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SOME RESULTS ON A DOUBLY TRUNCATED GENERALIZED DISCRIMINATION MEASURE

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Abstract. Doubly truncated data appear in some applications with survival and astrological data. Analogous to the doubly truncated discrimination measure defined by Misagh and Yari (2012), a generalized discrimination measure between two doubly truncated nonnegative random variables is proposed. Several bounds are obtained. It is remarked that the proposed measure can never be equal to a nonzero constant which is independent of the left and right truncated points. The effect of monotone transformations on the proposed measure is discussed. Finally, a simulation study is added to provide the estimates of the proposed discrimination measure.

Keywords: doubly truncated random variable; generalized discrimination measure; likelihood ratio order; stochastic order; proportional hazard model; proportional reversed hazard model; monotone transformation

MSC 2010: 62N05, 62B10

1. Introduction

Kullback and Leibler [9] were the first to introduce an information measure between two probability density functions (pdf's). It is dubbed as discrimination information, Kullback-Leibler (KL) divergence, relative information, Kullback's information. The discrimination information measure between two nonnegative absolutely continuous random variables X and Y with respective pdf's $f(x)$ and $g(x)$ is defined as (see Kullback and Leibler, [9])

$$J_{X,Y}^{\mathrm{KL}} = \int_0^\infty f(x) \ln \frac{f(x)}{g(x)} \,\mathrm{d}x. \tag{1.1}$$

Note that (1.1) is shift and scale invariant. It measures the similarity of two pdf's. It is nonnegative and equal to zero if and only if $f(x) = g(x)$ almost everywhere. The smaller value of (1.1) implies that the distributions corresponding to the random variables X and Y are more similar. Note that (1.1) is not a distance, since it is neither symmetric nor satisfies the triangle inequality. There is an extensive literature regarding criteria for evaluating the best statistical model. One of them is KL discrimination measure. For some results on discrimination information, we refer to Wang et al. [22], Misagh and Yari [12], Park [13], Park and Shin [14], and Al-Rahman and Kittaneh [1]. There have been several attempts on the generalizations of (1.1). Based on Renyi's entropy (see Renyi, [15]), the discrimination information measure (1.1) is generalized as

$$J_{X,Y}^{R} = \frac{1}{\alpha - 1} \ln \int_0^\infty f^\alpha(x) g^{1-\alpha}(x)\,\mathrm{d}x, \quad \alpha\ (\neq 1) > 0. \tag{1.2}$$

Note that as $\alpha \to 1$, (1.2) reduces to (1.1). Later, based on Varma's entropy (see Varma, [21]), the measure (1.2) is further generalized as

$$J_{X,Y}^{V} = \frac{1}{\alpha - \beta} \ln \int_0^\infty f^{\alpha+\beta-1}(x) g^{2-\alpha-\beta}(x)\,\mathrm{d}x, \quad 0 \leqslant \beta - 1 < \alpha < \beta. \tag{1.3}$$

The measure (1.3) reduces to (1.2) when $\beta = 1$. Also, for $\beta = 1$ and $\alpha \to 1$, (1.3) reduces to (1.1). In reliability and life testing studies, there are some experiments, where the current age of a system needs to be incorporated. Also, somebody may be interested in studying uncertainty which relies on the past lifetime of a component. Thus the measures (1.1), (1.2), and (1.3) are not appropriate to handle these situations. To overcome such difficulty, several authors have proposed discrimination information measures and their generalized versions between two residual and past lifetime distributions and studied their properties. In this direction, we refer to Ebrahimi and Kirmani [6], [7], Di Crescenzo and Longobardi [5], Asadi, Ebrahimi, Hamedant and Soofi [2], Asadi, Ebrahimi and Soofi [3], Sunoj and Linu [19], Kundu [10], and Kayal [8]. The generalized residual and past discrimination information measures between two nonnegative random variables X and Y are given by

$$J_{X,Y}^{V}(t) = \frac{1}{\alpha - \beta} \ln \int_t^\infty \frac{f^{\alpha+\beta-1}(x)}{\overline{F}^{\alpha+\beta-1}(t)} \frac{g^{2-\alpha-\beta}(x)}{\overline{G}^{2-\alpha-\beta}(t)}\,\mathrm{d}x \tag{1.4}$$

and

$$\overline{J}_{X,Y}^{V}(t) = \frac{1}{\alpha - \beta} \ln \int_0^t \frac{f^{\alpha+\beta-1}(x)}{F^{\alpha+\beta-1}(t)} \frac{g^{2-\alpha-\beta}(x)}{G^{2-\alpha-\beta}(t)}\,\mathrm{d}x, \tag{1.5}$$

where $F(x)$ and $G(x)$ are cumulative distribution functions (cdf's) and $\overline{F}(x)$ and $\overline{G}(x)$ are survival functions of X and Y, respectively. The measures (1.4) and (1.5) are known as the relative entropy of residual lifetimes $[X-t \mid X>t]$ and $[Y-t \mid Y>t]$, and past lifetimes $[t-X \mid X<t]$ and $[t-Y \mid Y<t]$, respectively.

Recently there have been growing interests in analyzing doubly truncated data in statistical analysis of survival data as well as in other fields like astronomy or economy. Doubly truncated failure time arises if an individual is observed and its failure time falls into a certain finite interval. An individual which is not observed in the interval means that the investigator does not have any information about the individual. Doubly truncated random variable appears in quasar survey, where an investigator assumes that the apparent magnitude is doubly truncated. Also, the times to progression for patients with certain disease who received chemotherapy, experienced tumor progression and subsequently died, are doubly truncated. For various results on doubly truncated random variable we refer to Ruiz and Navarro [16], Betensky and Martin [4], Sankaran and Sunoj [17], Sunoj, Sankaran and Maya [20], Misagh and Yari [11], [12]. In this paper we consider discrimination information measure between doubly truncated random variables $[X \mid u<X<v]$ and $[Y \mid u<Y<v]$. It is defined as

$$J^V_{X,Y}(u,v)=\frac{1}{\alpha-\beta}\ln\int_u^v\frac{f^{\alpha+\beta-1}(x)}{(\Delta F)^{\alpha+\beta-1}}\frac{g^{2-\alpha-\beta}(x)}{(\Delta G)^{2-\alpha-\beta}}\,\mathrm{d}x, \tag{1.6}$$

