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$\mathcal{D}_{n,r}$ IS NOT POTENTIALLY NILPOTENT FOR $n \geqslant 4r-2$

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Dedicated to the memory of Miroslav Fiedler

Abstract. An $n \times n$ sign pattern $\mathcal{A}$ is said to be potentially nilpotent if there exists a nilpotent real matrix B with the same sign pattern as $\mathcal{A}$. Let $\mathcal{D}_{n,r}$ be an $n \times n$ sign pattern with $2 \leqslant r \leqslant n$ such that the superdiagonal and the (n,n) entries are positive, the $(i,1)$ $(i=1,\dots,r)$ and $(i,i-r+1)$ $(i=r+1,\dots,n)$ entries are negative, and zeros elsewhere. We prove that for $r \geqslant 3$ and $n \geqslant 4r-2$, the sign pattern $\mathcal{D}_{n,r}$ is not potentially nilpotent, and so not spectrally arbitrary.

Keywords: sign pattern; potentially nilpotent pattern; spectrally arbitrary pattern

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1. Introduction

A *sign pattern* $\mathcal{A}$ is a matrix whose entries are from the set $\{+,-,0\}$. Associated with each sign pattern $\mathcal{A}=(a_{ij})$ is a class of real matrices, called the *qualitative class* of $\mathcal{A}$, defined by

$$Q(\mathcal{A})=\{B=(b_{ij})\colon\ B \text{ is an } n\times n \text{ real matrix, and sign } b_{ij}=a_{ij}\ \forall\, i,j\}.$$

An $n \times n$ sign pattern $\mathcal{A}$ is a *spectrally arbitrary sign pattern* (SAP) if for any given real monic polynomial $f(x)$ with degree n, there exists a real matrix $B \in Q(\mathcal{A})$ with characteristic polynomial $f(x)$. A sign pattern $\mathcal{A}$ is a minimal SAP (MSAP) if $\mathcal{A}$ is a SAP, but is not a SAP if one or more nonzero entries are replaced by zero.

An $n \times n$ sign pattern $\mathcal{A}$ is *potentially nilpotent* if there exists $B \in Q(\mathcal{A})$ such that B is nilpotent, i.e., there exists $B \in Q(\mathcal{A})$ with characteristic polynomial $f(x)=x^n$. In particular, each SAP must necessarily be potentially nilpotent.

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A *permutation sign pattern* is a square sign pattern with entries from the set $\{0, +\}$, where the entry $+$ occurs precisely once in each row and in each column. A *signature sign pattern* is a square diagonal sign pattern all of whose diagonal entries are nonzero. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two square sign patterns of the same order. A sign pattern $\mathcal{A}_1$ is said to be *permutationally similar* to $\mathcal{A}_2$ if there exists a permutation sign pattern $\mathcal{P}$ such that $\mathcal{A}_2 = \mathcal{P}^{\mathrm{T}}\mathcal{A}_1\mathcal{P}$. A sign pattern $\mathcal{A}_1$ is said to be *signature similar* to $\mathcal{A}_2$ if there exists a signature sign pattern $\mathcal{D}$ such that $\mathcal{A}_2 = \mathcal{D}\mathcal{A}_1\mathcal{D}$. The properties of being potentially nilpotent and spectrally arbitrary are preserved under negation, transposition, signature similarity and permutation similarity. Two sign patterns are said to be *equivalent* if one can be obtained from the other by any combination of these four operations.

Cavers and Vander Meulen [3] considered an interesting $n \times n$ sign pattern $\mathcal{D}_{n,r}$ with $2n$ nonzero entries defined as follows:

▷ $\mathcal{D}_{n,r}$ is an $n \times n$ sign pattern with $2 \leqslant r \leqslant n$ such that the superdiagonal and the (n, n) entries are positive, the $(i, 1)$, $i = 1, \ldots, r$, and $(i, i-r+1)$, $i = r+1, \ldots, n$, entries are negative, and zeros elsewhere.

They proved that if $n \leqslant 2r$, then $\mathcal{D}_{n,r}$ is a MSAP. If $r \geqslant 3$, then $\mathcal{D}_{2r+1,r}$ is not potentially nilpotent.

