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THE REAL SYMMETRIC MATRICES OF ODD ORDER WITH A P-SET OF MAXIMUM SIZE

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Dedicated to the memory of Miroslav Fiedler

Abstract. Suppose that A is a real symmetric matrix of order n. Denote by $m_A(0)$ the nullity of A. For a nonempty subset α of $\{1, 2, \ldots, n\}$, let $A(\alpha)$ be the principal submatrix of A obtained from A by deleting the rows and columns indexed by α. When $m_{A(\alpha)}(0) = m_A(0) + |\alpha|$, we call α a P-set of A. It is known that every P-set of A contains at most $\lfloor n/2 \rfloor$ elements. The graphs of even order for which one can find a matrix attaining this bound are now completely characterized. However, the odd case turned out to be more difficult to tackle. As a first step to the full characterization of these graphs of odd order, we establish some conditions for such graphs G under which there is a real symmetric matrix A whose graph is G and contains a P-set of size $(n-1)/2$.

Keywords: real symmetric matrix; graph; multiplicity of eigenvalues; P-set; P-vertices

MSC 2010: 15A18, 05C50

1. Introduction

Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix. The graph of A, denoted by G_A, is defined to be the graph whose vertex set is $\{1, 2, \ldots, n\}$ and edge set is $\{ij\colon\ i \neq j \text{ and } a_{ij} \neq 0\}$. When the associated graph G_A is a tree, then the matrix A is said to be *acyclic*. Obviously, G_A does not depend on the main diagonal entries of A. On the other hand, for a given graph G on n vertices, we define

$$\mathcal{S}(G) = \{A \in \mathbb{R}^{n\times n}\colon\ A \text{ is symmetric and } G_A = G\}.$$

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Therefore, matrices representing the underlying graph are not confined to the $(0,1)$ standard case. Indeed, graphs have become an important tool for the characterization of some spectral and combinatorial properties of matrices.

Given a nonempty subset α of $\{1,2,\dots,n\}$, denote by $A(\alpha)$ the principal submatrix of A obtained from A by deleting the rows and columns indexed by α, and by $A[\alpha]$ the principal submatrix of A consisting of both the rows and columns indexed by α.

Similarly, given a graph G with vertices $1,2,\dots,n$, and a nonempty subset α of $\{1,2,\dots,n\}$, denote by $G(\alpha)$ the subgraph of G obtained from G by deleting the vertices indexed by α and the corresponding edges, and by $G[\alpha]$ the subgraph of G induced by the vertices in α. Clearly, if $A\in\mathcal{S}(G)$, then $A(\alpha)\in\mathcal{S}(G(\alpha))$ and $A[\alpha]\in\mathcal{S}(G[\alpha])$ for any nonempty subset α of $\{1,2,\dots,n\}$. For $\alpha=\{i\}$, $A(\{i\})$ is written as $A(i)$, and $A[\{i\}]$ is written as $A[i]$.

Denote by $m_A(\lambda)$ the (algebraic) multiplicity of an eigenvalue λ of A. We assume that $m_A(\lambda)=0$ provided λ is not an eigenvalue of A. From Cauchy's interlacing theorem, see [12], Theorem 4.3.17, we can get that

$$m_A(\lambda)-1\leqslant m_{A(i)}(\lambda)\leqslant m_A(\lambda)+1. \tag{1.1}$$

In the case of equality on the right-hand side of (1.1), with $\lambda=0$, i.e.,

$$m_{A(i)}(0)=m_A(0)+1,$$

the index i is known as a *P-vertex* of A, see [13], [14]. Notice that $m_A(0)$ corresponds to the nullity of A.

The analysis of the P-vertices of a matrix is an active topic of contemporary research in combinatorial matrix theory (cf., e.g., [7], [13], [14], [15], [16]). Specifically, Kim and Shader [15] proposed several interesting questions on the maximum number of P-vertices which have been answered in [1], [2], [8], [9], [10]. The attention on this issue has resurged since then.

By using the right-hand side of (1.1) repeatedly, we can further get that

$$m_{A(\alpha)}(\lambda)\leqslant m_A(\lambda)+|\alpha| \tag{1.2}$$

for any nonempty subset α of $\{1,2,\dots,n\}$. When we have equality in (1.2), with $\lambda=0$, i.e.,

$$m_{A(\alpha)}(0)=m_A(0)+|\alpha|,$$

we call α a *P-set* of A, see [13].

So, both the concepts of P-vertex and P-set of real symmetric matrices are intimately related to Cauchy's interlacing theorem. From it, each vertex in a P-set of A

is a P-vertex of A, see [15], Proposition 5 (see also [13] for proofs). Further results on the relationship between P-vertices and P-set can be found in the recent papers by Nelson and Shader [18], [17] and Fernandes and da Cruz [11].

Extensive work has also been done in relation to the 0-1 standard adjacency matrix in this regard. In [3] the concept of the maximum multiplicity that an eigenvalue can reach for the 0-1 matrices is related to star partitions. Moreover, classes of extremal graphs with 0-1 adjacency matrix of nullity one exist. They are referred to as singular configurations in [19] when the subgraph $G(V)$ (equivalently $G[\overline{V}]$) is edgeless and as bipartite minimal configurations in [20] when both $G[V]$ and $G[\overline{V}]$ are edgeless. In this context, the P-vertices are called *upper core-forbidden vertices.*

Interestingly, also from Cauchy's interlacing theorem, we can have an obvious but important observation: every subset of a P-set of A is also a P-set of A. It is also worth mentioning that, when α is a P-set of A, for each proper subset $\beta \subset \alpha$ (which is also a P-set of A), $\alpha \setminus \beta$ is a P-set of $A(\beta)$.

The maximal size of a P-set of A is denoted by $P_s(A)$, see [15]. In 2009, Kim and Shader in [15] established an upper bound for $P_s(A)$.

