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GUARANTEED AND COMPUTABLE BOUNDS OF THE LIMIT LOAD FOR VARIATIONAL PROBLEMS WITH LINEAR GROWTH ENERGY FUNCTIONALS

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Abstract. The paper is concerned with guaranteed and computable bounds of the limit (or safety) load, which is one of the most important quantitative characteristics of mathematical models associated with linear growth functionals. We suggest a new method for getting such bounds and illustrate its performance. First, the main ideas are demonstrated with the paradigm of a simple variational problem with a linear growth functional defined on a set of scalar valued functions. Then, the method is extended to classical plasticity models governed by von Mises and Drucker-Prager yield laws. The efficiency of the proposed approach is confirmed by several numerical experiments.

Keywords: functionals with linear growth; limit load; truncation method; perfect plasticity

MSC 2010: 49M15, 74C05, 74S05, 90C25

1. Introduction

A class of physically important variational problems is represented by energy functionals with linear growth. Among others, it encompasses minimal and capillary surfaces [6], [10], [18], [9], [16], perfect plasticity [4], [27] and some other problems. These highly nonlinear problems differ substantially from the others with convex energy having superlinear growth with respect to derivatives of the solution function.

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Generally, functionals with linear growth are coercive only in non-reflexive spaces (typically in $W^{1,1}$). In order to obtain mathematically correct formulations of these problems, an appropriate extension of these spaces is required (this way leads to spaces like BD or BV, see, e.g., [11], [23], [27]).

In models of perfect plasticity, energy functionals consist of two parts: an inner energy functional (which has linear growth) and a linear functional formed by external loads. It is well-known that energy functionals may be unbounded from below for certain loads. Physically, unboundedness means that for the external loads there is no admissible stress-strain state (i.e., being subject to such loads the body must collapse). Within the framework of limit load analysis, volume and surface loads are multiplied by a scalar parameter λ, which is increased up to the moment when the energy functional becomes unbounded from below. Finding the respective value λ^* is one of the most important tasks in quantitative analysis of many problems (bearing capacity of strip-footing or slope stability in soil mechanics, e.g., [3], [5]).

There exist several approaches how to estimate λ^*. The classical one uses incremental techniques to enlarge λ up to its limit value λ^* [19], [29]. The load increments have to be chosen adaptively since the value of λ^* is not known. Recently, another physical quantity controlling the loading process has been proposed and analyzed in [2], [13], [26]. On the implementation level, incremental techniques are usually combined with finite element methods which may lead to an overestimation of the searched λ^*.

Another type of methods is based on the use of a specific variational problem characterizing directly the limit state. It can be formulated either in terms of displacements (kinematical approach) or in terms of stresses (static approach). The two approaches are mutually dual [4], [27]. As a method of computation, the static limit analysis has been used in [28], while the kinematic one in [1]. Various space discretizations are utilized for solving this variational problem: standard finite element methods [1], [22], mixed finite element methods [28] or discontinuous methods [12], [14], [17]. All these techniques lead to relatively complex problems from the computational point of view.

The present paper is concerned with guaranteed and computable bounds of the limit load. To this end, we distinguish whether functionals with linear growth have *purely linear* growth at infinity or not. In the former case, we show how to get a guaranteed and easily computable upper bound of λ^*. In the latter case, we propose its approximation by functionals with purely linear growth to get reliable estimates of λ^*. To demonstrate principal ideas we first focus on a model problem which can be viewed as a scalar counterpart of the classical Hencky model of plasticity. Then this approach in combination with results from [13] is extended to problems of elastoplasticity.

The paper is organized as follows. In Section 2 we present the above mentioned scalar problem. Section 3 deals with a class of scalar energy functions being the Fenchel transformation of the Euclidean norm of vectors which belong to a closed, convex subset $B \subset \mathbb{R}^d$. There are two rather different situations depending on whether B is bounded or not. For a bounded B the resulting energy function has purely linear growth at infinity and the new guaranteed upper bound of λ^* is derived. For B unbounded, we use the truncation approach to approximate the original B by a sequence $\{B_k\}$ of bounded, convex subsets of B. Particular attention is paid to two types of unbounded sets, namely to conical and cylindrical ones. The results of this section are straightforwardly transformed and implemented to a generalized Hencky model of plasticity in Section 4. The von Mises and Drucker-Prager yield functions serve as models of cylindrical and conical sets, respectively. Section 5 describes a finite element discretization of the generalized Hencky problem. In addition to known results from [2], [13], [26], some new ones are presented and extended for purposes of this paper. Section 6 describes strategy how to find computable and reliable lower and upper bounds of λ^*. Finally, lower and upper bounds of λ^* for several model examples with the above mentioned yield criteria are established.

2. A model variational problem with a linear growth energy functional

First, we discuss the main ideas of our method using a model variational problem for scalar valued functions. Consider a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with the Lipschitz continuous boundary $\partial\Omega = \bar{\Gamma}_N \cup \bar{\Gamma}_D$, where Γ_N, Γ_D are open in $\partial\Omega$, mutually disjoint, and $\Gamma_D \neq \emptyset$. On Γ_N, Γ_D, we prescribe the Neumann and homogeneous Dirichlet boundary condition, respectively. Let

$$\mathbb{V} = \{v \in H^1(\Omega)\colon\ v|_{\Gamma_D} = 0\}$$

be the space of kinematically admissible displacements. Define the functionals

$$\begin{aligned} L(v) &= \int_\Omega F v \,\mathrm{d}x + \int_{\Gamma_N} f v \,\mathrm{d}s, \quad v \in \mathbb{V}, \\ J(v) &= \int_\Omega j(\nabla v) \,\mathrm{d}x - L(v), \quad v \in \mathbb{V}, \end{aligned}$$

where $f \in L^2(\Gamma_N)$, $F \in L^2(\Omega)$ are such that

$$\|F\|_{L^2(\Omega)} + \|f\|_{L^2(\Gamma_N)} > 0 \tag{2.1}$$

and $j\colon \mathbb{R}^d \to \mathbb{R}_+$ is a *non-negative, continuous, convex function*, $j(0) = 0$, which satisfies the growth condition

$$\exists\, c_0 > 0,\ c_1 > 0,\ c_2 \geqslant 0\colon\ c_1|z| - c_2 \leqslant j(z) \leqslant c_0|z|^2 \quad \forall\, z \in \mathbb{R}^d. \tag{2.2}$$

Here, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$, $d = 2, 3$: $|z|^2 = z \cdot z$, $y \cdot z = y_i z_i$ for any $y, z \in \mathbb{R}^d$. The corresponding variational problem with a *guaranteed* linear growth reads as follows:

$$(\mathcal{P}) \qquad \text{minimize } J(v), \quad v \in \mathbb{V}.$$

Continuity of j and growth restrictions imposed by (2.2) guarantee that J is well-defined and continuous in $\mathbb{V}$ (see, e.g., [27], Proposition 2.4 or [8], [15], [10]). Further, it is readily seen that J is also convex in $\mathbb{V}$ and consequently, weakly lower semi-continuous in $\mathbb{V}$ (see, e.g., [10]). On the other hand, the lower bound in (2.2) does not guarantee coercivity of J in $\mathbb{V}$. For this reason, one cannot use the well-known existence results for $(\mathcal{P})$, see, e.g., [8], Proposition II.1.2.

Evidently, $\inf\limits_{v\in\mathbb{V}} J(v) > -\infty$ is only a necessary condition for solvability of $(\mathcal{P})$. On the other hand, if this condition holds, then the problem is meaningful in a certain sense, e.g., a dual problem to $(\mathcal{P})$ has a solution, see [8], Theorem III.4.1. The dual problem is introduced in Section 3. To decide whether J is bounded from below, the original problem is parametrized in a standard fashion:

$$(\mathcal{P})_\lambda \qquad \text{minimize } J_\lambda(v),\ v \in \mathbb{V}, \quad J_\lambda(v) = \int_\Omega j(\nabla v)\,\mathrm{d}x - \lambda L(v),$$

where $\lambda \geqslant 0$ is a load parameter. If $\lambda, \bar\lambda \geqslant 0$ are two parameters satisfying $\lambda \leqslant \bar\lambda$, then

$$\begin{aligned} \inf_{v\in\mathbb{V}} J_\lambda(v) &= \inf_{\substack{v\in\mathbb{V},\\ L(v)\geqslant 0}} J_\lambda(v) = \inf_{\substack{v\in\mathbb{V},\\ L(v)\geqslant 0}} \{J_{\bar\lambda}(v) - (\lambda - \bar\lambda)L(v)\} \\ &\geqslant \inf_{\substack{v\in\mathbb{V},\\ L(v)\geqslant 0}} J_{\bar\lambda}(v) = \inf_{v\in\mathbb{V}} J_{\bar\lambda}(v), \end{aligned} \tag{2.3}$$

i.e. the function $\lambda \mapsto \inf\limits_{v\in\mathbb{V}} J_\lambda(v)$, $\lambda \in \mathbb{R}_+$, is non-increasing in $\mathbb{R}_+$. Therefore, it is natural to define the limit load parameter by

$$\lambda^* = \sup\Big\{\lambda \geqslant 0\colon\ \inf_{v\in\mathbb{V}} J_\lambda(v) > -\infty\Big\}. \tag{2.4}$$

Clearly, λ^* is either finite non-negative or equal to ∞. If $\lambda = 1 > \lambda^*$ then the original functional $J := J_{\lambda=1}$ is unbounded from below in $\mathbb{V}$ and thus $(\mathcal{P})$ has no solution.

Therefore, the knowledge of λ^* is important to identify a range of admissible loads. To find reliable and easily computable estimates of λ^*, we introduce several auxiliary results.

Lemma 2.1. *Let $j_1, j_2\colon \mathbb{R}^d \to \mathbb{R}_+$, $0 \leqslant j_1 \leqslant j_2$, be two convex and continuous functions satisfying* (2.2). *Then the corresponding limit load parameters λ_1^*, λ_2^* satisfy $\lambda_1^* \leqslant \lambda_2^*$.*

The proof is straighforward.

Now, we introduce another variational problem giving an upper bound of λ^*. To this end, define the function

$$j_\infty\colon \mathbb{R}^d \to \overline{\mathbb{R}}_+,\ \overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{\infty\}, \quad j_\infty(z) = \lim_{\alpha\to\infty} \frac{1}{\alpha} j(\alpha z), \quad z \in \mathbb{R}^d.$$

Clearly, $j_\infty(0) = 0$ and j_∞ is a proper, convex function in $\mathbb{R}^d$ which is also positively 1-homogeneous (see [21], Theorem 8.5):

$$j_\infty(\alpha z) = \alpha j_\infty(z) \quad \forall\, z \in \mathbb{R}^d\ \forall\, \alpha \geqslant 0. \tag{2.5}$$

Since the function $\alpha \mapsto j(\alpha z)/\alpha$, $\alpha > 0$, is non-decreasing, Theorem 23.1 in [21] yields:

$$j(z) \leqslant \frac{1}{\alpha} j(\alpha z) \leqslant j_\infty(z) \quad \forall\, z \in \mathbb{R}^d\ \forall\, \alpha \geqslant 1 \tag{2.6}$$

and from (2.2) we obtain

$$j_\infty(z) = \lim_{\alpha\to\infty} \frac{1}{\alpha} j(\alpha z) \overset{(2.2)}{\geqslant} \lim_{\alpha\to\infty} \Big\{c_1|z| - \frac{c_2}{\alpha}\Big\} = c_1|z| \quad \forall\, z \in \mathbb{R}^d. \tag{2.7}$$

Although the function j_∞ need not be finite everywhere due to the upper bound in (2.2), one can extend the definition (2.4) and Lemma 2.1 to j_∞, as well. Therefore, the limit load parameter related to j_∞ is defined as follows:

$$\zeta^* = \sup\Big\{\lambda \geqslant 0\colon \inf_{v\in\mathbb{V}} \Big[\int_\Omega j_\infty(\nabla v)\,\mathrm{d}x - \lambda L(v)\Big] > -\infty\Big\}. \tag{2.8}$$

It is readily seen that

$$\lambda^* \leqslant \zeta^* \tag{2.9}$$

by making use of (2.6) and Lemma 2.1, i.e., ζ^* is an upper bound of the limit load parameter λ^*. The properties of j_∞ enable us to derive a more convenient definition of ζ^* than (2.8).

Lemma 2.2. *It holds:*

$$\zeta^* = \inf_{\substack{v\in\mathbb{V}\\ L(v)=1}} J_\infty(v), \quad J_\infty(v) = \int_\Omega j_\infty(\nabla v)\,\mathrm{d}x, \quad v\in\mathbb{V}. \tag{2.10}$$

P r o o f. Suppose that for some $\lambda > 0$ and $w \in \mathbb{V}$, $J_\infty(w) - \lambda L(w) < 0$. Then (2.5) entails that $J_\infty(kw) - \lambda L(kw) = k[J_\infty(w) - \lambda L(w)] \to -\infty$ as $k \to \infty$. Therefore, (2.8) can be rewritten as

$$\zeta^* = \sup\{\lambda \geqslant 0\colon\ J_\infty(v) - \lambda L(v) \geqslant 0 \quad \forall\, v \in \mathbb{V}\}. \tag{2.11}$$

Since j_∞ is nonnegative, the inequality $J_\infty(v) - \lambda L(v) \geqslant 0$ is automatically satisfied for any $v \in \mathbb{V}$ such that $L(v) \leqslant 0$. Thus,

$$\begin{aligned}\zeta^* &= \sup\{\lambda \geqslant 0\colon\ J_\infty(v) - \lambda L(v) \geqslant 0 \quad \forall\, v \in \mathbb{V},\ L(v) > 0\}\\ &\overset{(2.5)}{=} \sup\{\lambda \geqslant 0\colon\ J_\infty(v) - \lambda \geqslant 0 \quad \forall\, v \in \mathbb{V},\ L(v) = 1\} = \inf_{\substack{v\in\mathbb{V}\\ L(v)=1}} J_\infty(v).\end{aligned}$$

□

In the terminology accepted in perfect plasticity (e.g., see [4], [27]), the problem (2.10) is called the *limit analysis problem*. It leads to a non-smooth minimization problem with the isoperimetric condition $L(v) = 1$ and a possibly nonlinear constraint $v \in \operatorname{dom} J_\infty = \{w \in \mathbb{V}\colon\ J_\infty(w) < \infty\}$ which can be solved by methods presented, e.g., in [1], [4].