where $\Delta F=F(v)-F(u)$, $\Delta G=G(v)-G(u)$, $0<u<v$, and $0\leqslant\beta-1<\alpha<\beta$. It is easy to show that $J^V_{X,Y}(0,v)=\overline{J}^V_{X,Y}(v)$, $J^V_{X,Y}(u,\infty)=J^V_{X,Y}(u)$ and $J^V_{X,Y}(0,\infty)=J^V_{X,Y}$. Note that when $\beta=1$, (1.6) reduces to Renyi's doubly truncated discrimination information measure

$$J^R_{X,Y}(u,v)=\frac{1}{\alpha-1}\ln\int_u^v\frac{f^{\alpha}(x)}{(\Delta F)^{\alpha}}\frac{g^{1-\alpha}(x)}{(\Delta G)^{1-\alpha}}\,\mathrm{d}x,\quad 0<\alpha<1,$$

and (1.6) reduces to the doubly truncated discrimination information measure (see Misagh and Yari [12]), when $\beta=1$ and $\alpha\to1$. In this work our aim is to obtain some bounds and characterization results of the measure (1.6). Throughout the paper, we assume that the random variables are nonnegative and absolutely continuous. The terms increasing and decreasing are used in nonstrict sense. Henceforth, we denote $\gamma=\alpha+\beta-1$, where $\gamma>0$.

The paper is arranged as follows. In Section 2, we present some preliminary results. An example is presented to show that the measure given in (1.6) is not monotone in general. We obtain some bounds of the measure (1.6) in Section 3. In Section 4, we study the effect of the monotone transformations on (1.6). Monte-Carlo simulation is carried out for the purpose of estimation of the doubly truncated generalized discrimination measure given by (1.6) in Section 5. Finally, some concluding remarks are added in Section 6.

2. Preliminaries

In this section we present some definitions and theorems which are useful for obtaining some of our main results.

Definition 2.1. Let X and Y be two nonnegative random variables with pdf's $f(x)$ and $g(x)$, respectively. Then X is said to be less than or equal to Y in likelihood ratio ordering, denoted by $X \overset{\mathrm{lr}}{\leqslant} Y$, if $f(t)/g(t)$ is decreasing in $t > 0$.

Definition 2.2. Let X and Y be two nonnegative random variables with survival functions $\overline{F}(x)$ and $\overline{G}(x)$, respectively. Then X is said to be less than or equal to Y in usual stochastic ordering, denoted by $X \overset{\mathrm{st}}{\leqslant} Y$, if $\overline{F}(t) \leqslant \overline{G}(t)$, for all $t > 0$.

Theorem 2.1 (Shaked and Shanthikumar, [18]). *If X and Y are two nonnegative random variables such that $X \overset{\mathrm{lr}}{\leqslant} Y$, then for any measurable set $A \subseteq \mathbb{R}$ we have $[X \mid X \in A] \overset{\mathrm{lr}}{\leqslant} [Y \mid Y \in A]$.*

Theorem 2.2 (Shaked and Shanthikumar, [18]). *For two nonnegative random variables X and Y, $X \overset{\mathrm{lr}}{\leqslant} Y$ holds if and only if $[X \mid X \in A] \overset{\mathrm{st}}{\leqslant} [Y \mid Y \in A]$ for all measurable sets $A \subseteq \mathbb{R}$.*

Theorem 2.3 (Shaked and Shanthikumar, [18]). *If $X \overset{\mathrm{st}}{\leqslant} Y$, and φ is any increasing (decreasing) function, then $E[\varphi(X)] \leqslant (\geqslant)\ E[\varphi(Y)]$.*

For further reading on usual stochastic ordering one may refer to Shaked and Shanthikumar [18]. We consider the following example to show the role of (1.6) for comparison of life times.

Example 2.1. Suppose a nonnegative random variable X follows uniform distribution in the interval $(0, 1)$. Let Y and Z be two other nonnegative random variables with pdf's

$$g(x) = \begin{cases} \frac{2}{3}(1+x), & \text{if } 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

and

$$h(x) = \begin{cases} \frac{2}{3}(2-x), & \text{if } 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.2}$$

respectively. Then it is easy to obtain that

$$J^V_{X,Y} = \frac{1}{\alpha-\beta}\ln\Big[\Big(\frac{2}{3}\Big)^{1-\gamma}\frac{2^{2-\gamma}-1}{2-\gamma}\Big] = J^V_{X,Z}, \tag{2.3}$$

that is, the generalized discrimination information measure between X and Y, and that between X and Z are equal.

Moreover, for $0 < u < v < 1$,

$$J^V_{X,Y}(u,v) = \frac{1}{\alpha-\beta}\ln\Big[\frac{2^{1-\gamma}((1+v)^{2-\gamma}-(1+u)^{2-\gamma})}{(2-\gamma)(v-u)(2+u+v)^{1-\gamma}}\Big] \tag{2.4}$$

and

$$J^V_{X,Z}(u,v) = \frac{1}{\alpha-\beta}\ln\Big[\frac{2^{1-\gamma}((2-u)^{2-\gamma}-(2-v)^{2-\gamma})}{(2-\gamma)(v-u)(4-u-v)^{1-\gamma}}\Big]. \tag{2.5}$$

Now, from Fig. 1, it is not difficult to observe that in general the doubly truncated generalized discrimination information measures between X and Y, and X and Z are not equal.

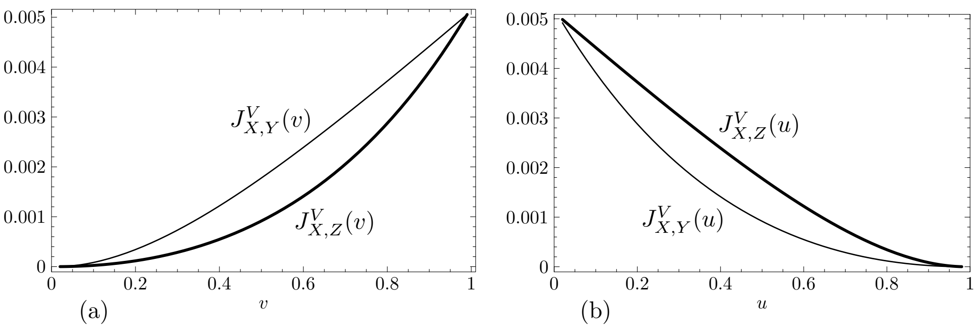

Figure 1. Fig. (a) represents the measures (2.4) and (2.5) for $v \in (0.02, .99)$, with $\beta = 1.2$, $\alpha = 0.3$, when $u = 0.01$. Figure (b) represents the measures (2.4) and (2.5) for $u \in (0.02, 0.98)$, with $\beta = 1.2, \alpha = 0.3$, when $v = 0.99$.