Lemma 1.1 ([15]). *For any real symmetric matrix A of order $n \geqslant 1$, we have*

$$P_s(A) \leqslant \left\lfloor \frac{n}{2} \right\rfloor.$$

Kim and Shader in [15] also verified that this upper bound $\lfloor \frac{1}{2}n \rfloor$ is tight for some nonsingular tridiagonal matrices and for a singular acyclic matrix of odd order. Motivated by Lemma 1.1, the authors in [6] completely characterized all the trees T on n vertices for which there exists a matrix $A \in \mathcal{S}(T)$ such that $P_s(A) = \lfloor \frac{1}{2}n \rfloor$. Later, since the upper bound $\lfloor \frac{1}{2}n \rfloor$ is not sharp for the singular acyclic matrices of even order, we showed in [5] that $\frac{1}{2}(n-2)$ is actually the tight bound for this case, and classified all of the underlying trees.

Recently, the first author in [4] extended all these analyses on maximal P-sets from acyclic matrices to general real symmetric matrices, investigating those matrices attaining the upper bound $\lfloor \frac{1}{2}n \rfloor$, and completely characterizing the associated graphs of even order. Such a characterization, given a nonempty proper vertex subset V of a graph G, $G[V, \overline{V}]$, was defined to be the subgraph of G induced by all the edges connecting the vertices of V with the vertices of $\overline{V}$.

Let K_n be the complete graph on n vertices. The notation sK_1 stands for an edgeless graph on s vertices, and sK_2 stands for s disjoint edges.

Theorem A ([4], Theorem 3)**.** *Let G be a graph on n vertices, where n is even. Then the following two conditions are equivalent:*

(a) *There exists a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}n$.*

(b) *There exists a vertex subset V, where $|V| = \frac{1}{2}n$, such that $G(V)$ is an edgeless graph, i.e., $G(V) = \frac{1}{2}nK_1$, and $\frac{1}{2}nK_2$ is a subgraph of $G[V, \overline{V}]$.*

We remark that $\frac{1}{2}nK_2$ being a subgraph of $G[V, \overline{V}]$ means that one can find $\frac{1}{2}n$ disjoint edges in $G[V, \overline{V}]$, which also means that G has a perfect matching.

Nonetheless, the odd case turned out to be particularly difficult to handle.

Definition 1.1. For odd n, let $\mathcal{G}_n$ be the set of graphs on n vertices, such that for each of these graphs, say G, there exists a vertex subset V, where $|V| = \frac{1}{2}(n-1)$, such that

(a) $G(V)$ is the disjoint union of a complete graph (possibly a singleton) and some isolated vertices, i.e., $G(V) = K_r \cup (\frac{1}{2}(n+1) - r)K_1$, where $r \geqslant 1$, and "$\cup$" represents the disjoint union of graphs;

(b) $\frac{1}{2}(n-1)K_2$ is a subgraph of $G[V, \overline{V}]$.

Observe that $G(V)$ is a particular case of a split graph, whose vertex set is $\overline{V}$. Therefore, we will call V a *split set* of G. Moreover, if E is the edge set of a subgraph $H := \frac{1}{2}(n-1)K_2$ of $G[V, \overline{V}]$, and v is the unique vertex in G that is not in H, then we say that v is the *outer vertex* of (V, E). Notice that $v \notin V$. Thus v is a vertex of $G(V)$, and for every edge in E, say $uw \in E$, one of u, w is in V, and the other is not in V.

For example, the graph depicted in Figure 1 is a graph in $\mathcal{G}_9$. In fact, denoting by G such a graph and setting $V = \{1, 2, 5, 6\}$ and $E = \{13, 24, 57, 68\}$, we have $G \in \mathcal{G}_9$, since V is a split set of G and E is the edge set of $4K_2$. In addition, 9 is the outer vertex of (V, E).

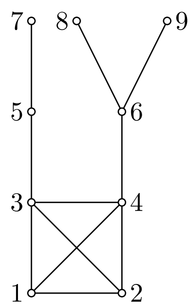

Figure 1. An example in $\mathcal{G}_9$.

The following properties were presented in [4] for the odd case.

Theorem B ([4], Theorem 2). *Let G be a graph on n vertices, where n is odd. If there exists a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$, then $G \in \mathcal{G}_n$. In particular, if V is such a P-set of $A \in \mathcal{S}(G)$ with $|V| = \frac{1}{2}(n-1)$, then V is a split set of G.*

In Theorem B, we only get a necessary condition for the existence of a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$. In order to get the complete characterization of the graphs G, we naturally wonder if the converse of Theorem B is also true. Actually, the answer is negative. A counterexample for that can be found with the graph in Figure 1, i.e., denoting by G the graph as shown in Figure 1, although $G \in \mathcal{G}_9$, there is no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = 4$; the detailed illustration will be presented in Proposition 4.4 of Section 4. More precisely, for every odd $n \geqslant 7$, we will there construct some counterexamples of order n to show that the converse of Theorem B is not true, including the connected and disconnected examples.

Our paper is organized as follows: in Section 2, some extra conditions will be added so that the converse of Theorem B comes true, i.e., we will present several sufficient conditions for the statement that there exists a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$; furthermore, in Section 3, we will present several necessary and sufficient conditions for that statement; as a consequence, in Section 4, we will construct a lot of counterexamples as many as we wish to show that the converse of Theorem B is not true. Finally, in Section 5, we will give a concluding remark on this topic.

2. Several sufficient conditions

We have seen before that the converse of Theorem B is not true, i.e., in general $G \in \mathcal{G}_n$ does not imply the existence of a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$. In this section, we add some extra conditions for $G \in \mathcal{G}_n$ in order to obtain a matrix $A \in \mathcal{S}(G)$ with $P_s(A) = \frac{1}{2}(n-1)$.