The limit analysis is trivial for functions with a quadratic growth like $j(z) = \frac{1}{2}|z|^2$, $z \in \mathbb{R}^d$. Then $\zeta^* = \lambda^* = \infty$. As we shall see later, the identity $\zeta^* = \lambda^*$ holds true for many other functions. In particular, it holds for functions j with *purely* linear growth, that is

$$\exists c_1 > 0,\ c_2 \geqslant 0,\ c_3 > 0\colon\ c_1|z| - c_2 \leqslant j(z) \leqslant c_3|z| \quad \forall\, z \in \mathbb{R}^d. \tag{2.12}$$

Even though (2.10) will not be solved directly in this paper, this problem is very useful for finding a new *guaranteed upper bound* of λ^*: given $\lambda > 0$ decide whether $\lambda > \lambda^*$ or not. The basic idea is very simple. Let us suppose that $\lambda^* < \infty$ and the functional J_{λ^*} is bounded from below:

$$\exists\, c > 0\colon\ J_{\lambda^*}(v) \geqslant -c \quad \forall\, v \in \mathbb{V}. \tag{2.13}$$

In view of (2.3), the functional J_λ is also bounded from below by the same constant c for all $\lambda < \lambda^*$. Suppose that $\bar\lambda > 0$ is such that there exists $w \in \mathbb{V}$ and $J_{\bar\lambda}(w) < -c$.

Then necessarily $\bar{\lambda} > \lambda^*$, i.e. $\bar{\lambda}$ is an upper bound of λ^*. The optimal value of c in (2.13) is $c_{\mathrm{opt}} = -\inf\limits_{v\in\mathbb{V}} J_{\lambda^*}(v)$, which is not usually available. For this reason, it is necessary to use in place of c_{opt} another quantity which is easily computable. In the next example, we show how to provide this quantity.

Example 2.1. Let

$$j(z) = \begin{cases} \frac{1}{2}|z|^2, & |z| \leqslant 1, \\ |z| - \frac{1}{2}, & |z| \geqslant 1, \end{cases} \qquad z \in \mathbb{R}^d.$$

This function has purely linear growth at infinity since

$$|z| - \frac{1}{2} \leqslant j(z) \leqslant |z| \quad \forall\, z \in \mathbb{R}^d. \tag{2.14}$$

Clearly, $j_\infty(z) = |z|$ and from (2.10) we see that $\zeta^* < \infty$ in view of (2.1). Further, from (2.11) we find that

$$\int_\Omega |\nabla v| \,\mathrm{d}x \geqslant \lambda L(v) \quad \forall\, v \in \mathbb{V},\ \forall\, 0 \leqslant \lambda < \zeta^*.$$

Since ζ^* is finite, this inequality also holds for $\lambda = \zeta^*$. Hence,

$$\begin{aligned} J_{\zeta^*}(v) &= \int_\Omega j(\nabla v)\,\mathrm{d}x - \zeta^* L(v) \overset{(2.14)}{\geqslant} \int_\Omega |\nabla v|\,\mathrm{d}x - \frac{1}{2}|\Omega| - \zeta^* L(v) \\ &\geqslant -\frac{1}{2}|\Omega| \quad \forall\, v \in \mathbb{V}. \end{aligned} \tag{2.15}$$

This estimate and (2.9) entail $\lambda^* = \zeta^* < \infty$. As a consequence of (2.15), we obtain the following simple condition for $\bar{\lambda}$ to be an upper estimate of λ^*:

$$\text{if } \exists\, w \in \mathbb{V}\colon\ J_{\bar{\lambda}}(w) < -\frac{1}{2}|\Omega| \quad \text{then } \bar{\lambda} > \lambda^*. \tag{2.16}$$

Similar considerations can be done for many other problems with linear growth functionals (see Section 3.1). Notice that (2.16) also provides an *easily computable upper bound of* λ^**. Indeed, one can construct a minimization sequence* $\{v_n\}$ *of* $J_{\bar{\lambda}}$ *in* $\mathbb{V}$*. Then from the numerical point of view, it is easier to verify that* $J_{\bar{\lambda}}(v_n) < -\frac{1}{2}|\Omega|$ *holds for some* n *than to show* $\lim\limits_{n\to\infty} J_{\bar{\lambda}}(v_n) = -\infty$.

3. Scalar problems defined by a special class of functions j

Consider a special class of functions j defined in such a way that the resulting problem $(\mathcal{P})_\lambda$ can be viewed as a simplified version of the classical variational problem of the Hencky plasticity (see, e.g., [7], [27]). Namely, we set

$$j(z) = \sup_{z^* \in B} \Big\{ z^* \cdot z - \frac{1}{2}|z^*|^2 \Big\} = \frac{1}{2}|z|^2 - \frac{1}{2} \inf_{z^* \in B} |z - z^*|^2, \quad z \in \mathbb{R}^d. \tag{3.1}$$

Here and later on, B is a *closed* and *convex* subset of $\mathbb{R}^d$ containing a neighborhood of the origin, i.e.,

$$\exists \varepsilon > 0\colon\ \forall\, z^* \in \mathbb{R}^d,\ |z^*| \leqslant \varepsilon \Longrightarrow z^* \in B. \tag{3.2}$$

From (3.1) it follows that

$$j(z) = \Pi_B(z) \cdot z - \frac{1}{2}|\Pi_B(z)|^2 \geqslant 0 \quad \forall\, z \in \mathbb{R}^d, \tag{3.3}$$

where Π_B is the projection of $\mathbb{R}^d$ on B:

$$|z - \Pi_B(z)| = \min_{z^* \in B} |z - z^*|$$

or equivalently,

$$(z - \Pi_B(z)) \cdot (z^* - \Pi_B(z)) \leqslant 0 \quad \forall\, z^* \in B. \tag{3.4}$$

Further, j is convex, continuously differentiable in $\mathbb{R}^d$,

$$\nabla j(z) = \Pi_B(z) \quad \forall\, z \in \mathbb{R}^d, \tag{3.5}$$

and $j(z) = \frac{1}{2}|z|^2$ for any $z \in B$.

The function j defined by (3.1) has a guaranteed linear growth, see the next lemma.

Lemma 3.1. *We have*

$$\frac{\varepsilon}{2}|z| - \frac{\varepsilon^2}{8} \leqslant j(z) \leqslant \frac{1}{2}|z|^2 \quad \forall\, z \in \mathbb{R}^d, \tag{3.6}$$

where ε is from (3.2).

P r o o f. The upper bound in (3.6) directly follows from (3.1).

Let $|z| \leqslant \varepsilon$. Then $z \in B$, $\Pi_B(z) = z$, and

$$j(z) = \frac{1}{2}|z|^2 = \frac{1}{2}\Big(|z| - \frac{\varepsilon}{2}\Big)^2 + \frac{\varepsilon}{2}|z| - \frac{\varepsilon^2}{8} \geqslant \frac{\varepsilon}{2}|z| - \frac{\varepsilon^2}{8}.$$

Let $|z| \geqslant \varepsilon$. Then $z^* = \varepsilon|z|^{-1}z \in B$. Inserting z^* into (3.4), we obtain

$$2\Pi_B(z) \cdot z \geqslant \Big(1 + \frac{\varepsilon}{|z|}\Big)\Pi_B(z) \cdot z \overset{(3.4)}{\geqslant} \varepsilon|z| + |\Pi_B(z)|^2.$$

Hence,

$$j(z) = \Pi_B(z) \cdot z - \frac{1}{2}|\Pi_B(z)|^2 \geqslant \frac{\varepsilon}{2}|z| \geqslant \frac{\varepsilon}{2}|z| - \frac{\varepsilon^2}{8}.$$

□

Lemma 3.2. *Let j be defined by* (3.1). *Then*

$$j_\infty(z) = \lim_{\alpha\to\infty} \frac{1}{\alpha} j(\alpha z) = \sup_{z^*\in B} z^* \cdot z \quad \forall\, z \in \mathbb{R}^d \tag{3.7}$$

and

$$\liminf_{\alpha\to\infty} \frac{1}{\alpha} j(\alpha z_\alpha) \geqslant j_\infty(z) \tag{3.8}$$

holds for any $z \in \mathbb{R}^d$ and any sequence $\{z_\alpha\}$ tending to z as $\alpha \to \infty$.

P r o o f. Since

$$\frac{1}{\alpha} j(\alpha z) = \sup_{z^*\in B} \Big\{ z^* \cdot z - \frac{1}{2\alpha}|z^*|^2 \Big\} \quad \forall\, z \in \mathbb{R}^d\ \forall\, \alpha > 0,$$

we have

$$\begin{aligned}
&\frac{1}{\alpha} j(\alpha z) \leqslant \sup_{z^*\in B} z^* \cdot z \quad \forall\, z \in \mathbb{R}^d,\ \forall\, \alpha > 0,\\
&\frac{1}{\alpha} j(\alpha z) \geqslant z^* \cdot z - \frac{1}{2\alpha}|z^*|^2 \to z^* \cdot z \quad \text{as } \alpha \to \infty\ \forall\, z \in \mathbb{R}^d,\ \forall\, z^* \in B.
\end{aligned}$$

Therefore, (3.7) holds.

If $z_\alpha \to z$ then

$$\frac{1}{\alpha} j(\alpha z_\alpha) \geqslant z^* \cdot z_\alpha - \frac{1}{2\alpha}|z^*|^2 \to z^* \cdot z \quad \forall\, z^* \in B.$$

This proves (3.8). □

Since j is convex and differentiable, problem $(\mathcal{P})_\lambda$ is equivalent to the nonlinear equation

$$\text{(3.9)} \qquad \text{find } u_\lambda \in \mathbb{V}\colon \quad \int_\Omega \sigma_\lambda \cdot \nabla v \, \mathrm{d}x = \lambda L(v) \quad \forall\, v \in \mathbb{V},$$

where $\sigma_\lambda = \Pi_B(\nabla u_\lambda)$, see (3.5). If the solution u_λ exists then $\sigma_\lambda \in L^2(\Omega; \mathbb{R}^d)$ solves the dual problem to $(\mathcal{P})_\lambda$:

$$(\mathcal{P}^*)_\lambda \qquad \text{find } \sigma_\lambda \in Q_{\lambda L} \cap P\colon \quad \mathcal{I}(\sigma_\lambda) \leqslant \mathcal{I}(\tau) \quad \forall\, \tau \in Q_{\lambda L} \cap P,$$

where

$$\begin{aligned}
\mathcal{I}(\tau) &= \int_\Omega \frac{1}{2}|\tau|^2 \, \mathrm{d}x, \ \ \tau \in L^2(\Omega; \mathbb{R}^d), \\
P &= \{\tau \in L^2(\Omega; \mathbb{R}^d)\colon\ \tau(x) \in B \text{ for a.a. } x \in \Omega\}, \\
Q_{\lambda L} &= \left\{\tau \in L^2(\Omega; \mathbb{R}^d)\colon\ \int_\Omega \tau \cdot \nabla v \, \mathrm{d}x = \lambda L(v) \quad \forall\, v \in \mathbb{V}\right\} \\
&= \{\tau \in L^2(\Omega; \mathbb{R}^d)\colon\ \operatorname{div} \tau + \lambda F = 0 \text{ in } \Omega,\ \tau \cdot \nu = \lambda f \text{ on } \Gamma_N\}.
\end{aligned}$$

Notice that the integrand $\frac{1}{2}|\tau|^2$ of $\mathcal{I}$ is the dual function to j in the sense of the Legendre-Fenchel transformation as follows from (3.1). Since P and $Q_{\lambda L}$ are closed and convex subsets of $L^2(\Omega; \mathbb{R}^d)$, problem $(\mathcal{P}^*)_\lambda$ has the unique solution σ_λ if and only if $Q_{\lambda L} \cap P \neq \emptyset$. Moreover,

$$\inf_{v \in \mathbb{V}} J_\lambda(v) = \sup_{\tau \in Q_{\lambda L} \cap P} \mathcal{I}(\tau) \quad \forall\, \lambda \geqslant 0.$$

If $Q_{\lambda L} \cap P = \emptyset$ then both sides of this equality are equal to $-\infty$. Hence, one can equivalently define the limit load parameter as follows:

$$\lambda^* \overset{(2.4)}{=} \sup\{\lambda \geqslant 0\colon\ Q_{\lambda L} \cap P \neq \emptyset\}.$$

This is the so-called *static principle* of limit analysis in terminology of [4] while (cf. (2.10))

$$\text{(3.10)} \qquad \zeta^* = \inf_{\substack{v \in \mathbb{V} \\ L(v)=1}} \int_\Omega j_\infty(\nabla v) \, \mathrm{d}x, \quad j_\infty(z) = \sup_{z^* \in B} z^* \cdot z \quad \forall\, z \in \mathbb{R}^d,$$

corresponds to the *kinematic principle* [4]. It is easy to see that ζ^* is dual to λ^* in the following sense:

$$\lambda^* = \sup_{\tau \in P} \inf_{\substack{v \in \mathbb{V}, \\ L(v)=1}} \int_\Omega \tau \cdot \nabla v \, \mathrm{d}x \leqslant \inf_{\substack{v \in \mathbb{V}, \\ L(v)=1}} \sup_{\tau \in P} \int_\Omega \tau \cdot \nabla v \, \mathrm{d}x = \zeta^*.$$

We only sketch the proof, since the analogous result is known from perfect plasticity [4], [27]:

$$\sup_{\tau\in P}\int_\Omega \tau\cdot\nabla v\,\mathrm{d}x \overset{(3.7)}{=} \int_\Omega j_\infty(\nabla v)\,\mathrm{d}x \quad \forall\, v\in\mathbb{V},$$

$$\inf_{\substack{v\in\mathbb{V}\\ L(v)=1}}\int_\Omega \tau\cdot\nabla v\,\mathrm{d}x = \inf_{v\in\mathbb{V}}\sup_{\lambda\in\mathbb{R}}\Bigl[\int_\Omega \tau\cdot\nabla v\,\mathrm{d}x - \lambda(L(v)-1)\Bigr]$$

$$= \sup_{\lambda\in\mathbb{R}}\Bigl\{\inf_{v\in\mathbb{V}}\Bigl[\int_\Omega \tau\cdot\nabla v\,\mathrm{d}x - \lambda L(v)\Bigr] + \lambda\Bigr\},$$

$$\inf_{v\in\mathbb{V}}\Bigl[\int_\Omega \tau\cdot\nabla v\,\mathrm{d}x - \lambda L(v)\Bigr] = \begin{cases} 0, & \tau\in Q_{\lambda L},\\ -\infty, & \tau\notin Q_{\lambda L}.\end{cases}$$

Notice that the duality between ζ^* and λ^* is sometimes useful for proving $\lambda^*=\zeta^*$ (see, e.g., [27]).