It is worthwhile to mention that the measure given in (1.6) is not monotone in general. We consider the following example.

Example 2.2. Let X and Y be two nonnegative random variables with pdf's

$$f(x) = \begin{cases} \frac{1}{4}(2x+1), & \text{if } 0 \leqslant x < 1,\\ \frac{1}{2}, & \text{if } 1 \leqslant x < 2,\\ 0, & \text{otherwise,}\end{cases} \tag{2.6}$$

and

$$g(x) = \begin{cases} \dfrac{1}{x^2}\exp\Bigl(\dfrac{1}{2}-\dfrac{1}{x}\Bigr), & \text{if } 0 \leqslant x < 2, \\ 0, & \text{otherwise,} \end{cases} \tag{2.7}$$

respectively. Then for $u \in (0,1)$ and $v \in (1,2)$ we have

$$\begin{aligned} &J^V_{X,Y}(u,v) \\ &= \frac{1}{\alpha-\beta}\ln\Bigl[\int_u^1 \Bigl(\frac{2x+1}{(v-u)(1+u+v)}\Bigr)^{\gamma}\Bigl(\frac{x^{-2}\exp(\frac{1}{2}-x^{-1})}{\exp(\frac{1}{2}-v^{-1})-\exp(\frac{1}{2}-u^{-1})}\Bigr)^{1-\gamma}\,\mathrm{d}x \\ &\quad+\int_1^v \Bigl(\frac{1}{v-u}\Bigr)^{\gamma}\Bigl(\frac{x^{-2}\exp(\frac{1}{2}-x^{-1})}{\exp(\frac{1}{2}-v^{-1})-\exp(\frac{1}{2}-u^{-1})}\Bigr)^{1-\gamma}\,\mathrm{d}x\Bigr], \end{aligned} \tag{2.8}$$

where $\beta-1<\alpha<\beta$, $\beta \geqslant 1$, and $\gamma \neq 1$. From Fig. 2, it is clear that $J^V_{X,Y}(u,v)$ is not monotone.

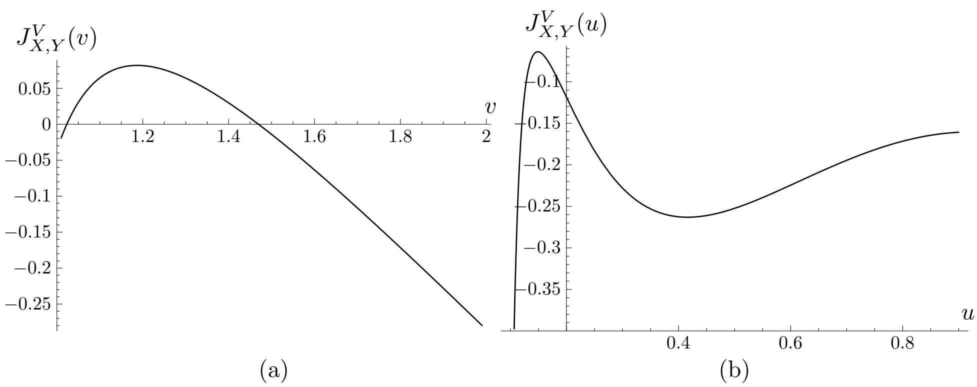

Figure 2. Fig. (a) and (b) represent the plot of the measure given in (2.8) for $u=0.8$ and $v=1.8$, respectively. We assume $\beta=2$ and $\alpha=1.5$.

3. Main results

Different kinds of bounds on the discrimination measures have been studied for truncated random variables recently. For some useful references, one may refer to Ebrahimi and Kirmani [6], Di Crescenzo and Longobardi [5], Sunoj and Linu [19], Kundu [10], and Kayal [8]. In this section we generalize some of their results based on the measure (1.6) and study several properties of it. First, we present the following theorem which provides upper and lower bounds of (1.6) in terms of density and distribution functions. The bounds of the discrimination information measure has various applications in different areas of science and technology such as sensor networks, regional segmentation, testing the order in a Markov chain and estimation theory. Here, $\alpha < \beta$ implies $\alpha - \beta < 0$.

Theorem 3.1. *Let $X \overset{\mathrm{lr}}{\leqslant} Y$. Then*

$$\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{f(u)/\Delta F}{g(u)/\Delta G}\Big) \leqslant (\geqslant)\ J^V_{X,Y}(u,v) \leqslant (\geqslant)\ \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{f(v)/\Delta F}{g(v)/\Delta G}\Big), \quad \textit{if } \gamma > (<) 1.$$

P r o o f. We obtain the inequality for $\gamma > 1$. Given $X \overset{\mathrm{lr}}{\leqslant} Y$ and $u < x$, then $f^{\gamma}(x)g^{1-\gamma}(x) \leqslant f^{\gamma-1}(u)g^{1-\gamma}(u)f(x)$. Hence, from (1.6) we obtain

$$J^V_{X,Y}(u,v) \geqslant \frac{1}{\alpha-\beta}\ln\int_u^v \frac{f^{\gamma-1}(u)}{(\Delta F)^{\gamma}}\frac{g^{1-\gamma}(u)}{(\Delta G)^{1-\gamma}}f(x)\,\mathrm{d}x = \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{f(u)/\Delta F}{g(u)/\Delta G}\Big).$$

Using $X \overset{\mathrm{lr}}{\leqslant} Y$ and $x < v$, we get $f^{\gamma}(x)g^{1-\gamma}(x) \geqslant g^{1-\gamma}(v)f^{\gamma-1}(v)f(x)$. Therefore, from (1.6)

$$J^V_{X,Y}(u,v) \leqslant \frac{1}{\alpha-\beta}\ln\int_u^v \frac{f^{\gamma-1}(v)}{(\Delta F)^{\gamma}}\frac{g^{1-\gamma}(v)}{(\Delta G)^{1-\gamma}}f(x)\,\mathrm{d}x = \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{f(v)/\Delta F}{g(v)/\Delta G}\Big).$$

When $\gamma < 1$, the inequality inside the parenthesis can be obtained similarly. Hence, the result follows. $\square$

The following example is an application of Theorem 3.1.