Let A be a real symmetric matrix of order n. Suppose that α is a nonempty proper subset of $\{1, 2, \ldots, n\}$. Let $A[\alpha, \overline{\alpha}]$ be the submatrix of A consisting of the rows indexed by α and the columns indexed by $\overline{\alpha}$.

By the definition of the graph $G \in \mathcal{G}_n$, we know that $G(V)$ is the disjoint union of a complete graph and some isolated vertices. Notice that the outer vertex of (V, E) is either a vertex of the complete subgraph of $G(V)$, or exactly an isolated vertex of $G(V)$.

First we consider the case when the outer vertex of (V, E) is a vertex of the complete subgraph of $G(V)$.

Theorem 2.1. *Let $G \in \mathcal{G}_n$. Suppose that V is a split set of G, E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V,\overline{V}]$, and v is the outer vertex of (V,E). If v is a vertex of the complete subgraph of $G(V)$, then there exists a matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A, implying that $P_s(A) = \frac{1}{2}(n-1)$.*

Proof. Without loss of generality, we may assume that $V = \{1,2,\ldots,\frac{1}{2}(n-1)\}$, and $\{\frac{1}{2}(n+1),\frac{1}{2}(n+3),\ldots,\frac{1}{2}(n-1)+r\}$ is the vertex subset of G corresponding to the vertices of the complete subgraph of $G(V)$, since $G(V) = K_r \cup (\frac{1}{2}(n+1)-r)K_1$, where $1 \leqslant r \leqslant \frac{1}{2}(n+1)$.

First, we construct a nonsingular matrix $A \in \mathcal{S}(G)$.

Since $G(V)$ is the disjoint union of a complete graph and some isolated vertices, i.e., $G(V) = K_r \cup (\frac{1}{2}(n+1)-r)K_1$, we may set A to be of the form

$$A = \begin{pmatrix} M & P & Q \\ P^{\mathrm{T}} & N & O \\ Q^{\mathrm{T}} & O & O \end{pmatrix},$$

where M is an $\frac{1}{2}(n-1)\times\frac{1}{2}(n-1)$ symmetric matrix, N is an $r \times r$ symmetric matrix with $1 \leqslant r \leqslant \frac{1}{2}(n+1)$, P is an $\frac{1}{2}(n-1)\times r$ matrix, and Q is an $\frac{1}{2}(n-1)\times(\frac{1}{2}(n+1)-r)$ matrix. In particular, $A[V] = M$,

$$A(V) = \begin{pmatrix} N & O \\ O & O \end{pmatrix},$$

and

$$A[V,\overline{V}] = (\,P \quad Q\,).$$

Since E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V,\overline{V}]$, we may choose $\frac{1}{2}(n-1)$ entries from

$$A[V,\overline{V}] = (\,P \quad Q\,)$$

corresponding to the $\frac{1}{2}(n-1)$ edges in E, such that no two entries come from the same row or column. Denote by β the collection of the indices of the rows and columns of the $\frac{1}{2}(n-1)$ chosen entries in

$$A[V,\overline{V}] = (\,P \quad Q\,).$$

Clearly, $|\beta| = n-1$.

In particular, in the matrix A,

(a) let the above $\frac{1}{2}(n-1)$ chosen entries in

$$A[V,\overline{V}] = (\,P \quad Q\,)$$

be 1;

(b) let every entry in N be x, where x is an undetermined constant;

(c) let the remaining nonzero entries of A be sufficiently close to 0.

Clearly, $|A|$ is a polynomial on x.

Observe that v is the outer vertex of (V, E). Thus $A(\beta) = A[v]$. Moreover, since v is a vertex of the complete subgraph of $G(V)$, we have

$$A(\beta) = A[v] = (\, x\,).$$

Using the above construction of A, we can deduce that the coefficient of x in $|A|$ is sufficiently close to 1 or -1, which implies that $|A|$ is a polynomial on x of degree at least 1. So we may assign a suitable nonzero value to x such that $|A| \neq 0$, i.e., $m_A(0) = 0$.

On the other hand, taking into account that the rank of

$$A(V) = \begin{pmatrix} N & O \\ O & O \end{pmatrix}$$

is 1, since every entry in N is $x \neq 0$, we have

$$m_{A(V)}(0) = \frac{n-1}{2} = 0 + \frac{n-1}{2} = m_A(0) + \frac{n-1}{2},$$

i.e., V is a P-set of A, which implies that $P_s(A) \geqslant \frac{1}{2}(n-1)$. Together with Lemma 1.1, we now get that $P_s(A) = \frac{1}{2}(n-1)$. $\square$

Let us take the graph in Figure 2 as an example for Theorem 2.1.

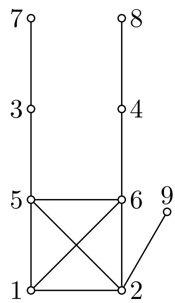

Figure 2. An example for Theorems 2.1 and 2.7.

Suppose that G is the graph as shown in Figure 2. By setting $V = \{1, 2, 3, 4\}$ and $E = \{15, 29, 37, 48\}$, we know that $G \in \mathcal{G}_9$, and 6 is the outer vertex of (V, E). Note that $G(V) = G[5,6,7,8,9]$ contains a complete subgraph on two vertices, i.e., the isolated edge 56, and three isolated vertices $7,8,9$. Let

$$A = \begin{pmatrix} 0 & 0.01 & 0 & 0 & 1 & 0.01 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0 & 0.01 & 0.01 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.01 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 1 & 0 \\ 1 & 0.01 & 0.01 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0.01 & 0.01 & 0 & 0.01 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}(G).$$

It is easily verified that $m_A(0) = 0$ and $m_{A(1,2,3,4)}(0) = 4$, which implies that $\{1,2,3,4\}$ is a P-set of A, and $P_s(A) = 4$.