More recently, Gao et al. in [4] proved that if $r \geqslant 3$ and $2r + 2 \leqslant n \leqslant 4r - 3$, then $\mathcal{D}_{n,r}$ is not potentially nilpotent and thus not a SAP. Garnett and Shader in [5] proved that $\mathcal{D}_{n,2}$ is a SAP for all $n \geqslant 2$.

To the best of our knowledge, for $r \geqslant 3$ and $n \geqslant 4r-2$, some questions about $\mathcal{D}_{n,r}$ are still open ([2], page 3084). For instance, "Is $\mathcal{D}_{n,r}$ a SAP?"

Our main result answers the above question:

Theorem 1.1. *If $r \geqslant 3$ and $n \geqslant 4r - 2$, then $\mathcal{D}_{n,r}$ is not potentially nilpotent and thus not a* SAP.

2. Preliminary

Let $A = (a_{ij})$ be an $n \times n$ matrix. The *directed graph (digraph)* $D(A)$ of A is the directed graph with vertex set $\{1, 2, \ldots, n\}$ such that there is a directed edge in $D(A)$ from i to j, denoted by $i \to j$, if and only if $a_{ij} \neq 0$.

A *directed path* of length $k-1$ in $D(A)$ is a sequence of $k-1$ edges $i_1 \to i_2 \to \ldots \to i_{k-1} \to i_k$ such that the vertices are distinct. A *simple cycle* of length k in $D(A)$ consists of a directed path as above together with the additional directed edge $i_k \to i_1$. A (*composite*) *k-cycle* is a set of simple cycles whose total length is k, and whose index sets are mutually disjoint.

A nonzero product of the form $\gamma = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1}$ in which the index set $\{i_1, i_2, \dots, i_k\}$ consists of distinct indices is called a *simple cycle of length k* of A. A *composite k-cycle* is a product of simple cycles whose total length is k, and whose index sets are mutually disjoint.

A cycle (simple or composite) of a matrix A just corresponds to $\pm$ a term in the principle minor of A based upon the indices appearing in the cycle.

Note that the cycles of a matrix A correspond exactly to the cycles of the digraph $D(A)$ (see [1]).

Let $D_{n,r}$ be a real matrix in $Q(\mathcal{D}_{n,r})$ with $r \geqslant 3$ and $n \geqslant 4r-3$. Up to equivalence, we may assume $D_{n,r}$ has the form

$$(2.1)\qquad D_{n,r} = \begin{bmatrix} -a_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ -a_2 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -a_r & 0 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & -a_{r+1} & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 1 \\ 0 & 0 & 0 & -a_n & \dots & 0 & b \end{bmatrix},$$

where b and all the a_i, $i = 1, 2, \dots, n$, are positive.

The digraph $D(D_{n,r})$ is as follows:

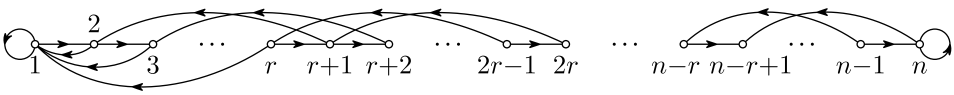

Figure 1. The digraph $D(D_{n,r})$.

3. The characteristic polynomial of $D_{n,r}$

The characteristic polynomial of a real matrix B of order n is given [6] by

$$p_B(t) = t^n - E_1(B)t^{n-1} + E_2(B)t^2 + \dots + (-1)^n E_n(B),$$

where $E_k(B)$ is the sum of the $k \times k$ principal minors of B, $k = 1, 2, \dots, n$. We use $B\{i_1, i_2, \dots, i_k\}$ to denote the $k \times k$ principal minor of B based on the indices $\{i_1, i_2, \dots, i_k\}$.

For $D_{n,r}$ in the form (2.1) with $r \geqslant 3$ and $n \geqslant 4r-3$, we have the following results, which are important in the proof of Theorem 1.1.

Lemma 3.1. $E_1(D_{n,r}) = b - a_1$, $E_2(D_{n,r}) = -ba_1 + a_2$.