E x a m p l e 3.1. Let X and Y be two nonnegative random variables with pdf's

$$f(x) = \begin{cases} \dfrac{ab^a}{x^{a+1}}, & \text{if } x > b > 0,\ a > 1, \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

and

$$g(x) = \begin{cases} \dfrac{(a-1)b^{a-1}}{x^a} & \text{if } x > b > 0,\ a > 1, \\ 0, & \text{otherwise,} \end{cases} \tag{3.2}$$

respectively. Then the generalized discrimination information measure between X and Y can be obtained as

$$J^V_{X,Y}(u,v) = \frac{1}{\alpha-\beta}\ln\Big[\frac{a^{\gamma}(a-1)^{1-\gamma}(u^{1-a-\gamma}-v^{1-a-\gamma})}{(u^{-a}-v^{-a})^{\gamma}(u^{1-a}-v^{1-a})^{1-\gamma}(a+\gamma-1)}\Big]. \tag{3.3}$$

Again,

$$\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{f(u)/\Delta F}{g(u)/\Delta G}\Big)=\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{av(v^{a-1}-u^{a-1})}{(a-1)(v^{a}-u^{a})}\Big)=\eta(u,v) \tag{3.4}$$

and

$$\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{f(v)/\Delta F}{g(v)/\Delta G}\Big)=\frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{au(v^{a-1}-u^{a-1})}{(a-1)(v^{a}-u^{a})}\Big)=\zeta(u,v). \tag{3.5}$$

From Fig. 3 and 4, the inequalities in Theorem 3.1 can be easily verified.

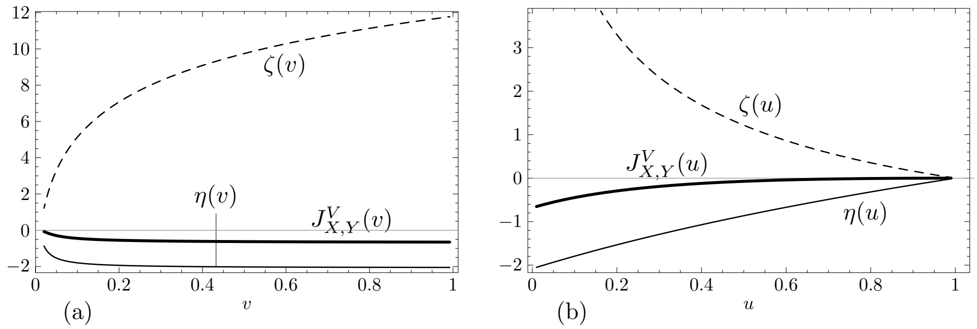

Figure 3. Figures (a) and (b) represent the plot of the functions given in (3.3), (3.4), and (3.5) for $\gamma > 1$. Assume $\alpha = 1.1$, $\beta = 1.2$, $a = 2$, and $u = 0.01$ in Fig. (a). In Fig. (b), we consider $\alpha = 1.1$, $\beta = 1.2$, $a = 2$, and $v = 1$.

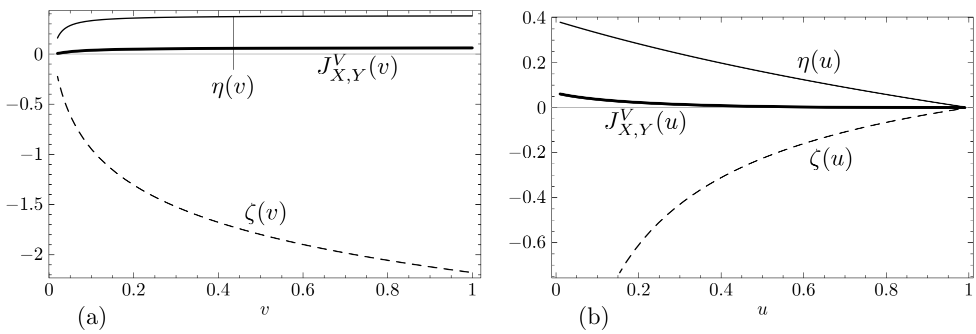

Figure 4. Figures (a) and (b) represent the plot of the functions given in (3.3), (3.4), and (3.5) for $\gamma < 1$. Assume $\alpha = 0.3$, $\beta = 1.2$, $a = 2$, and $u = 0.01$ in Fig. (a). In Fig. (b), we consider $\alpha = 0.3$, $\beta = 1.2$, $a = 2$, and $v = 1$.

Theorem 3.2. *Let $g(x)$ be decreasing in x. Then for*

$$J^V_X(u,v)=\frac{1}{\beta-\alpha}\ln\int_u^v\frac{f^{\gamma}(x)}{(\Delta F)^{\gamma}}\,\mathrm{d}x$$

we have

$$\frac{1-\gamma}{\alpha-\beta}\ln\Big(\frac{g(v)}{\Delta G}\Big) - J_X^V(u,v) \leqslant (\geqslant)\ J_{X,Y}^V(u,v)$$
$$\leqslant (\geqslant)\ \frac{1-\gamma}{\alpha-\beta}\ln\Big(\frac{g(u)}{\Delta G}\Big) - J_X^V(u,v), \quad \text{if } \gamma > (<)1.$$

P r o o f. The function $g(x)$ is decreasing in x and $u < x$. Therefore, for $\gamma > 1$ we have $g^{1-\gamma}(u) \leqslant g^{1-\gamma}(x)$. Hence,

$$J_{X,Y}^V(u,v) \leqslant \frac{1}{\alpha-\beta}\ln\int_u^v \frac{f^\gamma(x)}{(\Delta F)^\gamma}\frac{g^{1-\gamma}(u)}{(\Delta G)^{1-\gamma}}\,\mathrm{d}x = \frac{1-\gamma}{\alpha-\beta}\ln\Big(\frac{g(u)}{\Delta G}\Big) - J_X^V(u,v).$$