By Theorem 2.1, we get the following corollary immediately.

Corollary 2.2. *Let C_n be the cycle on $n \geqslant 3$ vertices, where n is odd. Then there exists a matrix $A \in \mathcal{S}(C_n)$ such that $P_s(A) = \frac{1}{2}(n-1)$.*

On the other hand, for the cycle C_n, where $n \geqslant 4$ is even, by Theorem A we can get that there exists a matrix $A \in \mathcal{S}(C_n)$ such that $P_s(A) = \frac{1}{2}n$.

By the definition of the graph $G \in \mathcal{G}_n$, we know that $G(V)$ is the disjoint union of a complete graph and some isolated vertices. As a consequence of Theorem 2.1, we now consider two extreme cases:

(i) the complete subgraph of $G(V)$ contains only one vertex, i.e., $G(V)$ is exactly the edgeless graph $\frac{1}{2}(n+1)K_1$.

(ii) $G(V)$ contains no isolated vertex, i.e., $G(V)$ is actually the complete graph $K_{(n+1)/2}$.

We start with the case (i), when $G(V)$ is the edgeless graph $\frac{1}{2}(n+1)K_1$.

Theorem 2.3. *Let $G \in \mathcal{G}_n$. Suppose that V is a split set of G. If $G(V)$ is an edgeless graph, i.e., $G(V) = \frac{1}{2}(n+1)K_1$, then there exists a matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A, implying that $P_s(A) = \frac{1}{2}(n-1)$.*

Proof. Suppose that v is the outer vertex of (V,E), where E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V,\overline{V}]$.

Since $G(V)$ is an edgeless graph, i.e., the complete subgraph of $G(V)$ contains only one vertex, every (isolated) vertex of $G(V)$ can be regarded as the (unique) vertex of the complete subgraph of $G(V)$. So we may assume that the outer vertex v is the (unique) vertex of the complete subgraph of $G(V)$.

Now by Theorem 2.1, the result follows. □

Let us take the graph G depicted in Figure 3 as an example for Theorem 2.3.

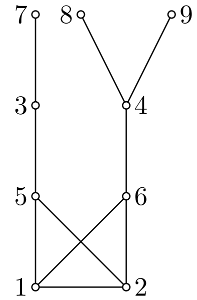

Figure 3. An example for Theorem 2.3.

By setting $V = \{1, 2, 3, 4\}$ and $E = \{15, 26, 37, 48\}$, we have $G \in \mathcal{G}_9$, and 9 is the outer vertex of (V, E). Note that $G(V) = G[5, 6, 7, 8, 9]$ is an edgeless graph. Let

$$A = \begin{pmatrix} 0 & 0.01 & 0 & 0 & 1 & 0.01 & 0 & 0 & 0 \\ 0.01 & 0 & 0 & 0 & 0.01 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.01 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.01 & 0 & 1 & 0.01 \\ 1 & 0.01 & 0.01 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.01 & 1 & 0 & 0.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \in \mathcal{S}(G).$$

It is easily verified that $m_A(0) = 0$ and $m_{A(1,2,3,4)}(0) = 4$, which implies that $\{1, 2, 3, 4\}$ is a P-set of A, and $P_s(A) = 4$.

Let $K_{r,s}$ be the complete bipartite graph with bipartition (X, Y), where $|X| = r$ and $|Y| = s$.

By Theorem 2.3, we get the following corollary.

Corollary 2.4. *For the complete bipartite graph $K_{(n-1)/2,(n+1)/2}$, where $n \geqslant 3$ is odd, there is a matrix $A \in \mathcal{S}(K_{(n-1)/2,(n+1)/2})$ such that $P_s(A) = \frac{1}{2}(n-1)$.*

On the other hand, for the complete bipartite graph $K_{n/2,n/2}$, where $n \geqslant 2$ is even, by Theorem A we can get that there exists a matrix $A \in \mathcal{S}(K_{n/2,n/2})$ such that $P_s(A) = \frac{1}{2}n$.

Now we consider the case (ii), when $G(V)$ is the complete graph $K_{(n+1)/2}$.

Theorem 2.5. *Let $G \in \mathcal{G}_n$. Suppose that V is a split set of G. If $G(V)$ is a complete graph, i.e., $G(V) = K_{(n+1)/2}$, then there exists a matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A, implying that $P_s(A) = \frac{1}{2}(n-1)$.*

P r o o f. Suppose that v is the outer vertex of (V, E), where E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V, \overline{V}]$.

Since $G(V)$ is a complete graph, every vertex of $G(V)$ is a vertex of the complete subgraph of $G(V)$. In particular, the outer vertex v is a vertex of the complete subgraph of $G(V)$.

Now by Theorem 2.1, the result follows. $\square$

Let us take the graph G in Figure 4 as an example for Theorem 2.5.

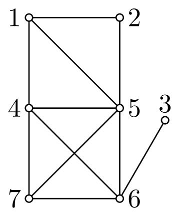

Figure 4. An example for Theorem 2.5.

By setting $V = \{1, 2, 3\}$ and $E = \{14, 25, 36\}$, we have $G \in \mathcal{G}_7$, and 7 is the outer vertex of (V, E). Note that $G(V) = G[4, 5, 6, 7]$ is the complete graph K_4. Let

$$A = \begin{pmatrix} 0 & 0.01 & 0 & 1 & 0.01 & 0 & 0 \\ 0.01 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0.01 & 1 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{pmatrix} \in \mathcal{S}(G).$$

It is easily verified that $m_A(0) = 0$ and $m_{A(1,2,3)}(0) = 3$, which implies that $\{1, 2, 3\}$ is a P-set of A, and $P_s(A) = 3$.

By Theorem 2.5, we can get the following corollary.