Proof. It is clear that $E_1(D_{n,r}) = \operatorname{tr}(D_{n,r}) = b - a_1$. The only two 2×2 nonzero principal minors of $D_{n,r}$ are $D_{n,r}\{1,2\} = a_2$ and $D_{n,r}\{1,n\} = -ba_1$. Thus, $E_2(D_{n,r}) = -ba_1 + a_2$. □

Lemma 3.2. $E_r(D_{n,r}) = (-1)^{r-1}ba_{r-1} + (-1)^r \sum\limits_{i=r}^{n} a_i$.

Proof. All r-cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$i \to i+1 \to \ldots \to i+r-1 \to i, \quad i = 1,2,\ldots,n-r+1,$$
$$(1 \to 2 \to \ldots \to r-1 \to 1) \cup (n \to n).$$

The corresponding nonzero $r \times r$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{i, i+1, \ldots, i+r-1\} = (-1)^r a_{i+r-1}, \quad i = 1,2,\ldots,n-r+1,$$
$$D_{n,r}\{1,2,\ldots,r-1,n\} = (-1)^{r-1} ba_{r-1}.$$

So, $E_r(D_{n,r}) = (-1)^{r-1}ba_{r-1} + (-1)^r \sum\limits_{i=r}^{n} a_i$. □

Lemma 3.3. $E_{r+1}(D_{n,r}) = (-1)^r b \sum\limits_{i=r}^{n-1} a_i - (-1)^r a_1 \sum\limits_{i=r+1}^{n} a_i$.

Proof. All $(r+1)$-cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$(i \to i+1 \to \ldots \to i+r-1 \to i) \cup (n \to n), \quad i = 1,2,\ldots,n-r,$$
$$(1 \to 1) \cup (i \to i+1 \to \ldots \to i+r-1 \to i), \quad i = 2,3,\ldots,n-r+1.$$

The corresponding nonzero $(r+1) \times (r+1)$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{i, i+1, \ldots, i+r-1, n\} = (-1)^r a_{i+r-1} b, \quad i = 1,2,\ldots,n-r,$$
$$D_{n,r}\{1, i, i+1, \ldots, i+r-1\} = -a_1(-1)^r a_{i+r-1}, \quad i = 2,3,\ldots,n-r+1.$$

So, $E_{r+1}(D_{n,r}) = (-1)^r b \sum\limits_{i=r}^{n-1} a_i - (-1)^r a_1 \sum\limits_{i=r+1}^{n} a_i$. □

Lemma 3.4. $E_{r+2}(D_{n,r}) = (-1)^{r+1} a_1 b \sum\limits_{i=r+1}^{n-1} a_i + (-1)^r a_2 \sum\limits_{i=r+2}^{n} a_i$.

Proof. All $(r+2)$-cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$(1 \to 1) \cup (i \to i+1 \to \ldots \to i+r-1 \to i) \cup (n \to n), \quad i = 2, 3, \ldots, n-r,$$
$$(1 \to 2 \to 1) \cup (i \to i+1 \to \ldots \to i+r-1 \to i), \quad i = 3, 4, \ldots, n-r+1.$$

The corresponding nonzero $(r+2) \times (r+2)$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{1, i, i+1, \ldots, i+r-1, n\} = (-1)^{r+1} a_1 b a_{i+r-1}, \quad i = 2, 3, \ldots, n-r,$$
$$D_{n,r}\{1, 2, i, i+1, \ldots, i+r-1\} = (-1)^{r} a_2 a_{i+r-1}, \quad i = 3, 4, \ldots, n-r+1.$$

So, $E_{r+2}(D_{n,r}) = (-1)^{r+1} a_1 b \sum\limits_{i=r+1}^{n-1} a_i + (-1)^r a_2 \sum\limits_{i=r+2}^{n} a_i.$ □

Lemma 3.5. $E_{2r}(D_{n,r}) = -a_{r-1} b \sum\limits_{i=2r-1}^{n-1} a_i + \sum\limits_{i=r}^{n-r} \sum\limits_{j=i+r}^{n} a_i a_j.$