Let $x < v$. Then for $\gamma > 1$ we have $g^{1-\gamma}(x) \leqslant g^{1-\gamma}(v)$. Consequently,

$$J_{X,Y}^V(u,v) \geqslant \frac{1}{\alpha-\beta}\ln\int_u^v \frac{f^\gamma(x)}{(\Delta F)^\gamma}\frac{g^{1-\gamma}(v)}{(\Delta G)^{1-\gamma}}\,\mathrm{d}x = \frac{1-\gamma}{\alpha-\beta}\ln\Big(\frac{g(v)}{\Delta G}\Big) - J_X^V(u,v).$$

The other part (when $\gamma < 1$) can be proved similarly. Hence the result follows. □

In environmental and ecological science, the observations under study may be nonexperimental or nonrandom. The theory of weighted distributions plays an important role in modeling these types of data. Somebody may be interested to measure discrimination between a distribution and the corresponding weighted distribution. The following remark provides the discrimination measure between f and its weighted distribution f_w in terms of the cdf and the conditional expectations.

R e m a r k 3.1. Let X be a nonnegative random variable with pdf $f(x)$ and cdf $F(x)$. Also, let X_w be a weighted random variable with respective pdf and cdf being $f_w(x) = w(x)\mu_w^{-1}f(x)$ and $F_w(x) = E(w(X)\,|\,X < x)\mu_w^{-1}F(x)$, where $w(x)$ is a nonnegative function with $\mu_w = E(w(X)) = \int_0^\infty w(x)f(x)\,\mathrm{d}x < \infty$. Now from (1.6) it can be shown that

$$J_{X,X_w}^V(u,v) = \frac{1-\gamma}{\alpha-\beta}\ln(\Delta F) + \frac{1}{\alpha-\beta}\ln E[w^{1-\gamma}(X)\,|\,u < X < v]$$
$$-\frac{1-\gamma}{\alpha-\beta}\ln[E(w(X)\,|\,X < v)F(v) - E(w(X)\,|\,X < u)F(u)].$$

Proposition 3.1. *Suppose X and Y are two nonnegative random variables with pdf's $f(x)$ and $g(x)$, respectively. The corresponding cdf's are $F(x)$ and $G(x)$. Let $J^V_{X,Y}(u,v)$ be increasing (decreasing) in u. Then*

$$J^V_{X,Y}(u,v) \geqslant (\leqslant) \frac{\gamma-1}{\beta-\alpha}\ln\Big(\frac{g(u)/\Delta G}{f(u)/\Delta F}\Big) + \frac{1}{\beta-\alpha}\ln\Big[(\alpha+\beta)\Big(1-\frac{g(u)/\Delta G}{f(u)/\Delta F}\Big)+2\Big(\frac{g(u)/\Delta G}{f(u)/\Delta F}\Big)-1\Big].$$

P r o o f. Differentiating (1.6) with respect to u, the inequality follows under the given conditions. □

Proposition 3.2. *Let X and Y be two nonnegative random variables as described in Proposition* 3.1. *Let $J^V_{X,Y}(u,v)$ be increasing (decreasing) in v. Then*

$$J^V_{X,Y}(u,v) \leqslant (\geqslant) \frac{\gamma-1}{\beta-\alpha}\ln\Big(\frac{g(v)/\Delta G}{f(v)/\Delta F}\Big) + \frac{1}{\beta-\alpha}\ln\Big[(\alpha+\beta)\Big(1-\frac{g(v)/\Delta G}{f(v)/\Delta F}\Big)+2\Big(\frac{g(v)/\Delta G}{f(v)/\Delta F}\Big)-1\Big].$$

P r o o f. Differentiating (1.6) with respect to v and using the given conditions, the inequality follows. □

As an application of Propositions 3.1 and 3.2, we consider the following example.

E x a m p l e 3.2. Consider two nonnegative random variables Y and Z with pdf's (2.1) and (2.2), respectively. Denote cdf's of Y and Z as $G(x)$ and $H(x)$ and $\Delta H = H(v) - H(u)$. It can be easily checked that

$$J^V_{Y,Z}(u,v) = \frac{1}{\alpha-\beta}\ln\int_u^v 2\Big(\frac{1+x}{(v-u)(2+v+u)}\Big)^{\gamma}\Big(\frac{2-x}{(v-u)(4-v-u)}\Big)^{1-\gamma}\,\mathrm{d}x. \tag{3.6}$$

Moreover,

$$\begin{aligned}\xi_1(u,v) &= \frac{\gamma-1}{\beta-\alpha}\ln\Big(\frac{h(u)/\Delta H}{g(u)/\Delta G}\Big) \\ &\quad+\frac{1}{\beta-\alpha}\ln\Big[(\alpha+\beta)\Big(1-\frac{h(u)/\Delta H}{g(u)/\Delta G}\Big)+2\frac{h(u)/\Delta H}{g(u)/\Delta G}-1\Big] \\ &= \frac{\gamma-1}{\beta-\alpha}\ln\Big(\frac{(2-u)(2+v+u)}{(4-v-u)(1+u)}\Big) \\ &\quad+\frac{1}{\beta-\alpha}\ln\Big[(\alpha+\beta)\Big(1-\frac{(2-u)(2+v+u)}{(1+u)(4-v-u)}\Big) \\ &\quad+2\Big(\frac{(2-u)(2+v+u)}{(1+u)(4-v-u)}\Big)-1\Big]\end{aligned} \tag{3.7}$$

and

$$\xi_2(u,v) = \frac{\gamma-1}{\beta-\alpha}\ln\Big(\frac{h(v)/\Delta H}{g(v)/\Delta G}\Big) + \frac{1}{\beta-\alpha}\ln\Big[(\alpha+\beta)\Big(1-\frac{h(v)/\Delta H}{g(v)/\Delta G}\Big)+2\frac{h(v)/\Delta H}{g(v)/\Delta G}-1\Big] = \frac{\gamma-1}{\beta-\alpha}\ln\Big(\frac{(2-v)(2+v+u)}{(4-v-u)(1+v)}\Big) + \frac{1}{\beta-\alpha}\ln\Big[(\alpha+\beta)\Big(1-\frac{(2-v)(2+v+u)}{(1+v)(4-v-u)}\Big) + 2\Big(\frac{(2-v)(2+v+u)}{(1+v)(4-v-u)}\Big)-1\Big]. \tag{3.8}$$

From Figs. 5 and 6, Propositions 3.1 and 3.2 can be easily verified.