Corollary 2.6. *For the complete graph K_n, where n is odd, there is a matrix $A \in \mathcal{S}(K_n)$ such that $P_s(A) = \frac{1}{2}(n-1)$.*

However, for the complete graph K_n, where $n \geqslant 4$ is even, by Theorem A, there exists no matrix $A \in \mathcal{S}(K_n)$ such that $P_s(A) = n/2$.

In the initial section we have mentioned that the graph in Figure 1 is a counterexample for the converse of Theorem B, i.e., denoting that graph by G and setting $V = \{1, 2, 5, 6\}$ and $E = \{13, 24, 57, 68\}$, we have $G \in \mathcal{G}_9$, and 9 is the outer vertex of (V, E), but there exists no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = 4$. Besides these facts, notice that the outer vertex 9 is an isolated vertex of $G(V)$. From this example we know that the outer vertex of (V, E) being an isolated vertex of $G(V)$ cannot imply the existence of matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$.

In the following theorem, in addition to the condition that the outer vertex of (V, E) is an isolated vertex of $G(V)$, we add some extra conditions. All together, they will lead to our desired result: there exists a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$.

Theorem 2.7. *Let $G \in \mathcal{G}_n$. Suppose that V is a split set of G, E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V, \overline{V}]$, and v is the outer vertex of (V, E). If v and some vertex, say u, of the complete subgraph of $G(V)$ possess a common neighbor, say w, and $uw \in E$, then there exists a matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A, implying that $P_s(A) = \frac{1}{2}(n-1)$.*

P r o o f. Clearly, $vw \notin E$ since v is the outer vertex of (V, E). Let $\widetilde{E} = E - \{uw\} + \{vw\}$. Note that $\widetilde{E}$ is also the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V, \overline{V}]$. This time, u is the outer vertex of $(V, \widetilde{E})$.

Notice that u is a vertex of the complete subgraph of $G(V)$, hence by Theorem 2.1, the result follows. □

In Theorem 2.7, the condition $uw \in E$ is necessary. For example, let us denote by G the graph of Figure 1. By setting $V = \{1, 2, 5, 6\}$ and $E = \{13, 24, 57, 68\}$, we know that $G \in \mathcal{G}_9$, and 9 is the outer vertex of (V, E). Recall that there exists no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = 4$ (the detailed account can be found later in Proposition 4.4). Although 4 is a vertex of the complete subgraph of $G(V)$ (i.e., the isolated edge 34 of $G(V)$), and the vertices 4 and 9 possess a common neighbor 6, there exists no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = 4$, just because $46 \notin E$.

Let us take the graph G of Figure 2 as an example for Theorem 2.7. By setting $V = \{1, 2, 3, 4\}$ and $E = \{15, 26, 37, 48\}$, we find that $G \in \mathcal{G}_9$, and 9 is the outer vertex of (V, E). Note that $G(V) = G[5, 6, 7, 8, 9]$ contains a complete graph on two vertices, i.e., the isolated edge 56, and three isolated vertices 7, 8, 9. Besides these, note that the vertices 6 and 9 possess a common neighbor 2, and $26 \in E$. In the example for Theorem 2.1, we have seen that there exists a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = 4$.

Next we generalize Theorem 2.7. Before that, we will introduce the following definition.

Definition 2.1. Let $G \in \mathcal{G}_n$. Suppose that V is a split set of G, and E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V, \overline{V}]$. Let v be a vertex outside the complete subgraph of $G(V)$, and let P be a path connecting v with some vertex of the complete subgraph of $G(V)$. Then P is said to be a path from v to the complete subgraph of $G(V)$. In particular, we say P is an *E-alternating path* from v to the complete subgraph of $G(V)$, if the edges of P are alternately in E and outside E.

Among the vertices of an E-alternating path from v to the complete subgraph of $G(V)$, it is not necessary that there is only one vertex lying on the complete subgraph of $G(V)$.

Suppose that G is the graph depicted in Figure 5. By setting $V = \{2, 4, 7\}$ and $E = \{23, 45, 67\}$, we know that $G \in \mathcal{G}_7$. Note that $G(V) = G[1, 3, 5, 6]$ contains a complete graph on three vertices, i.e., the triangle 356, and an isolated vertex 1. Then each of 123, 12345, and 123456 is an E-alternating path from the vertex 1 to the complete subgraph of $G(V)$ (i.e., the triangle 356).

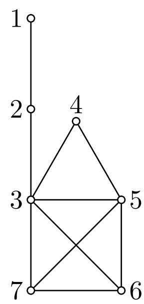

Figure 5. An example for Definition 2.1.

Theorem 2.8. *Let $G \in \mathcal{G}_n$. Suppose that V is a split set of G, E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V, \overline{V}]$, and v is the outer vertex of (V, E). If there exists an E-alternating path from v to the complete subgraph of $G(V)$, then there exists a matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A, implying that $P_s(A) = \frac{1}{2}(n-1)$.*

Proof. Let $v_0 = v$, and denote $P = v_0 v_1 \ldots v_k$, where v_k is a vertex of the complete subgraph of $G(V)$, and $k \geqslant 1$.

Without loss of generality, we may assume that v_k is the unique vertex of P lying on the complete subgraph of $G(V)$. As a consequence, there are no two consecutive vertices on P both outside V, otherwise, if v_i and v_{i+1} are two consecutive vertices on P, where $0 \leqslant i \leqslant k-1$, such that $v_i, v_{i+1} \notin V$, then $v_i v_{i+1}$ is an edge of $G(V)$, implying that v_i and v_{i+1} are both lying on the complete subgraph of $G(V)$, which is a contradiction.

Since $v_0 = v$ is the outer vertex of (V, E), we have $v_0 \notin V$. So $v_1 \in V$. Note that $v_0 v_1 \notin E$ also because v_0 is the outer vertex of (V, E). Now by the definition of an E-alternating path, we have $v_1 v_2 \in E$. Thus $v_2 \notin V$ follows from $v_1 \in V$.