Proof. All $2r$-cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$(1 \to 2 \to \ldots \to r-1 \to 1) \cup (i \to i+1 \to \ldots \to i+r-1 \to i) \cup (n \to n),$$
$$i = r, r+1, \ldots, n-r,$$
$$(i \to i+1 \to \ldots \to i+r-1 \to i) \cup (j \to j+1 \to \ldots \to j+r-1 \to j),$$
$$i = 1, 2, \ldots, n-2r+1, \quad j = i+r, i+r+1, \ldots, n-r+1.$$

The corresponding nonzero $2r \times 2r$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{1, 2, \ldots, r-1, i, i+1, \ldots, i+r-1, n\} = -a_{r-1} b a_{i+r-1},$$
$$i = r, r+1, \ldots, n-r,$$
$$D_{n,r}\{i, i+1, \ldots, i+r-1, j, j+1, \ldots, j+r-1\} = a_{i+r-1} a_{j+r-1},$$
$$i = 1, 2, \ldots, n-2r+1, \quad j = i+r, i+r+1, \ldots, n-r+1.$$

So, $E_{2r}(D_{n,r}) = -a_{r-1} b \sum\limits_{i=2r-1}^{n-1} a_i + \sum\limits_{i=r}^{n-r} \sum\limits_{j=i+r}^{n} a_i a_j.$ □

Lemma 3.6. $E_{2r+1}(D_{n,r}) = b \sum\limits_{i=r}^{n-r-1} \sum\limits_{j=i+r}^{n-1} a_i a_j - a_1 \sum\limits_{i=r+1}^{n-r} \sum\limits_{j=i+r}^{n} a_i a_j.$

Proof. All $(2r+1)$-cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$(i \to i+1 \to \ldots \to i+r-1 \to i) \cup (j \to j+1 \to \ldots \to j+r-1 \to j) \cup (n \to n),$$
$$i = 1, 2, \ldots, n-2r, \quad j = i+r, i+r+1, \ldots, n-r,$$
$$(1 \to 1) \cup (i \to i+1 \to \ldots \to i+r-1 \to i) \cup (j \to j+1 \to \ldots \to j+r-1 \to j),$$
$$i = 2, 3, \ldots, n-2r+1, \quad j = i+r, i+r+1, \ldots, n-r+1.$$

The corresponding nonzero $(2r+1)\times(2r+1)$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{i,i+1,\ldots,i+r-1,j,j+1,\ldots,j+r-1,n\} = ba_{i+r-1}a_{j+r-1},$$
$$i=1,2,\ldots,n-2r,\quad j=i+r,i+r+1,\ldots,n-r,$$
$$D_{n,r}\{1,i,i+1,\ldots,i+r-1,j,j+1,\ldots,j+r-1\} = -a_1a_{i+r-1}a_{j+r-1},$$
$$i=2,3,\ldots,n-2r+1,\quad j=i+r,i+r+1,\ldots,n-r+1.$$

So, $E_{2r+1}(D_{n,r}) = b\sum\limits_{i=r}^{n-r-1}\sum\limits_{j=i+r}^{n-1}a_ia_j - a_1\sum\limits_{i=r+1}^{n-r}\sum\limits_{j=i+r}^{n}a_ia_j.$ □

Lemma 3.7. $E_{2r+2}(D_{n,r}) = -a_1b\sum\limits_{i=r+1}^{n-r-1}\sum\limits_{j=i+r}^{n-1}a_ia_j + a_2\sum\limits_{i=r+2}^{n-r}\sum\limits_{j=i+r}^{n}a_ia_j.$

Proof. All $(2r+2)$-cycles in the digraph $D(\mathcal{D}_{n,r})$ are

$$(1\to 1)\cup(i\to i+1\to\ldots\to i+r-1\to i)\cup(j\to j+1\to\ldots\to j+r-1\to j)$$
$$\cup\,(n\to n),\qquad i=2,3,\ldots,n-2r,\quad j=i+r,i+r+1,\ldots,n-r,$$
$$(1\to 2\to 1)\cup(i\to i+1\to\ldots\to i+r-1\to i)\cup(j\to j+1\to\ldots\to j+r-1\to j),$$
$$i=3,4,\ldots,n-2r+1,\quad j=i+r,i+r+1,\ldots,n-r+1.$$