In addition to the properties of B formulated at the beginning of this section, we shall distinguish whether B is bounded or not.

3.1. Bounded sets B. Let B and $\bar\varrho>0$ be such that $|z|\leqslant\bar\varrho$ for any $z\in B$. Then

$$j_\infty(z) \overset{(3.7)}{=} \sup_{z^*\in B} z^*\cdot z \leqslant \bar\varrho|z| \quad \forall\, z\in\mathbb{R}^d.$$

Hence, $\operatorname{dom} j_\infty=\mathbb{R}^d$ and j, j_∞ satisfy (2.12):

$$\frac{\varepsilon}{2}|z| - \frac{\varepsilon^2}{8} \overset{(3.6)}{\leqslant} j(z) \overset{(2.6)}{\leqslant} j_\infty(z) \leqslant \bar\varrho|z| \quad \forall\, z\in\mathbb{R}^d,$$

i.e., both have purely linear growth. Morever,

$$\exists\, c_2>0\colon\ j_\infty(z)-c_2 \leqslant j(z) \overset{(2.6)}{\leqslant} j_\infty(z) \quad \forall\, z\in\mathbb{R}^d. \tag{3.11}$$

From (3.1) and (3.7) it follows that one can set $c_2=\max\limits_{z^*\in B}\frac12|z^*|^2$.

Remark 3.1. Notice that the function j from Example 2.1 corresponds to $B=\{z\in\mathbb{R}^d\colon\ |z|\leqslant 1\}$.

Using (3.11), one can straighforwardly extend the results of Example 2.1 to any bounded set B.

Theorem 3.1. *Let B be bounded and let j be defined by* (3.1). *Then $\lambda^* = \zeta^*$ and*

$$\text{if } \exists\, w \in \mathbb{V}\colon\ J_{\bar\lambda}(w) < -c_2|\Omega| \text{ for some } \bar\lambda > 0, \text{ then } \bar\lambda > \lambda^*, \tag{3.12}$$

where c_2 is the same as in (3.11).

The results of Example 2.1 and Theorem 3.1 are illustrated in the following 1D example with the known analytical solution.

Example 3.1. Let

$$B = [-1,1], \quad \Omega = (-1,1),$$
$$\mathbb{V} = \{v \in H^1((-1,1))\colon\ v(-1) = 0\}, \quad F = \text{const. in } \Omega,\ f(1) = 0.$$

Then

$$Q_{\lambda L} = \{\tau \in L^2((-1,1))\colon\ \tau' + \lambda F = 0 \text{ in } (-1,1),\ \tau(1) = 0\} = \{\sigma_\lambda\},$$

where $\sigma_\lambda(x) = \lambda F(1-x)$, $x \in (-1,1)$. Hence, $Q_{\lambda L} \cap P \neq \emptyset$ if and only if $2\lambda|F| \leqslant 1$. Thus $\lambda^* = (2|F|)^{-1}$ if $F \neq 0$, otherwise $\lambda^* = \infty$. The solution u_λ of the primal problem exists in the classical sense up to the limit load parameter and

$$u_\lambda(x) = \frac{1}{2}\lambda F[4 - (1-x)^2], \quad x \in [-1,1] \quad \forall\, \lambda \leqslant \lambda^*.$$

From now on, assume that $F \neq 0$. Then one can construct a minimizing sequence for the respective kinematic principle (3.10) of limit analysis:

$$\lambda^* \leqslant \zeta^* = \inf_{\substack{v \in \mathbb{V} \\ F\int_{-1}^1 v\,\mathrm{d}x = 1}} \int_{-1}^1 |v'|\,\mathrm{d}x \leqslant \lim_{n\to\infty} \int_{-1}^1 |v_n'|\,\mathrm{d}x = \frac{1}{2|F|},$$

where $v_n \in \mathbb{V}$, $F\int_{-1}^1 v_n\,\mathrm{d}x = 1$ and

$$v_n(x) = \begin{cases} \dfrac{1}{F}\dfrac{2n^2}{4n-1}(x+1) & \text{if } x \in [-1,-1+n^{-1}], \\[2mm] \dfrac{1}{F}\dfrac{2n}{4n-1} & \text{if } x \in [-1+n^{-1},1]. \end{cases}$$

Thus $\lambda^* = \zeta^* = (2|F|)^{-1}$. Since $j(u'_{\lambda^*}) = \frac{1}{2}(u'_{\lambda^*})^2$ in $\bar\Omega$, we have

$$\begin{aligned} J_{\lambda^*}(v) \geqslant J_{\lambda^*}(u_{\lambda^*}) &= \frac{1}{2}(\lambda^*)^2F^2\int_{-1}^1 (1-x)^2\,\mathrm{d}x - \frac{1}{2}(\lambda^*)^2F^2\int_{-1}^1 [4-(1-x)^2]\,\mathrm{d}x \\ &= -\frac{4}{3}(\lambda^*)^2F^2 = -\frac{1}{3} \end{aligned}$$

for any $v \in \mathbb{V}$. Hence,

$$\text{if } \exists\, w \in \mathbb{V}\colon\ J_{\bar\lambda}(w) < -\frac{1}{3} = c_{\text{opt}} \text{ for some } \bar\lambda > 0, \text{ then } \bar\lambda > \lambda^*,$$

which is a sharper estimate than (3.12), since $c_2|\Omega| = 1$. Finally, we sketch how the estimate (3.12) can be used for computing a “minimal” upper bound of λ^*. Consider the sequence $\{w_n\}$, where

$$w_n(x) = \begin{cases} \operatorname{sgn}(F)n^2(x+1) & \text{if } x \in [-1, -1+n^{-1}], \\ \operatorname{sgn}(F)n & \text{if } x \in [-1+n^{-1}, 1] \end{cases}$$

and set $\lambda_\varepsilon = (1+\varepsilon)\lambda^*$, $\varepsilon > 0$. Since

$$J_{\lambda_\varepsilon}(w_n) = -\varepsilon n - \frac{1}{2n} + \frac{1}{4}(1+\varepsilon) \quad \forall\, n \geqslant 1,$$

$\{w_n\}$ is a minimizing sequence of J_{λ_ε} in $\mathbb{V}$ for any $\varepsilon > 0$. Clearly, $J_{\lambda_\varepsilon}(w_n) \leqslant -1$ for $\varepsilon := \varepsilon(n) = 5/(4n)$. By (3.12), $\{\lambda_{\varepsilon(n)}\}_n$ is a sequence of upper bounds tending to λ^* from above. In Section 6.2, we introduce Algorithm 2 for finding the “minimal” upper bound being inspired by this idea.

3.2. Unbounded sets B, truncation approach. For a bounded set B we know that $\lambda^* = \zeta^*$ and the guaranteed upper bound (3.12) of λ^* holds as it follows from Theorem 3.1. If B is unbounded then such result is not at our disposal. For this reason, we construct its truncations using an appropriate system $\{B_k\}$, $k \to \infty$, of bounded sets.

Next, we assume that

$$\begin{cases} B_k \subset B \text{ is bounded, convex and satisfies (3.2)} \\ \text{for any } k > 0,\ \operatorname{diam} B_k = k; \\ B_k \subset B_l \text{ for any } k, l\colon\ 0 < k < l; \\ \overline{\bigcup_{k>0} B_k} = B, \text{ i.e., } \forall\, z \in B\ \exists \{z_k\},\ z_k \in B_k\colon\ z_k \to z,\ k \to \infty. \end{cases} \tag{3.13}$$

With any B_k we associate the functions j_k, $j_{k,\infty}$ and the limit parameters λ_k^*, ζ_k^* analogously to j, j_∞, and λ^*, ζ^* for unbounded B, respectively. The next theorem is an easy consequence of Lemma 2.1, (3.1), (3.7), and (3.13).

Theorem 3.2. *Let $B \subset \mathbb{R}^d$ be unbounded and let system $\{B_k\}$, $k \to \infty$, satisfy* (3.13). *Then*

$$
(3.14) \qquad \begin{cases} j_k \leqslant j_l \leqslant j,\ j_{k,\infty} \leqslant j_{l,\infty} \leqslant j_\infty & \forall\, k, l\colon\ 0 < k < l, \\ \zeta_k^* = \lambda_k^* \leqslant \zeta_l^* = \lambda_l^* \leqslant \lambda^* \leqslant \zeta^* & \forall\, k, l\colon\ 0 < k < l, \\ \lim\limits_{k\to\infty} j_{k,\infty}(z) = j_\infty(z) & \forall\, z \in \mathbb{R}^d, \\ \lim\limits_{k\to\infty} \lambda_k^* \leqslant \lambda^*. \end{cases}
$$

Notice that from $(3.14)_2$ it follows that λ_k^* is a lower bound of λ^* for any $k > 0$. Knowledge of a reliable lower bound of λ^* is important since it presents a safety parameter. A natural question arises, namely, under which conditions $\lambda_k^* \to \lambda^*$ as $k \to \infty$. Below, we present a sufficient condition ensuring this property. To this end, we define the following bounded sets of plastically admissible stress fields:

$$
P_k = \{\tau \in L^2(\Omega, \mathbb{R}^d)\colon\ \tau(x) \in B_k \text{ for a.a. } x \in \Omega\}.
$$

Lemma 3.3. *Let $\lambda > 0$ be such that there exists a sequence $\{\lambda_k\}$ with the following properties:*

$$
(3.15) \qquad P_k \cap Q_{\lambda_k L} \neq \emptyset \quad \forall\, k \in \mathbb{N},
$$

$$
(3.16) \qquad \lambda_k \to \lambda \quad \textit{as } k \to \infty.
$$

Then

$$
\lambda^* \geqslant \lim_{k\to\infty} \lambda_k^* \geqslant \lambda.
$$

P r o o f. The assertion easily follows from the chain of inequalities: $\lambda^* \geqslant \lambda_k^* \geqslant \lambda_k \to \lambda$ as $k \to \infty$. □

Corollary 3.1. *If $\lambda = \lambda^*$ and the sequence $\{\lambda_k\}$ satisfies* (3.15) *and* (3.16), *then $\lambda_k^* \to \lambda^*$ as $k \to \infty$.*

R e m a r k 3.2. Notice that the truncation approach can be also used if the set B itself is bounded. This fact will be used in Section 6.1 to get lower bounds of λ^*. Moreover, for bounded B it is easy to show that $P \cap Q_{\lambda^* L} \neq \emptyset$. Further, consider the sequences $\{B_k\}$, $B_k = \{(1 - 1/k)\tau\colon\ \tau \in B\}$, and $\{\lambda_k\}$, $\lambda_k = (1 - 1/k)\lambda^*$. Then (3.13), (3.15), and (3.16) are satisfied, and thus $\lambda_k^* \to \lambda^*$ as $k \to \infty$ by Corollary 3.1.

In the subsequent parts, we introduce two classes of unbounded B: cylindric and conic.

3.3. Unbounded cylindric sets B. Let $B = \widetilde{B} \oplus \mathcal{R}$, where $\mathcal{R}$ is a one-dimensional subspace of $\mathbb{R}^d = \mathcal{R}^\perp \oplus \mathcal{R}$, $d = 2, 3$, and $\widetilde{B} \subset \mathcal{R}^\perp$ is a closed, convex set such that

$$\exists\, \varepsilon, \varrho > 0\colon\ B(0;\varepsilon) = \{z^* \in \mathcal{R}^\perp\colon\ |z^*| \leqslant \varepsilon\} \subset \widetilde{B} \\ \subset B(0;\varrho) = \{z^* \in \mathcal{R}^\perp\colon\ |z^*| \leqslant \varrho\}. \tag{3.17}$$

Clearly, B is a closed, convex subset of $\mathbb{R}^d$ satisfying (3.2) and $\mathcal{R}$ represents the axis of the cylinder B.

Let $r\colon\ \mathbb{R}^d \to \mathcal{R}$ and $q\colon\ \mathbb{R}^d \to \mathcal{R}^\perp$ be the projections onto $\mathcal{R}$ and $\mathcal{R}^\perp$ with respect to the Euclidean scalar product, respectively. Then $z = r(z) + q(z)$ and $r(z)\cdot q(\widetilde{z}) = 0$ for any $z, \widetilde{z} \in \mathbb{R}^d$. Using (3.1), (3.7) and the orthogonality between $\mathcal{R}$ and $\mathcal{R}^\perp$, one can write

$$j(z) = \frac{1}{2}|r(z)|^2 + \sup_{z^*\in\widetilde{B}} \Big\{z^*\cdot q(z) - \frac{1}{2}|z^*|^2\Big\} \quad \forall\, z \in \mathbb{R}^d, \tag{3.18}$$

so that

$$j_\infty(z) = \begin{Bmatrix} \sup\limits_{z^*\in\widetilde{B}} z^*\cdot z, & z \in \mathcal{R}^\perp \\ \infty, & z \notin \mathcal{R}^\perp \end{Bmatrix} = \begin{Bmatrix} \sup\limits_{z^*\in\widetilde{B}} z^*\cdot q(z), & r(z) = 0 \\ \infty, & r(z) \neq 0 \end{Bmatrix}. \tag{3.19}$$

Hence,

$$\zeta^* = \inf_{\substack{w\in\mathbb{V}\\ L(w)=1}} \int_\Omega j_\infty(\nabla w)\,\mathrm{d}x = \inf_{\substack{w\in\mathbb{W}\\ L(w)=1}} \int_\Omega j_\infty(q(\nabla w))\,\mathrm{d}x,$$

where

$$\mathbb{W} = \{v \in \mathbb{V}\colon\ r(\nabla v) = 0 \text{ a.e. in } \Omega\}. \tag{3.20}$$

Thus the problem of limit analysis leads to a convex optimization problem with linear equality constraints.