Repeatedly using the definition of an E-alternating path as above, we can get that not only are the edges of P alternately in E and outside E, but also the vertices of P

are alternately in V and outside V. More precisely, $v_i \in V$ if i is odd, and $v_i \notin V$ if i is even. In particular, note that $v_k \notin V$ since v_k is a vertex of (the complete subgraph of) $G(V)$, implying that k is even. Now we know that

$$v_1v_2, v_3v_4, \ldots, v_{k-1}v_k \in E$$

and

$$v_0v_1, v_2v_3, \ldots, v_{k-2}v_{k-1} \notin E.$$

Let

$$\widetilde{E} = E - \{v_1v_2, v_3v_4, \ldots, v_{k-1}v_k\} + \{v_0v_1, v_2v_3, \ldots, v_{k-2}v_{k-1}\}.$$

Note that $\widetilde{E}$ is also an edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V, \overline{V}]$. At the same time, v_k is the outer vertex of $(V, \widetilde{E})$.

Taking into account that v_k is a vertex of the complete subgraph of $G(V)$, now the result follows from Theorem 2.1. □

3. Several necessary and sufficient conditions

In the previous section, we presented several sufficient conditions for the existence of matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$. Now in this section, we will further give some necessary and sufficient conditions for some particular structures of G.

First we focus on the graphs with isolated vertices.

Theorem 3.1. *Let $G = H \cup K_1$ be a graph in $\mathcal{G}_n$. Then the following two conditions are equivalent:*

(a) *There exists a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$.*
(b) *There exists a split set V of G such that $G(V)$ is an edgeless graph.*

Proof. Notice that (b) $\Rightarrow$ (a) follows from Theorem 2.3 clearly. So we need only to prove (a) $\Rightarrow$ (b).

Suppose that there exists a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$, and α is a P-set of $A \in \mathcal{S}(G)$ with $|\alpha| = \frac{1}{2}(n-1)$.

Let v be an isolated vertex of G such that $G = H \cup \{v\}$. Note that whether $A[v]$ is 0 or not, v cannot be a P-vertex of A, which implies that $v \notin \alpha$. So α is also a P-set of $A(v)$, i.e., $A(v)$ is a matrix in $\mathcal{S}(H)$ such that $P_s(A(v)) = \frac{1}{2}(n-1)$.

Finally, we point out that since $n-1$ is even, by virtue of Theorem A there exists a vertex subset V of H, where $|V| = \frac{1}{2}(n-1)$, such that $H(V)$ is an edgeless graph, i.e., $H(V) = \frac{1}{2}(n-1)K_1$, and $\frac{1}{2}(n-1)K_2$ is a subgraph of $H[V, \overline{V}]$. Equivalently, V is a vertex subset of G such that $G(V)$ is the edgeless graph $\frac{1}{2}(n+1)K_1$, and

$\frac{1}{2}(n-1)K_2$ is a subgraph of $G[V,\overline{V}]$. Therefore, V is exactly a split set of G such that $G(V)$ is an edgeless graph.

Now (a) $\Rightarrow$ (b) follows. $\square$

Next we consider the graphs in which there are two terminal vertices sharing a common neighbor.

Theorem 3.2. *Let G be a graph in $\mathcal{G}_n$ of the form shown in Figure 6. Suppose that V is a split set of G, and E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V,\overline{V}]$.*

(i) *Then either u or v is the outer vertex of (V,E).*

(ii) *Assume that v is the outer vertex of (V,E). Then the following two conditions are equivalent:*

  (a) *There exists a matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A.*

  (b) *Either $u \in V$ or $G(V)$ is an edgeless graph.*

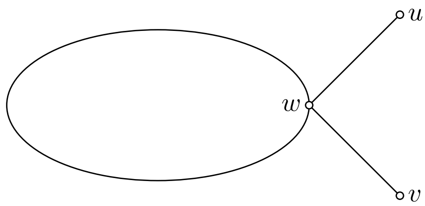

Figure 6. The graph in Theorem 3.2.

Proof. First we consider (i). Notice that both uw and vw are pendant edges. Together with the definition of $G \in \mathcal{G}_n$, we have either $uw \in E$ or $vw \in E$ (otherwise u and v are both outer vertices of (V,E), which is a contradiction). In particular, if $uw \in E$, then v is the outer vertex of (V,E), and if $vw \in E$, then u is the outer vertex of (V,E).

Next we will prove (b) $\Rightarrow$ (a) of (ii). Suppose that $u \in V$ or $G(V)$ is an edgeless graph.

First suppose that $u \in V$. Since v is the outer vertex of (V,E), we have $v \notin V$. Moreover, by the definition of $\mathcal{G}_n$, we know that $uw \in E$. So $w \notin V$ follows from $u \in V$. Thus vw is an edge of $G(V)$, i.e., v is a vertex of the complete subgraph of $G(V)$ (because every vertex of $G(V)$ is either a vertex of the complete subgraph of $G(V)$, or an isolated vertex of $G(V)$). Now by Theorem 2.1, (a) follows.

On the other hand, if $G(V)$ is an edgeless graph, then by Theorem 2.3, (a) follows.

Finally, we will prove (a) $\Rightarrow$ (b) of (ii). Suppose that $u \notin V$ and $G(V)$ is not an edgeless graph.

Clearly, in order to deduce (a) $\Rightarrow$ (b), we need only to show that there exists no matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A.

Suppose to the contrary that there exists a matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A.

Claim 1. $m_A(0) = 0$ and $m_{A(V)}(0) = \frac{1}{2}(n-1)$. Clearly, $A(V)$ is a symmetric matrix of order $\frac{1}{2}(n+1)$. If $m_{A(V)}(0) = \frac{1}{2}(n+1)$, then the rank of $A(V)$ is 0, i.e., $A(V) = O$, accordingly, $G(V)$ is an edgeless graph, which is a contradiction. So $m_{A(V)}(0) \leqslant \frac{1}{2}(n-1)$.