The corresponding nonzero $(2r+2)\times(2r+2)$ principal minors of $D_{n,r}$ are

$$D_{n,r}\{1,i,i+1,\ldots,i+r-1,j,j+1,\ldots,j+r-1,n\} = -a_1ba_{i+r-1}a_{j+r-1}$$
$$i=2,3,\ldots,n-2r,\quad j=i+r,i+r+1,\ldots,n-r,$$
$$D_{n,r}\{1,2,i,i+1,\ldots,i+r-1,j,j+1,\ldots,j+r-1\} = a_2a_{i+r-1}a_{j+r-1}$$
$$i=3,4,\ldots,n-2r+1,\quad j=i+r,i+r+1,\ldots,n-r+1.$$

So, $E_{2r+2}(D_{n,r}) = -a_1b\sum\limits_{i=r+1}^{n-r-1}\sum\limits_{j=i+r}^{n-1}a_ia_j + a_2\sum\limits_{i=r+2}^{n-r}\sum\limits_{j=i+r}^{n}a_ia_j.$ □

4. Proof of Theorem 1.1

We will prove the theorem by contradiction.

Let $r\geqslant 3$ and $n\geqslant 4r-2$. Suppose $\mathcal{D}_{n,r}$ is potentially nilpotent. Then there exists $D_{n,r}\in Q(\mathcal{D}_{n,r})$ in the form (2.1) such that $E_k(D_{n,r})=0$ for $k=1,2,\ldots,n$.

By Lemma 3.1, we have $E_1(D_{n,r}) = b-a_1 = 0$ and $E_2(D_{n,r}) = -ba_1+a_2 = 0$. Thus, $b=a_1$ and $a_2=ba_1$.

By Lemma 3.2, we have $E_r(D_{n,r}) = (-1)^{r-1}ba_{r-1} + (-1)^r \sum_{i=r}^{n} a_i = 0$. Thus,

$$ba_{r-1} = \sum_{i=r}^{n} a_i. \tag{4.1}$$

By Lemma 3.3, we have $E_{r+1}(D_{n,r}) = (-1)^r b \sum_{i=r}^{n-1} a_i - (-1)^r a_1 \sum_{i=r+1}^{n} a_i = 0$. Thus, $b \sum_{i=r}^{n-1} a_i = a_1 \sum_{i=r+1}^{n} a_i$. Since $b = a_1$, we get $a_r = a_n$.

By Lemma 3.4, we have $E_{r+2}(D_{n,r}) = (-1)^{r+1} a_1 b \sum_{i=r+1}^{n-1} a_i + (-1)^r a_2 \sum_{i=r+2}^{n} a_i = 0$. Thus, $a_1 b \sum_{i=r+1}^{n-1} a_i = a_2 \sum_{i=r+2}^{n} a_i$. Since $ba_1 = a_2$, we get $a_{r+1} = a_n$.

By Lemma 3.6, we have $E_{2r+1}(D_{n,r}) = b \sum_{i=r}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j - a_1 \sum_{i=r+1}^{n-r} \sum_{j=i+r}^{n} a_i a_j = 0$. Thus, $b \sum_{i=r}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j = a_1 \sum_{i=r+1}^{n-r} \sum_{j=i+r}^{n} a_i a_j$. Since $b = a_1$, we get

$$\sum_{i=r}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j = \sum_{i=r+1}^{n-r} \sum_{j=i+r}^{n} a_i a_j,$$

and so,

$$a_r(a_{2r} + a_{2r+1} + \ldots + a_{n-1}) = a_n(a_{r+1} + a_{r+2} + \ldots + a_{n-r}).$$

Noting that $a_r = a_n$, we have

$$a_{2r} + a_{2r+1} + \ldots + a_{n-1} = a_{r+1} + a_{r+2} + \ldots + a_{n-r}. \tag{4.2}$$

By Lemma 3.7, we have

$$E_{2r+2}(D_{n,r}) = -a_1 b \sum_{i=r+1}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j + a_2 \sum_{i=r+2}^{n-r} \sum_{j=i+r}^{n} a_i a_j = 0.$$