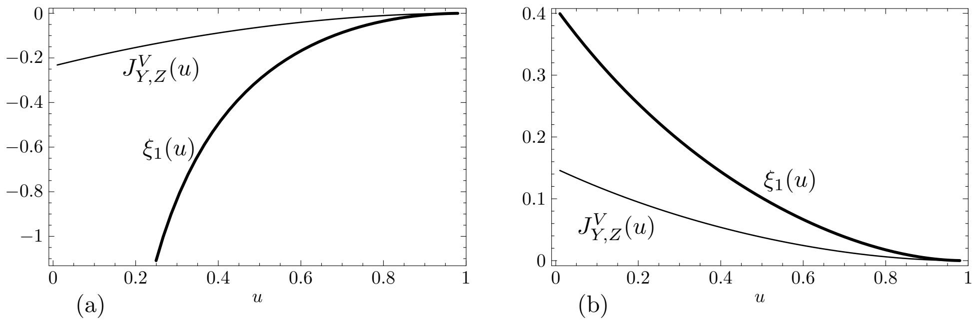

Figure 5. Figures (a) and (b) represent the plot of the functions given in (3.6) and (3.7). We consider $v = 0.99$. Assume $\alpha = 1.2$ and $\beta = 2$ in Fig. (a). In Fig. (b), we take $\alpha = 0.2$ and $\beta = 1$.

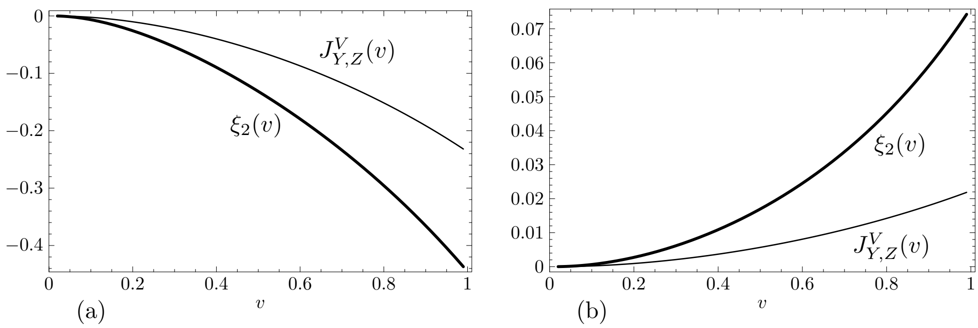

Figure 6. Figures (a) and (b) represent the plot of the functions given in (3.6) and (3.8). We fix $u = 0.01$. Assume $\alpha = 1.2$ and $\beta = 2$ in Fig. (a). In Fig. (b), we consider $\alpha = 0.3$ and $\beta = 1$.

In the next theorem we consider three random variables X_1, X_2, and X_3 and obtain lower and upper bounds of $J^V_{X_1,X_3}(u,v) - J^V_{X_2,X_3}(u,v)$.

Theorem 3.3. *Let X_1, X_2, and X_3 be three random variables with pdf's $f_1(x)$, $f_2(x)$, and $f_3(x)$, respectively. The corresponding cdf's are given by $F_1(x)$, $F_2(x)$ and $F_3(x)$. Also let $X_1 \overset{\mathrm{lr}}{\leqslant} X_2$. Then for $\Delta F_1 = F_1(v) - F_1(u)$ and $\Delta F_2 = F_2(v) - F_2(u)$,*

$$\frac{\gamma}{\alpha-\beta}\ln\Big(\frac{f_1(u)/\Delta F_1}{f_2(u)/\Delta F_2}\Big) \leqslant J^V_{X_1,X_3}(u,v) - J^V_{X_2,X_3}(u,v) \leqslant \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{f_1(v)/(\Delta F_1}{f_2(v)/\Delta F_2}\Big).$$

Proof. We have $X_1 \overset{\mathrm{lr}}{\leqslant} X_2$ and $u < x$. Therefore, $f_1^\gamma(x) \leqslant f_2^\gamma(x) f_1^\gamma(u)/f_2^\gamma(u)$. Now

$$\begin{aligned} J^V_{X_1,X_3}(u,v) &= \frac{1}{\alpha-\beta}\ln\int_u^v \frac{f_1^\gamma(x)}{(\Delta F_1)^\gamma}\frac{f_3^{1-\gamma}(x)}{(\Delta F_3)^{1-\gamma}}\,\mathrm{d}x \\ &\geqslant \frac{1}{\alpha-\beta}\ln\int_u^v \frac{f_2^\gamma(x) f_1^\gamma(u)}{f_2^\gamma(u)(\Delta F_1)^\gamma}\frac{f_3^{1-\gamma}(x)}{(\Delta F_3)^{1-\gamma}}\,\mathrm{d}x \\ &= J^V_{X_2,X_3}(u,v) + \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{f_1(u)/\Delta F_1}{f_2(u)/\Delta F_2}\Big), \end{aligned}$$

where $\Delta F_3 = F_3(v) - F_3(u)$. Hence,

$$J^V_{X_1,X_3}(u,v) - J^V_{X_2,X_3}(u,v) \geqslant \frac{\gamma}{\alpha-\beta}\ln\Big(\frac{f_1(u)/\Delta F_1}{f_2(u)/\Delta F_2}\Big).$$

The right-hand inequality can be obtained using $f_1^\gamma(x) \geqslant f_2^\gamma(x) f_1^\gamma(v)/f_2^\gamma(v)$ under the conditions $X_1 \overset{\mathrm{lr}}{\leqslant} X_2$ and $x < v$. This completes the proof of the theorem. $\square$