The truncation of B proposed in Section 3.2 can be defined as follows:

$$B_k = \widetilde{B} \oplus \mathcal{B}_k, \quad \mathcal{B}_k = \{z \in \mathcal{R}\colon\ |z| \leqslant k\}, \quad k > 0.$$

Then

$$j_k(z) = \begin{cases} \dfrac{1}{2}|r(z)|^2 + \sup\limits_{z^*\in\widetilde{B}} \Big\{z^*\cdot q(z) - \dfrac{1}{2}|z^*|^2\Big\}, & |r(z)| \leqslant k, \\ k|r(z)| - \dfrac{k^2}{2} + \sup\limits_{z^*\in\widetilde{B}} \Big\{z^*\cdot q(z) - \dfrac{1}{2}|z^*|^2\Big\}, & |r(z)| \geqslant k, \end{cases} \tag{3.21}$$

and

$$j_{\infty,k}(z) = j_\infty(q(z)) + k|r(z)| \quad \forall\, z \in \mathbb{R}^d,\ \forall\, k > 0.$$

It is worth noticing that the term $k|r(z)|$ is the penalty functional associated with the constraint $r(z) = 0$.

The limit analysis for a cylindric set is illustrated by the following 2D example.

Example 3.2. Let

- ▷ $\Omega = \{(x,y) \in \mathbb{R}^2\colon\ |x| \leqslant 1, |y| \leqslant 1\},\ \Gamma_D = (-1,1) \times \{-1\},\ \Gamma_N = \partial\Omega \setminus \bar\Gamma_D$,
- ▷ $f = 0$ on $\Gamma_N,\ F := F(x),\ (x,y) \in \Omega,\ \overline{F} = \frac{1}{2}\int_{-1}^{1} F(x)\,\mathrm{d}x$,
- ▷ $\mathbb{V} = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_D\}$,
- ▷ $B = \{z = (z_1, z_2) \in \mathbb{R}^2\colon\ |z_2| \leqslant 1\}$.

Then $\mathcal{R} = \{z = (z_1, 0),\ z_1 \in \mathbb{R}\}$, $\widetilde{B} = \{z = (0, z_2)\colon\ |z_2| \leqslant 1\}$, $r(z_1,z_2) = (z_1, 0)$, $q(z_1, z_2) = (0, z_2)$ and

$$B_k = \{z = (z_1, z_2) \in \mathbb{R}^2\colon\ |z_1| \leqslant k,\ |z_2| \leqslant 1\}, \quad k > 0.$$

From (3.18) and (3.19) we have:

$$j(z) = \begin{cases} \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2, & |z_2| \leqslant 1, \\ \frac{1}{2}z_1^2 + |z_2| - \frac{1}{2}, & |z_2| \geqslant 1, \end{cases} \tag{3.22}$$

$$j_\infty(z) = \begin{cases} |z_2|, & z_1 = 0, \\ \infty, & z_1 \neq 0 \end{cases} \qquad \forall\, z = (z_1, z_2) \in \mathbb{R}^2,$$

respectively. It follows from (3.20) that $\mathbb{W}$ consists of all functions from $\mathbb{V}$ which depend on y only:

$$\mathbb{W} = \{v \in \mathbb{V}\colon\ v := v(y),\ y \in (-1,1)\}. \tag{3.23}$$

The definition of ζ^*, (3.22) and (3.23) yield:

$$\begin{aligned}\zeta^* &= \inf_{\substack{v\in\mathbb{W}\\ L(v)=1}} \int_{-1}^{1}\int_{-1}^{1} j_\infty\Big(\Big|\frac{\partial v}{\partial y}\Big|\Big)\,\mathrm{d}x\,\mathrm{d}y = \inf_{\substack{v\in\mathbb{W}\\ \overline{F}\int_{-1}^{1} 2v\,\mathrm{d}y=1}} \int_{-1}^{1} 2\Big|\frac{\partial v}{\partial y}\Big|\,\mathrm{d}y \\ &= \inf_{\substack{w\in\overline{\mathbb{W}}\\ \overline{F}\int_{-1}^{1} w\,\mathrm{d}y=1}} \int_{-1}^{1} |w'(y)|\,\mathrm{d}y \overset{\text{Ex.3.1}}{=} \frac{1}{2|\overline{F}|},\end{aligned}$$

where $\overline{\mathbb{W}} = \{w \in H^1((-1,1))\colon\ w(-1) = 0\}$. Clearly, $\zeta^* = \infty$ for $\overline{F} = 0$. We prove the equality $\lambda^* = \zeta^*$ for two choices of F.

i) Let $F = \text{const.}$ in Ω. Then $\overline{F} = F$, $\zeta^* = 1/(2|F|)$ and problems $(\mathcal{P})_\lambda$, $(\mathcal{P}^*)_\lambda$ have the solutions

$$u_\lambda(x,y) = \frac{1}{2}\lambda F[4-(1-y)^2], \quad \sigma_\lambda(x,y) = (0, \lambda F(1-y)) \quad \forall\, \lambda \leqslant \frac{1}{2|F|},$$

respectively. Hence, $\lambda^* = \zeta^* = 1/(2|F|)$ by making use of Example 3.1. Since σ_λ, $\lambda \leqslant 1/(2|F|)$, also belongs to B_k everywhere in $\bar{\Omega}$, it is a solution to problem $(\mathcal{P}_k^*)_\lambda$ related to B_k. Consequently, $\lambda_k^* = \lambda^*$ for any $k > 0$.

ii) Let $F(x) = x$ in Ω. Then $\overline{F} = 0$, so $\zeta^* = \infty$ and

$$\inf_{v\in\mathbb{V}} J_\lambda(v) \geqslant \inf_{v\in\widetilde{\mathbb{V}}} J_\lambda(v) = J_\lambda(\widetilde{u}_\lambda) > -\infty, \quad \widetilde{u}_\lambda(x,y) = -\frac{\lambda}{6}x^3 + \frac{\lambda}{2}x, \quad \forall\, \lambda \geqslant 0,$$

where $\widetilde{\mathbb{V}} = \{v \in \mathbb{V}\colon \int_{\Gamma_D} v \,\mathrm{d}s = \int_{-1}^1 v(x,-1)\,\mathrm{d}x = 0\}$. Indeed, $\widetilde{u}_\lambda$ is a minimizer of J_λ on $\widetilde{\mathbb{V}}$ even in the classical sense since

$$\int_{-1}^1 \widetilde{u}_\lambda(x,-1)\,\mathrm{d}x = 0,$$

$$\nabla\widetilde{u}_\lambda(x,y) = \frac{\lambda}{2}(1-x^2,0)^{\mathrm{T}} = \Pi_B(\nabla\widetilde{u}_\lambda(x,y)) \quad \forall\,(x,y)\in\Omega,$$

$$\operatorname{div}\Pi_B(\nabla\widetilde{u}_\lambda) = -\lambda F \quad \text{in } \Omega,$$

$$\Pi_B(\nabla\widetilde{u}_\lambda)\cdot\nu = 0 \quad \text{on } \partial\Omega.$$

Hence, $\lambda^* = \zeta^* = \infty$. Since $\nabla\widetilde{u}_\lambda \in B_k$ for $\lambda \leqslant 2k$ everywhere in $\bar{\Omega}$, we have $\lambda_k^* \geqslant 2k$. Therefore, $\lambda_k^* \to \lambda^* = \infty$ as $k \to \infty$.

3.4. Unbounded conical sets B. Let $\mathcal{B} \subset \mathbb{R}^d$ be a *closed, convex cone* with vertex at 0, $z_0^* \in \mathbb{R}^d$ and $B = z_0^* + \mathcal{B}$. In order to satisfy (3.2) for such B we assume that $-z_0^* \in \operatorname{int}(\mathcal{B})$. Further, let $\mathcal{B}^-$ be the polar cone to $\mathcal{B}$:

$$\mathcal{B}^- = \{z \in \mathbb{R}^d\colon\ z\cdot z^* \leqslant 0 \quad \forall\, z^* \in \mathcal{B}\} = \{z\in\mathbb{R}^d\colon\ r(z) = 0\},$$

where $r\colon \mathbb{R}^d \to \mathcal{B}$ is the projection onto $\mathcal{B}$ with respect to the Euclidean scalar product, i.e.,

$$|z - r(z)| \leqslant |z - z^*| \quad \forall\, z^* \in \mathcal{B}. \tag{3.24}$$

We have

$$\begin{aligned} j(z) &= z_0^*\cdot z - \frac{1}{2}|z_0^*|^2 + \sup_{z^*\in\mathcal{B}}\Big\{z^*\cdot(z-z_0^*) - \frac{1}{2}|z^*|^2\Big\} \\ &= z_0^*\cdot z - \frac{1}{2}|z_0^*|^2 + \frac{1}{2}|z-z_0^*|^2 - \frac{1}{2}\inf_{z^*\in\mathcal{B}}|z - z_0^* - z^*|^2 \\ &\overset{(3.24)}{=} \frac{1}{2}|z|^2 - \frac{1}{2}|z - z_0^* - r(z-z_0^*)|^2 \end{aligned}$$

and consequently $j(0) = -\frac{1}{2}|-z_0^* - r(-z_0^*)|^2 = 0$ using that $-z_0^* \in \operatorname{int}(\mathcal{B})$. This fact follows from (3.2). The function j has a guaranteed linear growth:

$$j(z) = z_0^* \cdot z - \frac{1}{2}|z_0^*|^2 + \sup_{z^* \in \mathcal{B}} \Big\{ z^* \cdot (z - z_0^*) - \frac{1}{2}|z^*|^2 \Big\} \geqslant z_0^* \cdot z - \frac{1}{2}|z_0^*|^2 \quad \forall\, z \in \mathbb{R}^d$$

and

$$j_\infty(z) = z_0^* \cdot z + \sup_{z^* \in \mathcal{B}} \{z^* \cdot z\} = \begin{cases} z_0^* \cdot z, & r(z) = 0, \\ \infty, & r(z) \neq 0. \end{cases}$$

Then

$$\zeta^* = \inf_{\substack{v \in \mathbb{V} \\ L(v)=1}} \int_\Omega j_\infty(\nabla v)\,\mathrm{d}x = \inf_{\substack{v \in \mathbb{W} \\ L(v)=1}} \int_\Omega z_0^* \cdot \nabla v\,\mathrm{d}x = \inf_{\substack{v \in \mathbb{W} \\ L(v)=1}} \int_{\Gamma_N} (z_0^* \cdot \nu) v\,\mathrm{d}x,$$

where $\mathbb{W}$ is defined by (3.20) with r from (3.24). This is a minimization problem with a linear functional and nonlinear equality constraints. Notice that $\lambda^* = \zeta^* = 0$ if $\{v \in \mathbb{W}\colon\ L(v) = 1\} \neq \emptyset$ and $\Gamma_N = \emptyset$.

The truncation of the conical set $B = z_0^* + \mathcal{B}$ can be for example defined as follows:

$$B_k = z_0^* + \mathcal{B}_k, \quad \mathcal{B}_k = \mathcal{B} \cap B(0;k), \quad B(0;k) := \{z \in \mathbb{R}^d\colon\ |z| \leqslant k\}, \quad k > |z_0^*|.$$

Then

$$j_k(z) = \frac{1}{2}|z|^2 - \frac{1}{2}|z - z_0^* - r_k(z - z_0^*)|^2 \quad \forall\, k > 0,\ \forall\, z \in \mathbb{R}^d,$$

where

$$r_k(z) = \min\Big\{ \frac{k}{|r(z)|}, 1 \Big\} r(z) \quad \forall\, k > 0,\ \forall\, z \in \mathbb{R}^d,$$

and

$$j_{\infty,k}(z) = z_0^* \cdot z + k|r(z)| \quad \forall\, z \in \mathbb{R}^d,\ \forall\, k > 0.$$

Again, $j_{k,\infty}$ is a penalized form of j_∞ with respect to the constraint $r(z) = 0$.

The limit analysis for a conical set is illustrated by the following 1D example.

E x a m p l e 3.3. Let $B = (-\infty, 1]$, $\Omega = (-1, 1)$, $\mathbb{V} = \{v \in H^1(\Omega)\colon\ v(-1) = 0\}$, $F = \text{const.}$ and $f(1) = 0$. Then $B_k = [1-k, 1]$, $k > 1$, and similarly to Example 3.1 one can show that

$$\lambda^* = \zeta^* = \begin{cases} \dfrac{1}{2F}, & F > 0, \\ \infty, & F \leqslant 0, \end{cases} \qquad \lambda_k^* = \zeta_k^* = \begin{cases} \dfrac{1}{2F}, & F > 0, \\ \dfrac{k-1}{2|F|}, & F < 0, \\ \infty, & F = 0. \end{cases}$$

Hence, $\lambda_k^* \to \lambda^*$ as $k \to \infty$.

4. Generalized Hencky plasticity problem

The classical Hencky plasticity model of the deformation plasticity theory is based on the von Mises yield law. Since an abstract yield criterion is used within this section, we rather write "generalized" Hencky plasticity model in order to stress this fact. For more details, we refer to, e.g., [7], [13], [27].

This problems and the scalar one considered in Sections 2 and 3 have similar mathematical structures. Therefore we keep the same notation. Possible different meaning of notation in plasticity problems will be mentioned explicitly. From now on, we shall consider 3D problems. The space of admissible displacement fields has the form

$$\mathbb{V} = \{v \in H^1(\Omega; \mathbb{R}^3)\colon\ v|_{\Gamma_D} = 0\},$$

$f \in L^2(\Gamma_N; \mathbb{R}^3)$, $F \in L^2(\Omega; \mathbb{R}^3)$ denote the density of surface and volume forces, respectively, and

$$L(v) = \int_\Omega F \cdot v \,\mathrm{d}x + \int_{\Gamma_N} f \cdot v \,\mathrm{d}s, \quad v \in \mathbb{V}, \quad \|F\|_{L^2(\Omega;\mathbb{R}^3)} + \|f\|_{L^2(\Gamma_N;\mathbb{R}^3)} \neq 0.$$

Stress and strain tensors are represented locally by symmetric matrices, i.e., elements of $\mathbb{R}^{3\times3}_{\rm sym}$. The biscalar product and the corresponding norm in $\mathbb{R}^{3\times3}_{\rm sym}$ will be denoted by $e : \eta = e_{ij}\eta_{ij}$ and $\|e\|^2 = e : e$ for any $e, \eta \in \mathbb{R}^{3\times3}_{\rm sym}$, respectively.