On the other hand, we have

$$\frac{n-1}{2} \leqslant m_A(0) + \frac{n-1}{2} = m_{A(V)}(0) \leqslant \frac{n-1}{2},$$

which implies that $m_A(0) = 0$ and $m_{A(V)}(0) = \frac{1}{2}(n-1)$.

Claim 2. $G(V)$ is the disjoint union of a complete graph and some isolated vertices. By Claim 1, $A(V)$ is a symmetric matrix of order $\frac{1}{2}(n+1)$ with nullity $\frac{1}{2}(n-1)$, i.e., the rank of $A(V)$ is 1, which implies that $G(V)$ is the disjoint union of a complete graph and some isolated vertices.

Claim 3. $A[x] = (\,0\,)$ if x is an isolated vertex of $G(V)$. If x is an isolated vertex of $G(V)$, and $A[x] \neq (\,0\,)$, then together with Claim 1, we have

$$m_{A(V \cup \{x\})}(0) = m_{A(V)}(0) = \frac{n-1}{2}.$$

Note that $A(V \cup \{x\})$ is a symmetric matrix of order $\frac{1}{2}(n-1)$, which implies that the rank of $A(V \cup \{x\})$ is 0, i.e., $A(V \cup \{x\}) = O$. Accordingly, $G(V \cup \{x\})$ is an edgeless graph, implying that $G(V)$ is also an edgeless graph, which is a contradiction.

Claim 4. $w \in V$. Since u is not the outer vertex of (V, E), uw is an edge of E. So $w \in V$ follows from $u \notin V$. As a consequence, we have that w is a P-vertex of A, i.e., $m_{A(w)}(0) = m_A(0) + 1$.

From $u, v \notin V$ and $w \in V$, we have that u and v are two isolated vertices of $G(V)$. Now by Claim 3, we have

$$A[u] = A[v] = (\,0\,).$$

Thus

$$m_A(0) + 1 = m_{A(w)}(0) \geqslant 2,$$

i.e., $m_A(0) \geqslant 1$, which is a contradiction to Claim 1.

Therefore, there is no matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A. So (a) $\Rightarrow$ (b) follows. $\square$

Actually, we can generalize the graph G considered in Theorem 3.2 to the graphs of the following form: Given a graph H, arbitrarily choosing a vertex subset $S = \{u_1, u_2, \ldots, u_s\}$ from H, furthermore, let G be a graph obtained from H by adding vertices $v_1, v_2, \ldots, v_{s+1}$, where $\widetilde{S} = \{v_1, v_2, \ldots, v_{s+1}\}$, and joining edges between every vertex of S and every vertex of $\widetilde{S}$. In particular, $G[S]$ possibly contains some edges, while $G[\widetilde{S}]$ must be an edgeless graph, and the subgraph of G induced by all the edges connecting the vertices in S with the vertices in $\widetilde{S}$ is actually the complete bipartite graph $K_{s,s+1}$. For example, when $s = 2$, the graph G is of the form depicted in Figure 7, where $S = \{u_1, u_2\}$ and $\widetilde{S} = \{v_1, v_2, v_3\}$.

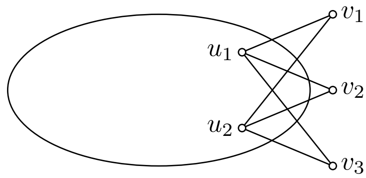

Figure 7. An example for the graphs in Theorem 3.3 when $s = 2$.

Based on the graphs G defined as above, similarly to the proof of Theorem 3.2, we can get a generalized version of Theorem 3.2.

Theorem 3.3. *Let G be a graph in $\mathcal{G}_n$ with the form defined as above. Suppose that V is a split set of G, and E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V, \overline{V}]$.*

(i) *Then one of the vertices in $\widetilde{S}$ is the outer vertex of (V, E).*

(ii) *Assume that v_{s+1} is the outer vertex of (V, E). Then the following two conditions are equivalent:*
   (a) *There exists a matrix $A \in \mathcal{S}(G)$ such that V is a P-set of A.*
   (b) *Either $\{v_1, v_2, \ldots, v_s\} \cap V \neq \emptyset$ or $G(V)$ is an edgeless graph.*

4. On the converse of Theorem B

We have mentioned that the converse of Theorem B is not true, i.e., $G \in \mathcal{G}_n$ does not imply that there is a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$. In this section we will construct several types of such graphs.

The first type of such graphs is based on a complete graph of even order.

Proposition 4.1. *Let $\widetilde{K}_n$ be the disjoint union of the complete graph K_{n-1} and an isolated vertex, i.e., $\widetilde{K}_n = K_{n-1} \cup K_1$, where $n \geqslant 5$ is odd. Then there is no matrix $A \in \mathcal{S}(\widetilde{K}_n)$ such that $P_s(A) = \frac{1}{2}(n-1)$.*

Proof. It is easily verified that $\widetilde{K}_n \in \mathcal{G}_n$. On the other hand, notice that $\widetilde{K}_n(V)$ cannot be an edgeless graph for any split set V of $\widetilde{K}_n$. Now from Theorem 3.1 (a) $\Rightarrow$ (b), the result follows easily. $\square$

It is worth mentioning that, as an application of Theorems 3.2 and 3.3, we can construct a lot of counterexamples for the converse of Theorem B, i.e., the graphs $G \in \mathcal{G}_n$ for which there exists no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$. For example, in this section we use the particular structure as depicted in Theorem 3.2.

Let G be a graph in $\mathcal{G}_n$ with the form as shown in Figure 6. In view of Theorem 3.2 (i), in the following two corollaries we may assume that v is the outer vertex of (V, E), for every split set V of G, and E is the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G[V, \overline{V}]$.