Theorem 3.4. *Let X_1, X_2, and X_3 be three random variables as described in Theorem 3.3. Denote $\Delta F_2 = F_2(v) - F_2(u)$ and $\Delta F_3 = F_3(v) - F_3(u)$. Let $X_2 \overset{\mathrm{lr}}{\leqslant} X_3$. Then for $\gamma < (>) 1$*

$$\begin{aligned} \frac{1-\gamma}{\alpha-\beta}\ln\Big(\frac{f_2(u)/\Delta F_2}{f_3(u)/\Delta F_3}\Big) &\leqslant (\geqslant)\ J^V_{X_1,X_2}(u,v) \\ &- J^V_{X_1,X_3}(u,v) \leqslant (\geqslant)\ \frac{1-\gamma}{\alpha-\beta}\ln\Big(\frac{f_2(v)/\Delta F_2}{f_3(v)/\Delta F_3}\Big). \end{aligned}$$

Proof. The proof is omitted as it follows along the arguments of that of Theorem 3.3. $\square$

Theorem 3.5. *Let X_1, X_2, and X_3 be three random variables as described in Theorem* 3.4. *Also let $X_1 \overset{\mathrm{lr}}{\leqslant} X_2$ and $X_1 \overset{\mathrm{lr}}{\geqslant} X_3$. Then,*

$$J^V_{X_1,X_3}(u,v) - J^V_{X_2,X_3}(u,v) \geqslant (\leqslant)\, \frac{\gamma-1}{\alpha-\beta} \ln\Big(\frac{f_1(u)/\Delta F_1}{f_2(u)/\Delta F_2}\Big), \text{ if } \gamma > (<) 1.$$

Proof. Let $\gamma > 1$. From (1.6), we have

$$\begin{aligned}
&J^V_{X_1,X_3}(u,v) - J^V_{X_2,X_3}(u,v) \\
&\quad = \frac{1}{\alpha-\beta}\Big[\ln \int_u^v \frac{f_1^\gamma(x)}{(\Delta F_1)^\gamma} \frac{f_3^{1-\gamma}(x)}{(\Delta F_3)^{1-\gamma}}\,\mathrm{d}x - \ln \int_u^v \frac{f_2^\gamma(x)}{(\Delta F_2)^\gamma} \frac{f_3^{1-\gamma}(x)}{(\Delta F_3)^{1-\gamma}}\,\mathrm{d}x\Big] \\
&\quad = \frac{1}{\alpha-\beta}\Big[\ln \int_u^v \Big(\frac{f_1(x)}{f_3(x)}\Big)^{\gamma-1} \frac{f_1(x)}{\Delta F_1}\,\mathrm{d}x \\
&\qquad - \ln \int_u^v \Big(\frac{f_1(x)}{f_3(x)}\Big)^{\gamma-1} \Big(\frac{f_2(x)}{f_1(x)}\Big)^{\gamma-1} \frac{f_2(x)}{\Delta F_2}\,\mathrm{d}x + \ln\Big(\frac{\Delta F_2}{\Delta F_1}\Big)^{\gamma-1}\Big] \\
&\quad \geqslant \frac{1}{\alpha-\beta}\Big[\Big\{\ln \int_u^v \Big(\frac{f_1(x)}{f_3(x)}\Big)^{\gamma-1} \frac{f_1(x)}{\Delta F_1}\,\mathrm{d}x - \ln \int_u^v \Big(\frac{f_1(x)}{f_3(x)}\Big)^{\gamma-1} \frac{f_2(x)}{\Delta F_2}\,\mathrm{d}x\Big\} \\
&\qquad + (\gamma-1)\ln\Big(\frac{f_1(u)/\Delta F_1}{f_2(u)/\Delta F_2}\Big)\Big] \geqslant \frac{\gamma-1}{\alpha-\beta}\ln\Big(\frac{f_1(u)/\Delta F_1}{f_2(u)/\Delta F_2}\Big),
\end{aligned}$$

where the first inequality is due to the fact that $X_1 \overset{\mathrm{lr}}{\leqslant} X_2$ and the second inequality comes from the fact that the expression within the braces is nonpositive due to $E(\psi(X_1 \,|\, u < X_1 < v)) \leqslant E(\psi(X_2 \,|\, u < X_2 < v))$ for $\psi(x) = (f_1(x)/f_3(x))^{\gamma-1}$. This completes the proof of one part of the theorem. The other part can be proved similarly. Hence, the result follows. □

Theorem 3.6. *Let X_1, X_2, and X_3 be three random variables as described in Theorem* 3.4. *Also let $X_1 \overset{\mathrm{lr}}{\leqslant} X_2$ and $X_1 \overset{\mathrm{lr}}{\leqslant} X_3$. Then,*

$$J^V_{X_1,X_3}(u,v) - J^V_{X_2,X_3}(u,v) \geqslant (\leqslant)\, \frac{\gamma-1}{\alpha-\beta} \ln\Big(\frac{f_1(v)/\Delta F_1}{f_2(v)/\Delta F_2}\Big), \quad \text{if } \gamma > (<) 1.$$

Proof. Proof of this theorem is similar to that of Theorem 3.5, hence omitted. □

It is a natural task for a reliability engineer to identify statistical models for which generalized discrimination information measures in residual and past lifetimes are independent of the time variable. In this regards, Kayal (in press) showed that the generalized residual discrimination measure $J^V_{X,Y}(t)$ is independent of t for $\gamma\theta -$

$\gamma + 1 > 0$ if and only if X and Y satisfy the proportional hazard rate model, that is, the survival functions of X and Y are related by $\overline{F}(t) = [\overline{G}(t)]^\theta$, where $t > 0$ and $\theta > 0$. The constant θ is known as the proportionality constant. The author also showed that the generalized past discrimination measure $\overline{J}^V_{X,Y}(t)$ is independent of t for $\gamma\theta - \gamma + 1 > 0$ if and only if X and Y satisfy the proportional reversed hazard rate model, that is, the cumulative distribution functions of X and Y satisfy the relation $F(t) = [G(t)]^\theta$, where $t > 0$ and $\theta > 0$. In the following we study the above facts for the doubly truncated generalized discrimination information measure. Let

$$J^V_{X,Y}(u, v) = A, \tag{3.9}$$

where the constant A is independent of both u and v. Let $u \to 0+$. Then we get $\overline{J}^V_{X,Y}(v) = A$ which implies that the random variables X and Y satisfy the proportional reversed hazards model. Let $v \to \infty$. Then from (3.7) we get $J^V_{X,Y}(u) = A$ which implies that X and Y satisfy the proportional hazards model. Hence, the random variables X and Y have identical probability distribution, that is, $A = 0$. Therefore, we conclude that $J^V_{X,Y}(u, v)$ cannot be equal to a nonzero constant which is independent of u and v.