Let B be a closed, convex subset of $\mathbb{R}^{3\times3}_{\rm sym}$ containing a vicinity of the origin. This set represents *plastically admissible stresses* and mostly can be defined as follows:

$$B = \{\tau \in \mathbb{R}^{3\times3}_{\rm sym}\colon\ \Phi(\tau) \leqslant \gamma\}, \tag{4.1}$$

where $\Phi\colon\ \mathbb{R}^{3\times3}_{\rm sym} \to \mathbb{R}$, $\gamma > 0$ are a yield function and an initial yield stress, respectively. We shall suppose that Φ is convex and $\Phi(0) = 0$.

Unlike in Section 3, the mapping Π_B now denotes a *generalized* projection of $\mathbb{R}^{3\times3}_{\rm sym}$ onto B (in the sense of [24]) and represents the constitutive stress-strain relation:

$$\Pi_B\colon\ e \mapsto \Pi_B(e), \quad \|\mathbb{C}e - \Pi_B(e)\|_{\mathbb{C}^{-1}} = \min_{\tau\in B} \|\mathbb{C}e - \tau\|_{\mathbb{C}^{-1}}, \quad e \in \mathbb{R}^{3\times3}_{\rm sym},$$

where $\mathbb{C}\colon\ \mathbb{R}^{3\times3}_{\rm sym} \to \mathbb{R}^{3\times3}_{\rm sym}$ is a linear, positive definite, fourth order elasticity tensor characterizing the elastic material response, $\mathbb{C}^{-1}$ is the corresponding inverse and $\|\tau\|^2_{\mathbb{C}^{-1}} := \mathbb{C}^{-1}\tau\colon\ \tau$ for any $\tau \in \mathbb{R}^{3\times3}_{\rm sym}$. The potential $j\colon\ \mathbb{R}^{3\times3}_{\rm sym} \to \mathbb{R}_+$ of Π_B is defined by

$$j(e) = \sup_{\tau\in B} \Big\{\tau : e - \frac{1}{2}\|\tau\|^2_{\mathbb{C}^{-1}}\Big\}, \quad e \in \mathbb{R}^{3\times3}_{\rm sym}. \tag{4.2}$$

Again,

$$j(e) = \Pi_B(e) \colon e - \frac{1}{2}\|\Pi_B(e)\|^2_{\mathbb{C}^{-1}} \geqslant 0, \quad \nabla j(e) = \Pi_B(e) \quad \forall\, e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \tag{4.3}$$

and

$$\frac{\varepsilon}{2}\|e\|_{\mathbb{C}} - \frac{\varepsilon^2}{8} \leqslant j(e) \leqslant \frac{1}{2}\|e\|^2_{\mathbb{C}}, \quad \|e\|^2_{\mathbb{C}} := \mathbb{C}e : e, \quad \forall\, e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}, \tag{4.4}$$

where $\varepsilon > 0$ is such that the ball $\{\tau \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \colon \|\tau\|_{\mathbb{C}^{-1}} \leqslant \varepsilon\}$ belongs to B. Thus, j has a guaranteed linear growth at infinity.

The generalized Hencky plasticity problem (in terms of displacements) for a given value of the load parameter $\lambda \geqslant 0$ reads as follows:

$$(\mathcal{P})_\lambda \qquad \inf_{v \in V} J_\lambda(v), \quad J_\lambda(v) = \int_\Omega j(\varepsilon(v))\,\mathrm{d}x - \lambda L(v), \quad \varepsilon(v) = \frac{1}{2}(\nabla v + \nabla^{\mathrm{T}} v).$$

As in Section 2 and 3, one can introduce the static and kinematic principle of limit analysis:

$$\lambda^* = \sup\Big\{\lambda \geqslant 0 \colon \inf_{v \in \mathbb{V}} J_\lambda(v) > -\infty\Big\} \leqslant \zeta^* = \inf_{\substack{v \in \mathbb{V} \\ L(v)=1}} \int_\Omega j_\infty(\varepsilon(v))\,\mathrm{d}x,$$

where

$$j_\infty(e) = \lim_{\alpha \to \infty} \frac{1}{\alpha} j(\alpha e) = \sup_{\tau \in B} e : \tau \quad \forall\, e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}.$$

Analogously to (3.8), we have

$$\liminf_{\alpha \to \infty} \frac{1}{\alpha} j(\alpha e_\alpha) \geqslant j_\infty(e) \tag{4.5}$$

for any $e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ and any sequence $\{e_\alpha\}$ tending to e as $\alpha \to \infty$.

As we have already mentioned, the set B is usually defined by a yield criterion (4.1). For example, the Cam-Clay and capped Drucker-Prager criteria lead to bounded sets, while the von Mises or Tresca criteria lead to unbounded cylindric and Drucker-Prager or Mohr-Coulomb criteria to unbounded conical sets. The mentioned yield criteria and many others are presented in [5].

For bounded B, we have the following elastoplastic counterpart of Theorem 3.1.

Theorem 4.1. *Let B be bounded and let j be defined by* (4.2). *Then* $\lambda^* = \zeta^*$,

$$j_\infty(e) - c_2 \leqslant j(e) \leqslant j_\infty(e) \quad \forall\, e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}},\ c_2 = \frac{1}{2}\sup_{\tau\in B}\|\tau\|^2_{\mathbb{C}^{-1}} \tag{4.6}$$

and

$$\text{if } \exists\, w \in \mathbb{V}\colon\ J_{\bar\lambda}(w) < -c_2|\Omega| \text{ for some } \bar\lambda > 0, \text{ then } \bar\lambda > \lambda^*. \tag{4.7}$$

In the subsequent parts of this section, we introduce the von Mises and Drucker-Prager yield criteria.

4.1. The von Mises yield criterion. The set B of admissible stresses for the von Mises yield criterion is defined by

$$B = \{\tau \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}\colon\ \|\tau^D\| \leqslant \gamma\}, \tag{4.8}$$

where $\tau^D = \tau - \frac{1}{3}(\operatorname{tr}\tau)\iota$ is the deviatoric part of τ, $\operatorname{tr}\tau = \tau_{ii}$ is the trace of τ, $\iota = \operatorname{diag}(1,1,1)$ is the unit matrix and $\gamma > 0$ represents the initial yield stress. Notice that the (hydrostatic) axis $\mathcal{R}$ of this cylindric set is

$$\mathcal{R} = \{\tau \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}\colon\ \tau = a\iota,\ a \in \mathbb{R}\}.$$

If the elastic stress-strain relation is isotropic and expressed in terms of the bulk ($K > 0$) and shear ($G > 0$) moduli, i.e.,

$$\tau = \mathbb{C}e = K(\operatorname{tr} e)\iota + 2Ge^D \quad \forall\, e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}, \tag{4.9}$$

then the function j can be written as

$$j(e) = \begin{cases} \dfrac{1}{2}K(\operatorname{tr} e)^2 + G\|e^D\|^2, & 2G\|e^D\| \leqslant \gamma, \\[2mm] \dfrac{1}{2}K(\operatorname{tr} e)^2 + \gamma\|e^D\| - \dfrac{\gamma^2}{4G}, & 2G\|e^D\| > \gamma, \end{cases} \qquad e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}},$$

see e.g. [27]. It is readily seen that

$$j_\infty(e) = \lim_{\alpha\to\infty}\frac{1}{\alpha}j(\alpha e) = \begin{cases} \gamma\|e^D\|, & \operatorname{tr} e = 0, \\ \infty, & \operatorname{tr} e \neq 0 \end{cases} \qquad \forall\, e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$$

and the corresponding problem of limit analysis (2.10) becomes:

$$\zeta^* = \inf_{\substack{v\in\mathbb{V},\ \operatorname{div} v=0 \\ L(v)=1}} \int_\Omega \gamma\|\varepsilon(v)\|\,\mathrm{d}x. \tag{4.10}$$

It is known that $\lambda^* = \zeta^*$ (see [27]).

One can easily extend the truncation method introduced in Sections 3.2 and 3.3. To this end, consider the system of bounded subsets of B:

$$B_k = \Big\{\tau \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}\colon\ \frac{1}{3}|\operatorname{tr}\tau| \leqslant k\gamma,\ \|\tau^D\| \leqslant \gamma\Big\}, \quad k > 0. \tag{4.11}$$

The functions j_k and $j_{k,\infty}$ associated with B_k are given by

$$j_k(e) = \begin{cases} \dfrac{1}{2}K(\operatorname{tr} e)^2 + G\|e^D\|^2, & K|\operatorname{tr} e| \leqslant k\gamma,\ 2G\|e^D\| \leqslant \gamma, \\ \dfrac{1}{2}K(\operatorname{tr} e)^2 + \gamma\|e^D\| - \dfrac{\gamma^2}{4G}, & K|\operatorname{tr} e| \leqslant k\gamma,\ 2G\|e^D\| > \gamma, \\ \gamma k|\operatorname{tr} e| - \dfrac{\gamma^2k^2}{2K} + G\|e^D\|^2, & K|\operatorname{tr} e| > k\gamma,\ 2G\|e^D\| \leqslant \gamma, \\ \gamma k|\operatorname{tr} e| - \dfrac{\gamma^2k^2}{2K} + \gamma\|e^D\| - \dfrac{\gamma^2}{4G}, & K|\operatorname{tr} e| > k\gamma,\ 2G\|e^D\| > \gamma \end{cases} \quad e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}},$$

and

$$j_{k,\infty}(e) = \gamma(\|e^D\| + k|\operatorname{tr} e|), \quad e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}. \tag{4.12}$$

For the sake of brevity, we skip their derivation. We know that $\lambda_k^* \leqslant \lambda^*$ for any $k \geqslant 0$. Further, from Theorem 4.1 we know that

$$\lambda_k^* = \zeta_k^*,$$
$$j_{k,\infty}(e) - \frac{\gamma^2}{2}\Big(\frac{k^2}{K} + \frac{1}{2G}\Big) \leqslant j_k(e) \leqslant j_{k,\infty}(e) \quad \forall\, e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}},$$

and

$$\text{if } \exists\, w \in \mathbb{V}\colon \int_\Omega j_k(\varepsilon(w))\,\mathrm{d}x - \bar\lambda L(w) < -\frac{\gamma^2}{2}\Big(\frac{k^2}{K} + \frac{1}{2G}\Big)|\Omega|, \quad \text{then } \bar\lambda > \lambda_k^*. \tag{4.13}$$

These results hold for any $k > 0$.

4.2. The Drucker-Prager yield criterion. The set of the admissible stresses for the Drucker-Prager yield criterion reads as follows:

$$B = \Big\{\tau \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}\colon\ \frac{a}{3}\operatorname{tr}\tau + \|\tau^D\| \leqslant \gamma\Big\} = \frac{\gamma}{a}\iota + \mathcal{B}, \quad a, \gamma > 0, \tag{4.14}$$

where

$$\mathcal{B} = \{\tau \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \colon \tfrac{1}{3} a \operatorname{tr} \tau + \|\tau^D\| \leqslant 0\}$$

is a cone containing the hydrostatic axis. For the shape of the yield surface in the Haigh-Westergaard coordinates, we refer to [5].

Assume that $\mathbb{C}$ is the same as in (4.9) and denote

$$q_s(e) := Ka(\operatorname{tr} e) + 2G\|e^D\| - \gamma, \quad q_a(e) := Ka(\operatorname{tr} e) - Ka^2\|e^D\| - \gamma, \quad e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}.$$

Notice that $q_s \geqslant q_a$. Then

$$\begin{aligned} j(e) &= \frac{K}{2}(\operatorname{tr} e)^2 + G\|e^D\|^2 - \frac{1}{2(Ka^2+2G)}\Big\{[(q_s(e))^+]^2 + \frac{2G}{Ka^2}[(q_a(e))^+]^2\Big\} \\ &= \begin{cases} \dfrac{K}{2}(\operatorname{tr} e)^2 + G\|e^D\|^2, & \text{if } q_s(e) \leqslant 0, \\ -\dfrac{\gamma^2}{2Ka^2} + \dfrac{\gamma}{a}\operatorname{tr} e + \dfrac{G}{Ka^2(Ka^2+2G)} q_a(e)^2, & \text{if } q_s(e) \geqslant 0 \geqslant q_a(e), \\ -\dfrac{\gamma^2}{2Ka^2} + \dfrac{\gamma}{a}\operatorname{tr} e, & \text{if } q_a(e) \geqslant 0, \end{cases} \end{aligned}$$

where g^+ denotes the positive part of g. The second form of j can be found in [20], as well as the proof of the equality $\lambda^* = \zeta^*$ which holds for sufficiently small values of the parameter a and under appropriate assumptions.

For purposes of Section 6, we define the truncation of B as follows:

$$B_k = \{\tau \in B \colon \tfrac{1}{3} a \operatorname{tr} \tau \geqslant -k\gamma\}, \quad k \geqslant 1. \tag{4.15}$$

Clearly, B_k is a bounded subset of B. If we denote

$$\begin{aligned} q_{1,k}(e) &:= Ka(\operatorname{tr} e) - Ka^2\|e^D\| + \gamma\Big[\frac{(1+k)Ka^2}{2G} + k\Big], \\ q_{2,k}(e) &:= Ka(\operatorname{tr} e) + k\gamma, \quad q_{3,k}(e) := 2G\|e^D\| - (1+k)\gamma, \end{aligned}$$

then

$$j_k(e) = \begin{cases} \dfrac{K}{2}(\operatorname{tr} e)^2 + G\|e^D\|^2, \quad \text{if } q_s(e) \leqslant 0,\ q_{2,k}(e) \geqslant 0, \\ -\dfrac{\gamma^2}{2Ka^2} + \dfrac{\gamma}{a}\operatorname{tr} e + \dfrac{G}{Ka^2(Ka^2+2G)} q_a(e)^2, \\ \qquad\qquad\qquad\qquad \text{if } q_s(e) \geqslant 0 \geqslant q_a(e),\ q_{1,k}(e) \geqslant 0, \\ -\dfrac{\gamma^2}{2Ka^2} + \dfrac{\gamma}{a}\operatorname{tr} e, \qquad \text{if } q_a(e) \geqslant 0, \\ -\dfrac{k^2\gamma^2}{2Ka^2} - \dfrac{k\gamma}{a}\operatorname{tr} e - \dfrac{(1+k)^2\gamma^2}{4G} + (1+k)\gamma\|e^D\|, \\ \qquad\qquad\qquad\qquad \text{if } q_{1,k}(e) \leqslant 0,\ q_{3,k}(e) \geqslant 0, \\ -\dfrac{k^2\gamma^2}{2Ka^2} - \dfrac{k\gamma}{a}\operatorname{tr} e + G\|e^D\|^2, \quad \text{if } q_{2,k}(e) \leqslant 0,\ q_{3,k}(e) \leqslant 0, \end{cases}$$

$$
(4.16) \qquad j_{k,\infty}(e) = \begin{cases} \dfrac{\gamma}{a}\operatorname{tr} e, & \text{if } \operatorname{tr} e \geqslant a\|e^D\|, \\ -\dfrac{k\gamma}{a}\operatorname{tr} e + (1+k)\gamma\|e^D\|, & \text{if } \operatorname{tr} e \leqslant a\|e^D\|, \end{cases}
$$

and

$$
(4.17) \qquad \text{if } \exists\, w \in \mathbb{V}\colon \int_\Omega j_k(\varepsilon(w))\,\mathrm{d}x - \bar\lambda L(w) < -\frac{\gamma^2}{2}\Big(\frac{k^2}{Ka^2} + \frac{(1+k)^2}{2G}\Big)|\Omega|, \\ \text{then } \bar\lambda > \lambda_k^*.
$$

These results hold for any $k \geqslant 1$. For the sake of brevity, we skip their technical proofs.

