and $J'(v_0)v_0 = 0$, it follows from (3.4) that $J'(v_0)v = 0$ for every $v \in E$ and thus $J'(v_0) = 0$ in E^*. So, v_0 is a critical point for J and then v_0 is a weak solution of (2.14) in E, and so $u = f(v_0)$ is a weak solution of (1.1). Since $J(v_0) = J(|v_0|) = d > 0$, we can assume that v_0 is nonnegative and nontrivial in $\mathbb{R}^N$ and so is $u_0(x) = f(v_0(x))$ in $\mathbb{R}^N$. On the other hand, it follows from Remark 2.8 that $v_0(x) \to 0$ as $|x| \to \infty$. Since $|u_0(x)| = |f(v_0(x))| \leqslant |v_0(x)|$ in $\mathbb{R}^N$, we have $u_0(x) \to 0$ as $|x| \to \infty$. Then the proof of Theorem 1.4 is completed.

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