By Theorem 3.2 and Theorem B, we can easily get the following corollary.

Corollary 4.2. *Let G be a graph in $\mathcal{G}_n$ with the form as shown in Figure 6. Then the following two conditions are equivalent:*

(a) *There exists a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$.*

(b) *There exists a split set V of G such that either $u \in V$ or $G(V)$ is an edgeless graph.*

In particular, we can get a method for distinguishing the existence of a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$.

Corollary 4.3. *Let G be a graph in $\mathcal{G}_n$ as shown in Figure 6. If for every split set V of G, we have $u \notin V$ and $G(V)$ is not an edgeless graph, then there is no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$.*

Now we use Corollary 4.3 to show that the graph G in Figure 1 is actually a counterexample for the converse of Theorem B, i.e., although $G \in \mathcal{G}_9$, there is no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = 4$.

Proposition 4.4. *Let G be the graph in Figure 1. Then there is no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = 4$.*

Proof. Let V be a split set of G, and E the edge set of a subgraph $4K_2$ of $G[V, \overline{V}]$. Clearly, $|V| = 4$.

By Theorem 3.2 (i), we may assume that 9 is the outer vertex of (V, E), so $9 \notin V$. Furthermore, by Corollary 4.3, if $8 \notin V$ and $G(V)$ is not an edgeless graph, then the result follows.

First suppose that $8 \in V$. Clearly, $68 \in E$ since 9 is the outer vertex of (V, E). So $6 \notin V$ follows from $8 \in V$. Note that 69 is an edge of $G(V)$, more precisely, 69 is an edge of the complete subgraph of $G(V)$, implying that $G(V)$ consists of an isolated edge 69 and three isolated vertices. In order to obtain this configuration, we must delete at least four vertices in $G[1, 2, 3, 4, 5, 7]$, which would result in V contain at least five vertices since $8 \in V$, which is a contradiction. So $8 \notin V$.

Next suppose that $G(V)$ is an edgeless graph. As above, we need to delete at least four vertices in $G[1,2,3,4,5,7]$ and at least one vertex in $G[6,8,9]$. But then one would get that V containing at least five vertices, which is a contradiction again. So $G(V)$ is not an edgeless graph.

Finally, by Corollary 4.3, there is no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = 4$. $\square$

Similarly, we can construct such examples for every odd $n \geqslant 7$.

Proposition 4.5. *Let G_n be the graph obtained by attaching two terminal vertices u, v to some vertex, say w, of the complete graph K_{n-2}, where $n \geqslant 7$ is odd. Then there exists no matrix $A \in \mathcal{S}(G_n)$ such that $P_s(A) = \frac{1}{2}(n-1)$.*

Proof. It is readily verified that $G_n \in \mathcal{G}_n$. Let V be a split set of G_n, and E the edge set of a subgraph $\frac{1}{2}(n-1)K_2$ of $G_n[V, \overline{V}]$.

By Theorem 3.2 (i), we may assume that v is the outer vertex of (V, E), so $v \notin V$. Furthermore, by Corollary 4.3, it is easily seen that if we can show that $u \notin V$ and $G_n(V)$ is not an edgeless graph, then it follows that there exists no matrix $A \in \mathcal{S}(G_n)$ such that $P_s(A) = \frac{1}{2}(n-1)$.

First suppose that $u \in V$. Clearly, $uw \in E$ since v is the outer vertex of (V, E). So $w \notin V$ follows from $u \in V$. Note that vw is an edge of $G_n(V)$, more precisely, vw is an edge of the complete subgraph of $G_n(V)$. Thus $G_n(V)$ consists of an isolated edge vw and some isolated vertices. Note that this is impossible since w has $n-1$ neighbors in G_n. So $u \notin V$.

On the other hand, since G_n contains the complete graph K_{n-2} as a subgraph, $G_n(V)$ cannot be an edgeless graph.

Finally, by Corollary 4.3, there is no matrix $A \in \mathcal{S}(G_n)$ such that $P_s(A) = \frac{1}{2}(n-1)$. $\square$

The above results are established, based on Theorem 3.2 and the particular structure as depicted in Figure 6. Actually, by using Theorem 3.3 and the particular structure as depicted in Figure 7, we can also construct many other graphs G for which there is no matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$.

5. Concluding remarks

For the graphs G for which there is a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \lfloor n/2 \rfloor$, when n is even, the completed characterization for the graphs G can be found in Theorem A, but when n is odd, we need a completely different approach. For example, in Section 2, we have seen that for the complete graph K_n, when n is odd, there is a matrix $A \in \mathcal{S}(K_n)$ such that $P_s(A) = \frac{1}{2}(n-1)$, but when n is even, there is no matrix $A \in \mathcal{S}(K_n)$ such that $P_s(A) = n/2$.

Besides, the case n is odd turns out to be much harder than the case n is even. From Theorem A, we know that when n is even, the graphs G are uniquely determined by the structure of $G(V)$ (i.e., $G(V)$ is an edgeless graph), and the existence of a subgraph $\frac{1}{2}nK_2$ in $G[V,\overline{V}]$. However, for the case n is odd, we may take the graphs in Figures 1 and 2 as examples. Although the two graphs possess the same $G(V)$ (i.e., $G(V)$ always contains an isolated edge and three isolated vertices), and there is a subgraph $4K_2$ in $G[V,\overline{V}]$, the existence of a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$ holds for one graph but not for the other.

In this paper, we presented some sufficient conditions, and some necessary and sufficient conditions for the graphs G for which there exists a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$. Using these results, we can get a partial characterization for certain graphs G for which there is a matrix $A \in \mathcal{S}(G)$ such that $P_s(A) = \frac{1}{2}(n-1)$, and some forbidden structures for such graphs.

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