5. Finite element approximation

In this section, classical finite element approximations are considered in order to compute bounds of the limit load in the generalized Hencky plasticity. A similar approach can also be used for the scalar problem introduced in Sections 2 and 3.

For the sake of simplicity, we suppose that $\Omega \subset \mathbb{R}^3$ is a *polyhedral* domain. Let $\{\mathcal{T}_h\}$, $h \to 0_+$, be a system of regular partitions of $\overline{\Omega}$ into tetrahedrons Δ, $\operatorname{diam}\Delta \leqslant h$ for any $\Delta \in \mathcal{T}_h$, which are consistent with the decomposition of $\partial\Omega$ into Γ_D and Γ_N. With any $\mathcal{T}_h$ we associate the finite-dimensional space $\mathbb{V}_h$:

$$
\mathbb{V}_h = \{v_h \in C(\overline{\Omega};\mathbb{R}^3)\colon\ v_h|_\Delta \in P_1(\Delta;\mathbb{R}^3)\ \forall\,\Delta \in \mathcal{T}_h,\ v_h = 0 \text{ on } \Gamma_D\},
$$

i.e., $\mathbb{V}_h$ consists of all continuous, piecewise linear vector functions $v_h\colon \overline{\Omega} \to \mathbb{R}^3$ vanishing on Γ_D. The space $\mathbb{V}_h$ is the simplest conformal finite element discretization of $\mathbb{V}$.

We arrive at the following discrete form of $(\mathcal{P})_\lambda$, $\lambda > 0$:

$$
(\mathcal{P}_h)_\lambda \qquad \text{minimize } J_\lambda(v_h),\ v_h \in \mathbb{V}_h, \quad J_\lambda(v_h) = \int_\Omega j(\varepsilon(v_h))\,\mathrm{d}x - \lambda L(v_h).
$$

Since $\mathbb{V}_h$ is finite dimensional, the mapping $|\!|\!|\cdot|\!|\!|\colon \mathbb{V}_h \to \mathbb{R}_+$,

$$
|\!|\!|v_h|\!|\!| = \int_\Omega \|\varepsilon(v_h)\|_{\mathbb{C}}\,\mathrm{d}x \quad \forall\, v_h \in \mathbb{V}_h,
$$

defines a norm in $\mathbb{V}_h$ and there exist positive constants c_1, c_2 such that

$$
(5.1) \qquad \int_\Omega j(\varepsilon(v_h))\,\mathrm{d}x \overset{(4.4)}{\geqslant} c_1|\!|\!|v_h|\!|\!| - c_2 \quad \forall\, v_h \in \mathbb{V}_h.
$$

Analogously to Section 2, we define the *discrete static limit load parameter* λ_h^*:

$$\lambda_h^* = \sup\{\lambda \geqslant 0\colon \inf_{v_h \in \mathbb{V}_h} J_\lambda(v_h) > -\infty\}, \tag{5.2}$$

and the *discrete kinematic limit load parameter* ζ_h^*:

$$\zeta_h^* = \inf_{\substack{v_h \in \mathbb{V}_h,\\ L(v_h)=1}} J_\infty(v_h), \tag{5.3}$$

where

$$J_\infty(v_h) = \int_\Omega j_\infty(\varepsilon(v_h))\,\mathrm{d}x \quad \forall\, v_h \in \mathbb{V}_h.$$

Again, $\lambda_h^* \leqslant \zeta_h^*$ for any $h > 0$.

In [13], [26], the following result was proven.

Theorem 5.1. *Let $\lambda > 0$ be given. Then the following statements are equivalent:*

(i) $\lambda < \lambda_h^*$;

(ii) *J_λ is coercive on $\mathbb{V}_h$;*

(iii) *the solution set to $(\mathcal{P}_h)_\lambda$ is nonempty and bounded.*

Parallel to $(\mathcal{P}_h)_\lambda$, the following minimization problem was introduced in [26]: given $\alpha > 0$,

$$(\mathcal{P}_h)^\alpha \qquad \text{minimize} \quad \int_\Omega j(\varepsilon(v_h))\,\mathrm{d}x, \quad v_h \in \mathbb{V}_h^\alpha,$$

where

$$\mathbb{V}_h^\alpha = \{v_h \in \mathbb{V}_h\colon\ L(v_h) = \alpha\}.$$

Owing to (5.1), problem $(\mathcal{P}_h)^\alpha$ has at least one solution $u_{h,\alpha}$ for any $\alpha > 0$ and $h > 0$. Problems $(\mathcal{P}_h)^\alpha$ and $(\mathcal{P}_h)_\lambda$ are linked to each other as follows from the next theorem (for the proof see [2]).

Theorem 5.2. *Let $u_{h,\lambda}$ be a solution to $(\mathcal{P}_h)_\lambda$. Then $u_{h,\lambda}$ solves $(\mathcal{P}_h)^\alpha$ for $\alpha = L(u_{h,\lambda})$. Conversely, let $u_{h,\alpha}$ be a solution to $(\mathcal{P}_h)^\alpha$. Then $u_{h,\alpha}$ solves $(\mathcal{P}_h)_\lambda$, where*

$$\lambda = \frac{1}{\alpha}\int_\Omega \Pi_B(\varepsilon(u_{h,\alpha}))\colon\ \varepsilon(u_{h,\alpha})\,\mathrm{d}x. \tag{5.4}$$

Moreover, λ does not depend on the choice of $u_{h,\alpha}$ solving $(\mathcal{P}_h)^\alpha$.

On the basis of Theorem 5.2 one can define the function $\psi_h\colon \mathbb{R}_+ \to \mathbb{R}_+$ by

$$(5.5)\qquad \begin{cases} \psi_h(\alpha) = \dfrac{1}{\alpha}\displaystyle\int_\Omega \Pi_B(\varepsilon(u_{h,\alpha})) : \varepsilon(u_{h,\alpha})\,\mathrm{d}x, & \alpha > 0,\\ \psi_h(0) = 0 \end{cases}$$

with $u_{h,\alpha}$ being a solution to $(\mathcal{P}_h)^\alpha$. The properties of ψ_h are listed in the next theorem. For the proof we refer to [13], [26].

Theorem 5.3. *We have:*

(j) ψ_h *is continuous and nondecreasing in* $\mathbb{R}_+$*;*

(jj) $\psi_h(\alpha) \to \lambda_h^*$ *as* $\alpha \to \infty$.

From (jj), we see that $\psi_h(\alpha)$ is an approximation of λ_h^* from below for $\alpha > 0$ large enough. The function ψ_h, however, provides also an information on λ^* defined by (2.4). This is a consequence of the following result [2], [13].

Theorem 5.4. *There exists a continuous, nondecreasing function* $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\psi(0) = 0$, *such that*

$$\psi(\alpha) \to \lambda^* \quad \text{as } \alpha \to \infty;$$
$$\psi_h(\alpha) \to \psi(\alpha) \quad \text{as } h \to 0_+ \ \forall\, \alpha > 0.$$

Since $\lambda_h^* \leqslant \zeta_h^*$, one can ask whether $\lambda_h^* = \zeta_h^*$ or not. Notice that the minimizer in (5.3) exists provided that the set of admissible functions is nonempty, i.e., there exists $v_h \in \mathbb{V}_h$ such that $L(v_h) = 1$ and $j_\infty(\varepsilon(v_h)|_\Delta) < \infty$ for any $\Delta \in \mathcal{T}_h$. It was shown in [2] that $\lambda_h^* = \zeta_h^*$ for the Mises yield criterion. This result will be now extended to the whole class of yield functions Ψ defining the set B of plastically admissible stresses (see (4.1)).

Theorem 5.5. *Let* $j\colon \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}_+$ *be defined by* (4.2) *and let* $u_{h,\alpha}$, $\alpha > 0$, *be a solution to* $(\mathcal{P}_h)^\alpha$. *If* $\lambda_h^* < \infty$ *then the sequence* $\{\alpha^{-1}u_{h,\alpha}\}_\alpha$ *is bounded in* $\mathbb{V}_h$. *In addition, any accumulation point* $u_{h,\infty}$ *of this sequence belongs to* $\operatorname{dom} J_\infty$, $L(u_{h,\infty}) = 1$, *and*

$$(5.6)\qquad \lambda_h^* = \zeta_h^* = \inf_{\substack{w_h \in \mathbb{V}_h,\\ L(w_h)=1}} \int_\Omega j_\infty(\varepsilon(w_h))\,\mathrm{d}x = \int_\Omega j_\infty(\varepsilon(u_{h,\infty}))\,\mathrm{d}x.$$

P r o o f. Let $\lambda_h^* < \infty$, $\alpha > 0$ be arbitrary and let $u_{h,\alpha} \in \mathbb{V}_h^\alpha$ be a solution to $(\mathcal{P}_h)^\alpha$. From Theorem 5.2 we know that there exists $\lambda_\alpha \in [0, \lambda_h^*]$ such that $u_{h,\alpha}$ solves $(\mathcal{P}_h)_{\lambda_\alpha}$. Therefore,

$$(5.7)\qquad \int_\Omega j(\varepsilon(u_{h,\alpha}))\,\mathrm{d}x = \inf_{v_h \in \mathbb{V}_h} J_{\lambda_\alpha}(v_h) + \lambda_\alpha\alpha \leqslant J_{\lambda_\alpha}(0) + \lambda_\alpha\alpha = \lambda_\alpha\alpha \leqslant \lambda_h^*\alpha.$$

Boundedness of $\{\alpha^{-1}u_{h,\alpha}\}$ in $\mathbb{V}_h$ follows from (5.7) and (5.1). Let $\{{\alpha'}^{-1}u_{h,\alpha'}\}$ be a subsequence such that ${\alpha'}^{-1}u_{h,\alpha'} \to u_{h,\infty}$ in $\mathbb{V}_h$ as $\alpha' \to \infty$. Clearly, $L(u_{h,\infty}) = 1$. Further,

$$\liminf_{\alpha'\to\infty} \frac{1}{\alpha'} j(\varepsilon(u_{h,\alpha'})|_\Delta) \overset{(4.5)}{\geqslant} j_\infty(\varepsilon(u_{h,\infty})|_\Delta) \quad \forall\, \Delta \in \mathcal{T}_h, \tag{5.8}$$

so that

$$\begin{aligned} \int_\Omega j_\infty(\varepsilon(u_{h,\infty}))\,\mathrm{d}x &\leqslant \int_\Omega \liminf_{\alpha'\to\infty} \frac{1}{\alpha'} j(\varepsilon(u_{h,\alpha'}))\,\mathrm{d}x \\ &\leqslant \liminf_{\alpha'\to\infty} \int_\Omega \frac{1}{\alpha'} j(\varepsilon(u_{h,\alpha'}))\,\mathrm{d}x \overset{(5.7)}{\leqslant} \lambda_h^* \end{aligned} \tag{5.9}$$

owing to Fatou's lemma and nonnegativeness of j. From (5.9) we may conclude that $u_{h,\infty} \in \operatorname{dom} J_\infty$ and $\lambda_h^* = \zeta_h^*$, proving (5.6). $\square$

R e m a r k 5.1. If $\lambda_h^* = \infty$ then the equality $\lambda_h^* = \zeta_h^*$ is automatically satisfied.

Corollary 5.1. *The following statements are equivalent:*

(k) $\lambda_h^* < \infty$;

(kk) *there exists* $w_h \in \mathbb{V}_h \cap \operatorname{dom} J_\infty$ *such that* $L(w_h) = 1$.

It is known from [13] that if B is bounded and the system $\{\mathbb{V}_h\}$ is limit dense in $\mathbb{V}$ then

$$\lambda_h^* \to \lambda^*, \quad h \to 0_+. \tag{5.10}$$

If B is unbounded, then (5.10) does not hold, in general. It is easy to show that a *necessary* condition for (5.10) to be satisfied is that $\lambda^* = \zeta^*$. Indeed, suppose that $\lambda^* < \zeta^*$ and (5.10) is satisfied. Since $\mathbb{V}_h \subset \mathbb{V}$ for any $h > 0$, we have

$$\lambda^* \leqslant \lambda_h^*, \quad \zeta^* \leqslant \zeta_h^* \quad \forall\, h > 0, \tag{5.11}$$

and at the same time $\lambda_h^* = \zeta_h^*$ as follows from Theorem 5.5. Then also $\zeta_h^* \to \lambda^* < \zeta^*$, which contradicts the second inequality in (5.11).

For unbounded B one can apply the truncation technique from Section 3.2 to the discretized problem. Let $\{B_k\}$ be a system of bounded, closed and convex subsets of B that satisfies (3.13). As in Section 3.2 we associate with any B_k the functions j_k, $J_{k,\lambda}$, $j_{k,\infty}$, $J_{k,\infty}$ and the discrete limit load parameters $\lambda_{k,h}^*$ and $\zeta_{k,h}^*$. Then $\lambda_{k,h}^* = \zeta_{k,h}^*$ and the following analogue of Theorem 4.1 holds:

$$\text{if } \exists\, \overline{w}_h \in \mathbb{V}_h\colon\ J_{k,\bar\lambda}(\overline{w}_h) < -c_k|\Omega| \quad \text{then } \bar\lambda > \lambda_{k,h}^*, \tag{5.12}$$

where

$$c_k = \frac{1}{2} \sup_{\tau \in B_k} \|\tau\|^2_{\mathbb{C}^{-1}}. \tag{5.13}$$

R e m a r k 5.2. From (4.12) and (4.16) we see that $j_{k,\infty}$ is a penalized form of j_k associated with a specific constraint. In this case, it is very easy to prove that

$$\lambda^*_{k,h} = \zeta^*_{k,h} \to \lambda^*_h = \zeta^*_h, \quad k \to \infty,$$

using standard techniques and the fact that $\mathbb{V}_h$ is finite dimensional.