Thus $a_1 b \sum_{i=r+1}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j = a_2 \sum_{i=r+2}^{n-r} \sum_{j=i+r}^{n} a_i a_j$. Since $ba_1 = a_2$, we get

$$\sum_{i=r+1}^{n-r-1} \sum_{j=i+r}^{n-1} a_i a_j = \sum_{i=r+2}^{n-r} \sum_{j=i+r}^{n} a_i a_j,$$

and so,

$$a_{r+1}(a_{2r} + a_{2r+1} + \ldots + a_{n-1}) = a_n(a_{r+2} + a_{r+2} + \ldots + a_{n-r}).$$

Noting that $a_{r+1} = a_n$, we have

$$a_{2r+1} + a_{2r+2} + \ldots + a_{n-1} = a_{r+2} + a_{r+3} + \ldots + a_{n-r}. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain $a_{r+1} = a_{2r}$.

By Lemma 3.5 and (4.1), that is, $ba_{r-1} = \sum_{i=r}^{n} a_i$, we have

$$E_{2r}(D_{n,r}) = -a_{r-1}b \sum_{i=2r-1}^{n-1} a_i + \sum_{i=r}^{n-r} \sum_{j=i+r}^{n} a_i a_j = -\sum_{i=r}^{n} a_i \sum_{j=2r-1}^{n-1} a_j + \sum_{i=r}^{n-r} a_i \sum_{j=i+r}^{n} a_j.$$

By the assumption $n \geqslant 4r-2$, we have $n-r-1 \geqslant 3r-3 > r$. Thus the above expression can be written as

$$\begin{aligned}
E_{2r}(D_{n,r}) &= \sum_{i=r}^{n-r} a_i \sum_{j=i+r}^{n} a_j - \sum_{i=r}^{n} a_i \sum_{j=2r-1}^{n-1} a_j \\
&= \sum_{i=r}^{n-r-1} a_i \Bigg(\sum_{j=i+r}^{n} a_j - \sum_{j=2r-1}^{n-1} a_j \Bigg) + a_{n-r}a_n \\
&\quad - \sum_{i=n-r}^{n-1} a_i \sum_{j=2r-1}^{n-1} a_j - a_n \sum_{j=2r-1}^{n-1} a_j \\
&= \sum_{i=r}^{n-r-1} a_i \Bigg(a_n - \sum_{j=2r-1}^{i+r-1} a_j \Bigg) + a_{n-r}a_n - \sum_{i=n-r}^{n-1} a_i \sum_{j=2r-1}^{n-1} a_j - a_n \sum_{j=2r-1}^{n-1} a_j \\
&= a_n \sum_{i=r}^{n-r} a_i - \sum_{i=r}^{n-r-1} a_i \sum_{j=2r-1}^{i+r-1} a_j - \sum_{i=n-r}^{n-1} a_i \sum_{j=2r-1}^{n-1} a_j - a_n \sum_{j=2r-1}^{n-1} a_j.
\end{aligned}$$

By (4.2),

$$a_n \sum_{i=r}^{n-r} a_i - a_n \sum_{j=2r-1}^{n-1} a_j = a_n(a_r - a_{2r-1}).$$

Then according to the assumption that $a_i > 0$ for $i = 1, 2, \ldots, n$ and the known result $a_r = a_{r+1} = a_{2r} = a_n$, we have

$$\begin{aligned}
E_{2r}(D_{n,r}) &= a_n(a_r - a_{2r-1}) - \sum_{i=r}^{n-r-1} a_i \sum_{j=2r-1}^{i+r-1} a_j - \sum_{i=n-r}^{n-1} a_i \sum_{j=2r-1}^{n-1} a_j \\
&< a_n(a_r - a_{2r-1}) - a_{r+1}(a_{2r-1} + a_{2r}) \\
&= a_n(a_n - a_{2r-1}) - a_n(a_{2r-1} + a_n) = -2a_n a_{2r-1} < 0,
\end{aligned}$$

contradicting the identity $E_{2r}(D_{n,r}) = 0$. So $\mathcal{D}_{n,r}$ is not potentially nilpotent. □

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