6. Computable bounds of λ^* and numerical experiments

6.1. Computable bounds of λ^*. Since λ^* is a safety parameter, reliable computable bounds of this quantity are important. The first approach how to get them is based on the estimate

$$\psi(\alpha) \leqslant \lambda^* \leqslant \lambda^*_h \quad \forall\, \alpha > 0\ \forall\, h > 0, \tag{6.1}$$

and on convergence properties of $\psi_h(\alpha)$ for $\alpha \to \infty$ and/or $h \to 0_+$. To this end, we construct numerically the functions $\psi_{h_1}, \psi_{h_2}, \ldots, \psi_{h_N}$ on an interval $[0, \alpha_{\max}]$ for N different partitions $\mathcal{T}_{h_i}$, $i = 1, \ldots, N$, and $h_1 > h_2 > \ldots > h_N > \ldots > 0$ approaching zero. If $\alpha \in [0, \alpha_{\max}]$ is fixed and the values $\psi_{h_i}(\alpha)$ are visually the same for i large then $\psi_{h_i}(\alpha) \approx \psi(\alpha)$ due to Theorem 5.4. This yields an estimate of the lower bound in (6.1). In our experiments, the values $\psi_{h_i}(\alpha)$ almost coincide for α small enough and all $i = 1, \ldots, N$, while for α large convergence can be slow (see Figure 2). Further, if $i \in \{1, \ldots, N\}$ is fixed and the function ψ_{h_i} becomes almost constant for α large then $\psi_{h_i}(\alpha_{\max}) \approx \lambda^*_{h_i}$ by Theorem 5.3. This yields an estimate of the upper bound in (6.1). This way of estimating λ^* was suggested in [13] and can be used for both, bounded and unbounded sets B.

The second approach is based on the guaranteed upper bound (5.12) which is valid for *bounded* B. We proceed as follows: we try to find a parameter $\bar{\lambda}$ such that

(i) $\exists w_h \in \mathbb{V}_h\colon\ J_{\bar{\lambda}}(w_h) < -c_2|\Omega|$, where $c_2 > 0$ is known (cf. (3.11));
(ii) $\bar{\lambda}$ satisfying (i) is as small as possible.

From (i) it follows that $\lambda^* \leqslant \lambda^*_h < \bar{\lambda}$. From (ii) we obtain that $\bar{\lambda}$ is close to λ^*_h. In the next subsection, we present two strategies how to compute $\bar{\lambda}$ satisfying (i) and (ii). Notice that $\bar{\lambda}$ is a guaranteed upper bound of λ^* which can be established (at least theoretically) with an arbitrary accuracy using that $\lambda^*_h \to \lambda^*$, $h \to 0_+$ for bounded B. The upper bound (5.12) in combination with the truncation method can be also helpful to establish a lower bound of λ^*. We proceed as follows: the original set B of plastically admissible stresses (bounded or unbounded) is replaced by a bounded set $B_k \subset B$. Let $\underline{\lambda}_{k,h}$ denote a lower bound of $\lambda^*_{k,h}$ computed using the function $\psi_{k,h}$, and let $\overline{\lambda}_{k,h}$ be the guaranteed upper bound of $\lambda^*_{k,h}$. If $\underline{\lambda}_{k,h}$ and $\overline{\lambda}_{k,h}$: (j) are close to each other; (jj) remain almost constant for h small enough, then $\underline{\lambda}_{k,h} \approx \lambda^*_k \approx \overline{\lambda}_{k,h}$, i.e. $\underline{\lambda}_{k,h}$ is a *reliable* lower bound of λ^*. In our numerical experiments, (j) is always observed and (jj) occurs for sufficiently small k.

These approaches are used in two model examples with the von Mises and the Drucker-Prager yield criterion, respectively.

6.2. Numerical methods. The functions ψ_h, $h > 0$, can be assessed solving either $(\mathcal{P}_h)_\lambda$ or $(\mathcal{P}_h)^\alpha$. To this end, we use the semismooth Newton method with damping or, as a case may be, with regularized tangent stiffness matrices. In the latter case the tangent stiffness matrix is replaced by a convex combination of the tangent and elastic stiffness matrices to get positive definiteness. Damped parameters belong to $(0, 1]$ and guarantee the decrease of minimized functions in the Newton direction. For convergence analysis and numerical experiments with the variants of the semismooth Newton method we refer to [2].

Let $h > 0$ be fixed. Since the modified semismooth Newton method with damping generates a minimization sequence of J_λ in $\mathbb{V}_h$ for any $\lambda > 0$ we use this fact for finding the "minimal" guaranteed upper bound $\bar{\lambda}$ of λ^*_h for bounded B. We present two strategies.

Algorithm 1

1: Choose $\alpha_{\max} > 0$ sufficiently large.
2: Set $\bar{\lambda}^0 = \psi_h(\alpha_{\max})$, $n = 0$, and choose $\delta\lambda > 0$ sufficiently small.
3: Repeat
 ▷ $n := n + 1$;
 ▷ $\bar{\lambda}^n = \bar{\lambda}^{n-1} + \delta\lambda$;
 ▷ construct a minimization sequence of $J_{\bar{\lambda}^n}$ in $\mathbb{V}_h$;
4: Until $\bar{\lambda}^n$ satisfies (i).
5: Set $\bar{\lambda} := \bar{\lambda}^n$.

Notice that we approach $\bar{\lambda}$ from below in Algorithm 1 starting from $\bar{\lambda}^0 \leqslant \lambda^*_h$. Since $\bar{\lambda} - \lambda^*_h \leqslant \delta\lambda$, a smaller value of $\delta\lambda$ leads to a closer upper bound but a larger number of iterations is needed. The choice of $\alpha_{\max}$ depends on the particular problem.

Algorithm 2

1: Choose $\bar{\lambda}^0$ sufficiently large so that $\lambda^*_h < \bar{\lambda}^0$ can be expected.
2: Choose $u^0 \in \mathbb{V}_h$ and set $n = 0$.

3: For $m = 1, \ldots, m_{\max}$ do:

▷ Find $u^m \in \mathbb{V}_h$ such that $J_{\bar\lambda^n}(u^m) < J_{\bar\lambda^n}(u^{m-1})$.

▷ If $J_{\bar\lambda^n}(u^m) < -c_2|\Omega|$ then:

– compute $\delta\lambda = -(J_{\bar\lambda^n}(u^m) + c_2|\Omega|)/L(u^m)$;

– set $n := n + 1$;

– set $\bar\lambda^n := \bar\lambda^{n-1} - \delta\lambda$.

▷ End. (If)

4: Set $\bar\lambda := \bar\lambda^n$.

Algorithm 2 generates the sequence satisfying $\bar\lambda^0 > \bar\lambda^1 > \ldots > \bar\lambda^n =: \bar\lambda > \lambda_h^*$, unlike Algorithm 1. Although Algorithm 2 is more straightforward than Algorithm 1, it yields only a rough upper bound when the minimizing sequence is constructed by the Newton-like method. Therefore, we will present only results obtained by Algorithm 1 for the sake of brevity.

The computational experiments described below were implemented in MatLab.

6.3. Numerical example with the von Mises criterion. We consider a plane strain problem with Ω depicted in Figure 1: Ω is a quarter of the 10×10 (m) square with the circular hole of radius 1 in its center. The constant traction of density $f = (0, 450)$ and $(0, 0)$ (MPa) acts on the upper and the right vertical side, respectively. The volume forces are neglicted. This load corresponds to $\lambda = 1$. On the rest of $\partial\Omega$ the symmetry boundary conditions are prescribed. The material parameters are set as follows: $E = 206900\,\text{MPa}$ (Young's modulus), $\nu = 0.29$ (Poisson ratio), and $\gamma = 450\sqrt{2/3}$ MPa. Hence, the values of K and G needed in (4.9) are $K = \frac{1}{3}E/(1-2\nu)$ and $G = \frac{1}{2}E/(1+\nu)$.

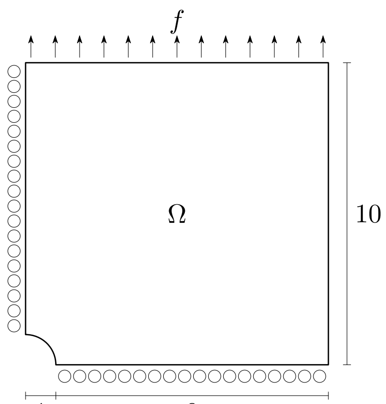

Figure 1. Geometry of the plane strain problem.

To estimate the bounds in (6.1), the first approach is used. As we have already mentioned, this approach is based on the numerical construction of the function

ψ_h defined by (5.5). To this end, it is necessary to solve problems $(\mathcal{P}_h)^\alpha$ for $\alpha \in \{\alpha_i\}_{i=1}^m$, $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n = \alpha_{\max}$. In computations, we use $\alpha_i = i\Delta\alpha$, $i = 1, \ldots, m$, where $\Delta\alpha > 0$ is an increment. The following values of $\Delta\alpha$ are used: $\Delta\alpha = 5$, 100, 1000 for $\alpha_i \in [0, 300]$, $[300, 10000]$, and $[10000, 100000]$, respectively. The loading paths represented by the graphs of $\psi_h\colon\ \alpha \mapsto \lambda_h$ are compared for seven different $\mathcal{T}_h$ with 1080, 2072, 3925, 10541, 23124, 41580, and 92120 nodes.

Zoom of the resulting loading path in a vicinity of the limit load for $\mathcal{T}_h$ mentioned above is depicted in Figure 2 (cf. [13]).

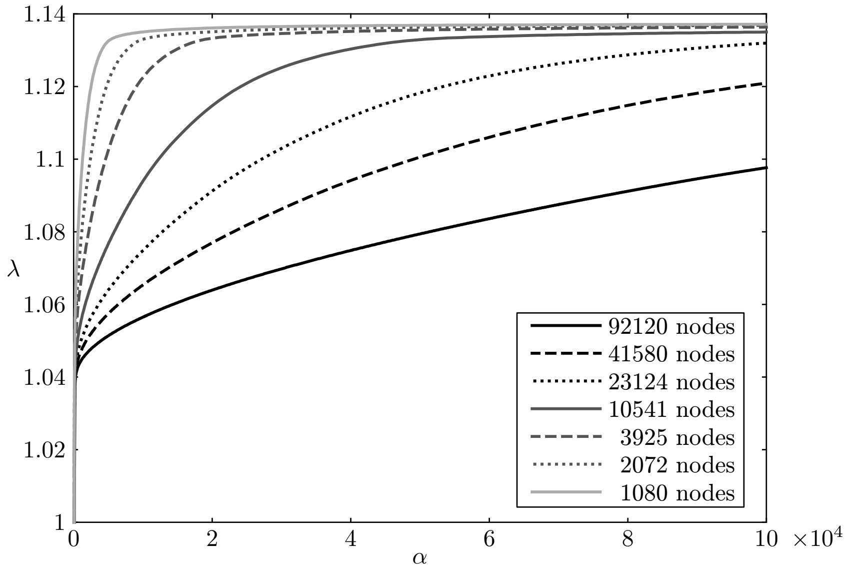

Figure 2. Loading paths (zoom).

One can observe that for any $\alpha > 0$ fixed, the sequence $\{\psi_h(\alpha)\}_{h\to 0_+}$ is decreasing but convergence of $\psi_h(\alpha)$ to $\psi(\alpha)$, $h \to 0_+$ becomes very slow for α large. In [13], the values 1.00 and 1.14 were found as reliable lower and upper bounds of λ^*, respectively, using this numerical experiment and (6.1).

To verify and possibly improve the lower bound we use the truncation approach for B_k defined by (4.11) with $k = 5$ and $k = 1$. The respective results are seen from Figures 3, 4 and Tables 1 and 2.

No. of nodes	1080	2072	3925	10541	23124	41580	92120
lower bound	1.1362	1.1331	1.1234	1.1003	1.0836	1.0729	1.0620
upper bound	1.1362	1.1332	1.1237	1.1019	1.0850	1.0768	1.0681

Table 1. Lower and upper bound of $\lambda^*_{k,h}$, $k = 5$.

	1080	2072	3925	10541	23124	41580	92120
lower bound	1.0590	1.0542	1.0505	1.0466	1.0446	1.0434	1.0421
upper bound	1.0591	1.0544	1.0508	1.0472	1.0450	1.0437	1.0427

Table 2. Lower and upper bound of $\lambda^*_{k,h}$, $k = 1$.

For $k = 5$, one can observe again the strong dependence of convergence on the mesh sizes (Figure 3). On the other hand, the curves are practically almost constant for α large enough even for finer meshes unlike Figure 2. Therefore, one can expect that the computed values $\psi_{k,h}(\alpha_{\max})$ for $\alpha_{\max} = 100000$ are close to $\lambda^*_{k,h}$ from below, see Table 1, row "lower bound". To find guaranteed upper bounds of $\lambda^*_{k,h}$ we use Algorithm 1 with $\delta\lambda = 1\mathrm{e}-4$. The lowest values of the upper bound found for different meshes are summarized in Table 1. We see that the lower and upper estimates of $\lambda^*_{k,h}$ are close to each other. This confirms a relatively high accuracy of the results. The values in Table 1 decrease almost linearly for finer meshes and thus $\lambda^*_{k=5}$ cannot be reliably estimated from these data.

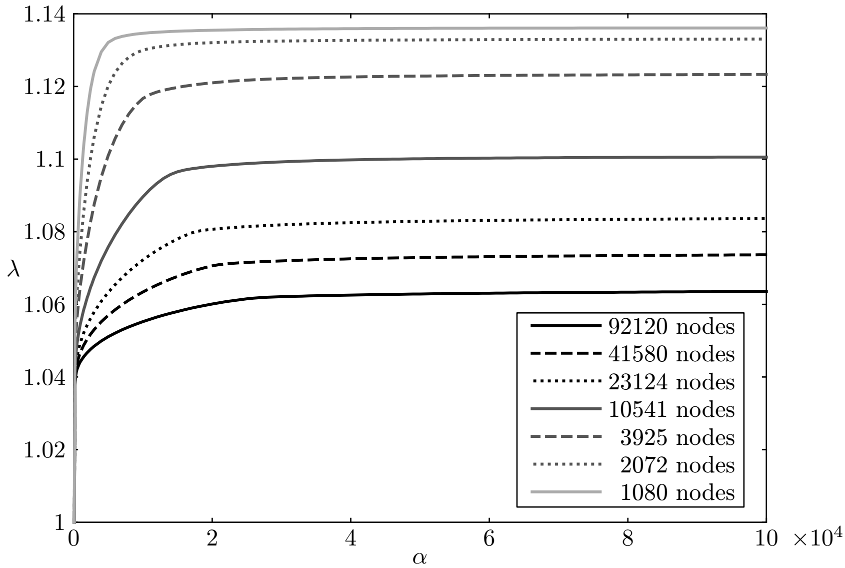

Figure 3. Loading paths for $k = 5$ (zoom).

For $k = 1$ (Figure 4), the curves uniformly converge to a limit curve and for $\alpha > 1\mathrm{e}4$ practically remain constant. Table 2 displays lower and guaranteed upper bounds of $\lambda^*_{k,h}$, $k = 1$, which were found in the same way as for $k = 5$. Notice that the values in Table 2 decrease for $h \to 0_+$ and remain close to 1.04 for finer meshes. Moreover, the curves in Figure 4 almost coincide up to $\lambda = 1.04$. Therefore, $\lambda^*_{k=1} \approx 1.04$.

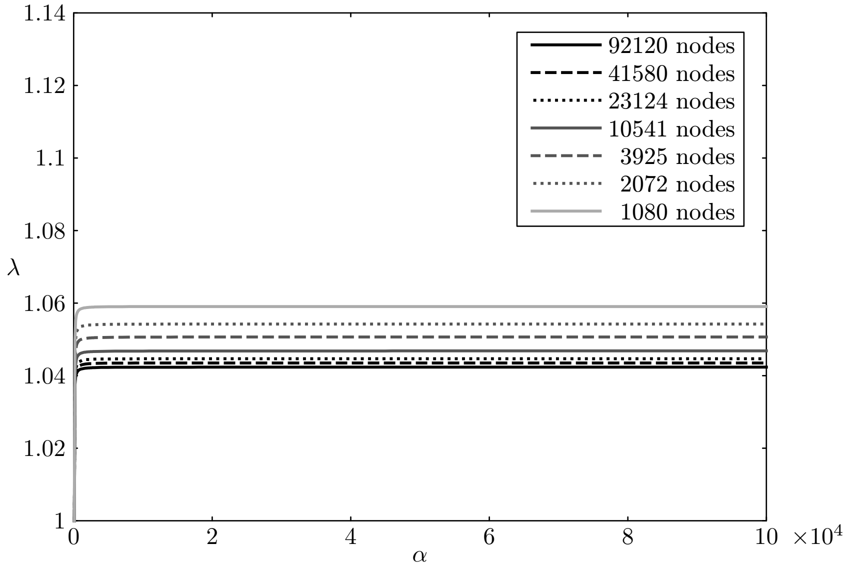

Figure 4. Loading paths for $k = 1$ (zoom).

Finally, one can easily check that the value 1.14 is a reliable upper bound of λ^* by using the linear function $w(x, y) = (-x, y)/45000$. Clearly, $w \in \mathbb{V}$, $\operatorname{div} w = 0$ in Ω and $L(w) = 1$, where $\mathbb{V}$ is the space of kinematically admissible displacements which satisfy the symmetry boundary conditions on the respective parts of $\partial\Omega$. From (4.10), the guaranteed upper bound $\lambda^* \leqslant 1.1456$ follows. Since $w \in \mathbb{V}_h$ for any triangulation $\mathcal{T}_h$, $h > 0$, it is also an upper bound of λ_h^*.

In conclusion: The values 1.04 and 1.14 could serve as reliable lower and upper bounds of λ^*, respectively. We were not able to improve them within $P1$ elements. The observed strong dependence on mesh sizes was likely due to the presence of the divergence-free constraint appearing in (4.10). It would be interesting to compare our results with the ones obtained by using mixed finite element methods directly in (4.10) as discussed in [1]. However, this is beyond the scope of this paper.

6.4. Numerical example with the Drucker-Prager criterion. The second example is a slope stability benchmark considered as a plane strain problem [3], [5], [25]. We use the same geometry and material properties as in [5]. The shape and sizes of 2D domain Ω with a uniform triangular mesh are shown in Figure 5.

The slope inclination is 45°. We assume that Ω is fixed on the bottom and the zero normal displacements are prescribed on both vertical sides. The remaining part of Ω is free. The load L is represented by the gravity force F. We set the specific weight $\varrho g = 20\,\mathrm{kN/m^3}$ with ϱ being the mass density and g the gravitational acceleration. The Drucker-Prager parameters a and γ appearing in (4.14) are computed from the

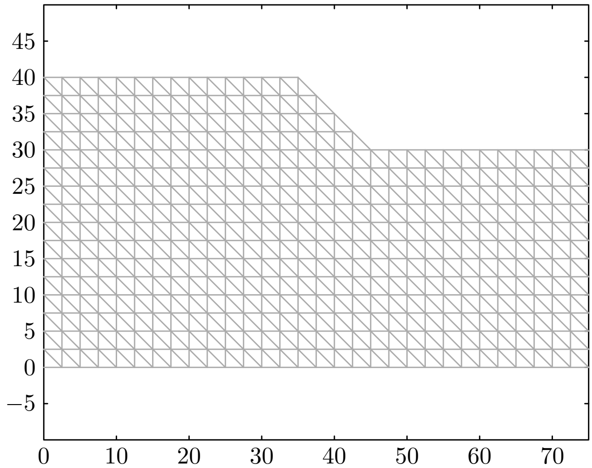

Figure 5. Geometry of the problem.

friction angle φ and the cohesion c as follows [5]:

$$a = \frac{3\sqrt{2}\tan\varphi}{\sqrt{9+12(\tan\varphi)^2}}, \quad \gamma = \frac{3\sqrt{2}c}{\sqrt{9+12(\tan\varphi)^2}}.$$

Finally, we set $E = 20\,000\,\mathrm{kPa}$, $\nu = 0.49$, $\varphi = 20^\circ$ and $c = 50\,\mathrm{kPa}$. The bulk and shear moduli are computed as in Section 6.3. The discretization of $(\mathcal{P}_h)^\alpha$ is done by $P1$ elements using four uniform triangulations $\mathcal{T}_h$ of $\bar{\Omega}$ with $h = 0.1$, 0.2, 0.5, 1.0 meters, where h stands for the length of the leg of the isosceles right triangles creating $\mathcal{T}_h$.

The loading paths for all meshes mentioned above are depicted in Figures 6 and 7 for $\alpha \in [0, 1\mathrm{e}5]$. In Figure 6 we see the loading paths for the original set B defined by (4.14). Again the strong dependence of the results on h is visible. However, one can observe that the curves converge to some limit curve. Since the paths are almost constant for $\alpha > 2\mathrm{e}4$ the respective values at $\alpha = 1\mathrm{e}5$ can be considered to be equal to λ_h^*. Consequently, $\lambda_{h=0.1}^* = 4.5$ is a reliable upper bound of λ^*.

To get a lower bound of λ^* we use the truncation approach with B_k defined by (4.15) for $k = 15$ (below we justify this choice of k). Figure 7 depicts the resulting loading paths for $B_{k=15}$. It is worth noticing that this time the paths practically coincide for all h and their values at $\alpha = 1\mathrm{e}5$ give reliable lower bounds of $\lambda_{k,h}^*$. To verify these results, Algorithm 1 with $\delta\lambda = 1\mathrm{e}-3$ was used to get guaranteed upper bounds of $\lambda_{k,h}^*$. They are summarized in Table 3 together with the lower bounds of $\lambda_{k,h}^*$ computed by means of the functions $\psi_{k,h}$.

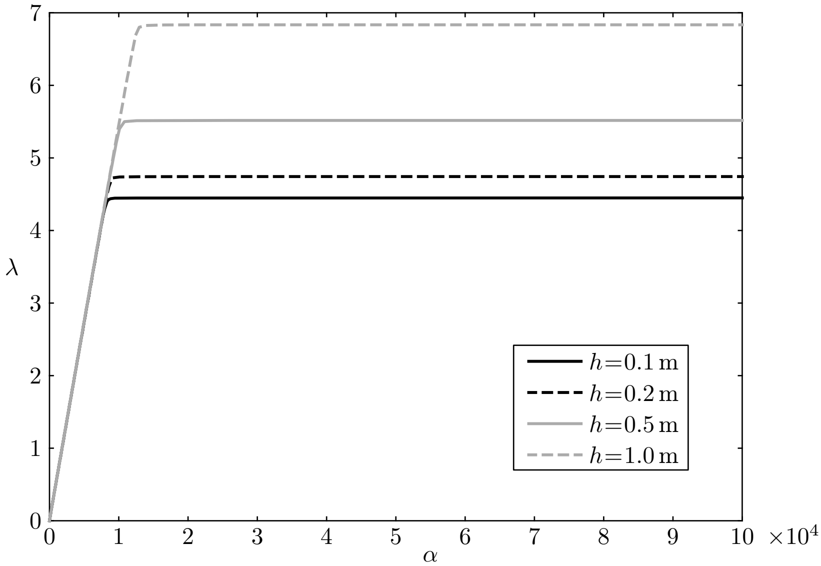

Figure 6. Loading paths for B.

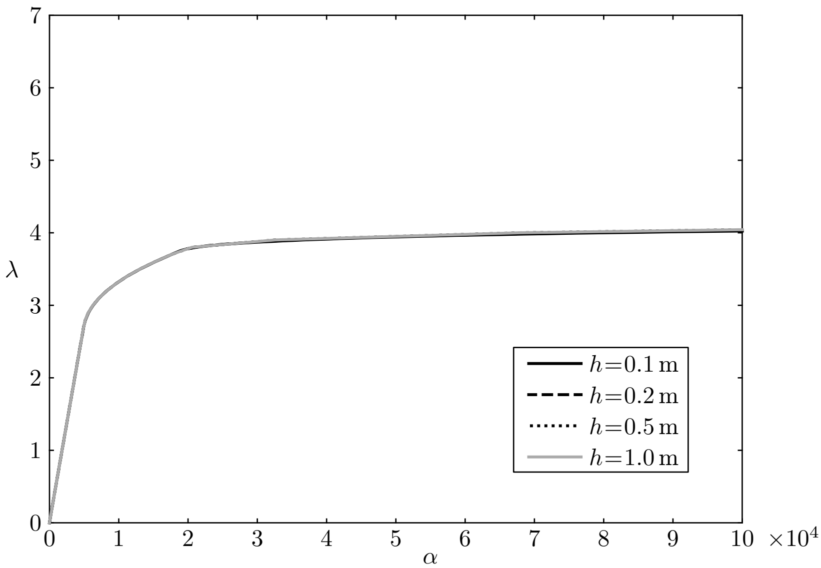

Figure 7. Loading paths for B_k, $k = 15$.

h (m)	0.1	0.2	0.5	1.0
lower bound	4.123	4.129	4.148	4.177
upper bound	4.126	4.131	4.149	4.178

Table 3. Lower and upper bounds of $\lambda^*_{k,h}$, $k = 15$.

Moreover, we are able to find another guaranteed upper bound of λ_k^* using the piecewise linear function

$$w_\varepsilon(x,y) = \begin{cases} (0, -\frac{b}{\varepsilon}y), & y \in [0,\varepsilon], \\ (0,-b), & y \in [\varepsilon, 40], \end{cases} \qquad \varepsilon > 0,\ b = (53000 - 750\varepsilon)^{-1}.$$

It is readily seen that w_ε satisfies all kinematic boundary conditions on $\partial\Omega$ and $L(w_\varepsilon) = 1$ due to the choice of b. Further, $(4.16)_2$ yields

$$\lambda_k^* = \zeta_k^* \leqslant \int_\Omega j_{k,\infty}(\varepsilon(w_\varepsilon))\,\mathrm{d}x\,\mathrm{d}y = \frac{75\gamma}{53000 - 750\varepsilon}\Big[\frac{k}{a} + (1+k)\frac{\sqrt{6}}{3}\Big]. \tag{6.2}$$

Thus for $k = 14, 15, 16$ we obtain from (6.2) the following upper bounds of the corresponding λ_k^*: 3.851, 4.121, and 4.391, respectively, when $\varepsilon \to 0$. This justifies our choice of k. In addition, choosing $\varepsilon = h$, the function w_ε belongs to the $P1$ finite element discretization of the space of all kinematic admissible displacements. Hence, for $k = 15$ and $h = 0.1$, 0.2, 0.5, and 1.0, the upper bound (6.2) gives 4.127, 4.133, 4.150, 4.180, respectively. These values are slightly higher than the bounds computed by Algorithm 1 as follows from Table 3.

Based on this experiment we may conclude that the values 4.1 and 4.5 could serve as reliable lower and upper bounds to λ^*, respectively. The bound 4.5 likely overestimates λ^* and could be possibly improved using a locally refined mesh in a vicinity of the slope in $\partial\Omega$, see [5], [25]. The analytical estimate to this problem for the Mohr-Coulomb yield function presented in [3] gives the value $\lambda^* \approx 4.045$ which is close to our estimate of the lower bound.

7. Conclusion

The paper deals with guaranteed and computable bounds of the limit load parameter λ^* in variational problems for functionals with linear growth. The new guaranteed upper bound for functionals with purely linear growth and the truncation technique for the ones with linear growth are the main results of the paper. These results when used in elasto-plasticity read as follows: functionals with purely linear growth arise in models with bounded yield surfaces so that the upper bound of λ^* is directly at our disposal. If the yield surface is unbounded, then one uses the truncation approach first to approximate the original surface by a sequence of bounded ones. Further, we combine these techniques with the incremental method from [13] which uses the function ψ_h defined by (5.5). Such an approach is also simple from the computational point of view. It requires to solve only a smooth convex optimization problem subject to at most one linear equality constraint. Although only linear simplicial finite elements are used in the paper, the results concerning the guaranteed upper bound, the truncation approach and the function ψ (see Theorem 5.4) are independent of the particular choice of the finite element space.

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