4. The effect of monotone transformations on $J^V_{X,Y}(u, v)$

In this section we confine our attention to show how monotone transformation effects the discrimination information measure (1.6).

Theorem 4.1. *Let X and Y be two random variables with common support $(0, \infty)$ having pdf's $f(x)$ and $g(x)$, and cdf's $F(x)$ and $G(x)$, respectively. Also let φ be a bijective transformation from $(0, \infty)$ to $(0, \infty)$. Then*
(i) $J^V_{\varphi(X),\varphi(Y)}(u, v) = J^V_{X,Y}(\varphi^{-1}(u), \varphi^{-1}(v))$, *if φ is strictly increasing and*
(ii) $J^V_{\varphi(X),\varphi(Y)}(u, v) = J^V_{X,Y}(\varphi^{-1}(v), \varphi^{-1}(u))$, *if φ is strictly decreasing.*

Proof. (i): Note that when φ is strictly increasing the pdf and cdf of $\varphi(X)$ are given by

$$f_\varphi(x) = \frac{f(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))} \quad \text{and} \quad F_\varphi(x) = F(\varphi^{-1}(x)), \tag{4.1}$$

respectively, and those of $\varphi(Y)$ are given by

$$g_\varphi(x) = \frac{g(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))} \quad \text{and} \quad G_\varphi(x) = G(\varphi^{-1}(x)). \tag{4.2}$$

Denote $\Delta F_\varphi = F(\varphi^{-1}(v)) - F(\varphi^{-1}(u))$ and $\Delta G_\varphi = G(\varphi^{-1}(v)) - G(\varphi^{-1}(u))$. From (1.6),

$$
\begin{aligned}
(4.3) \qquad J^V_{\varphi(X),\varphi(Y)}(u,v) &= \frac{1}{\alpha-\beta}\ln\int_u^v \frac{f^\gamma_\varphi(x)}{(\Delta F_\varphi)^\gamma}\frac{g^{1-\gamma}_\varphi(x)}{(\Delta G_\varphi)^{1-\gamma}}\,\mathrm{d}x \\
&= \frac{1}{\alpha-\beta}\ln\int_u^v \frac{f^\gamma(\varphi^{-1}(x))}{(\varphi'(\varphi^{-1}(x)))^\gamma(\Delta F_\varphi)^\gamma}\frac{g^{1-\gamma}(\varphi^{-1}(x))}{(\varphi'(\varphi^{-1}(x)))^{1-\gamma}(\Delta G_\varphi)^{1-\gamma}}\,\mathrm{d}x \\
&= \frac{1}{\alpha-\beta}\ln\int_{\varphi^{-1}(u)}^{\varphi^{-1}(v)} \frac{f^\gamma(w)}{(\Delta F_\varphi)^\gamma}\frac{g^{1-\gamma}(w)}{(\Delta G_\varphi)^{1-\gamma}}\,\mathrm{d}w \\
&= J^V_{X,Y}(\varphi^{-1}(u),\varphi^{-1}(v)),
\end{aligned}
$$

where the third equality is a result of using the transformation $w = \varphi^{-1}(x)$.

(ii): To prove the second part we assume that φ is strictly decreasing. Therefore, the pdf and cdf of $\varphi(X)$ are given by

$$
(4.4) \qquad f_\varphi(x) = -\frac{f(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))} \quad \text{and} \quad F_\varphi(x) = \overline{F}(\varphi^{-1}(x)),
$$

respectively, and those of $\varphi(Y)$ are given by

$$
(4.5) \qquad g_\varphi(x) = -\frac{g(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))} \quad \text{and} \quad G_\varphi(x) = \overline{G}(\varphi^{-1}(x)).
$$

Denote $\Delta\overline{F}_\varphi = \overline{F}(\varphi^{-1}(v)) - \overline{F}(\varphi^{-1}(u))$ and $\Delta\overline{G}_\varphi = \overline{G}(\varphi^{-1}(v)) - \overline{G}(\varphi^{-1}(u))$. From (1.6),

$$
\begin{aligned}
&J^V_{\varphi(X),\varphi(Y)}(u,v) = \frac{1}{\alpha-\beta}\ln\int_u^v \frac{f^\gamma_\varphi(x)}{(\Delta\overline{F}_\varphi)^\gamma}\frac{g^{1-\gamma}_\varphi(x)}{(\Delta\overline{G}_\varphi)^{1-\gamma}}\,\mathrm{d}x \\
&= \frac{1}{\alpha-\beta}\ln\int_u^v \frac{(-1)^\gamma f^\gamma(\varphi^{-1}(x))}{(\varphi'(\varphi^{-1}(x)))^\gamma(\Delta\overline{F}_\varphi)^\gamma}\frac{(-1)^{1-\gamma}g^{1-\gamma}(\varphi^{-1}(x))}{(\varphi'(\varphi^{-1}(x)))^{1-\gamma}(\Delta\overline{G}_\varphi)^{1-\gamma}}\,\mathrm{d}x \\
&= \frac{1}{\alpha-\beta}\ln\int_{\varphi^{-1}(v)}^{\varphi^{-1}(u)} \frac{f^\gamma(w)}{(F(\varphi^{-1}(u)) - F(\varphi^{-1}(v)))^\gamma}\frac{g^{1-\gamma}(w)}{(G(\varphi^{-1}(u)) - G(\varphi^{-1}(v)))^{1-\gamma}}\,\mathrm{d}w \\
&= J^V_{X,Y}(\varphi^{-1}(v),\varphi^{-1}(u)),
\end{aligned}
$$

where the third equality is a result of using the transformation $w = \varphi^{-1}(x)$. This completes the proof of the theorem. □

The next remarks immediately follow from Theorem 4.1. First and second remarks show the effect of the cdf and the survival functions of a nonnegative random variable X on (1.6), respectively. The third remark presents the effect of the scale and shift transformations on (1.6).

R e m a r k 4.1. Let $\varphi(x) = F(x)$, where $F(x)$ is the cdf of X. Assume that $F(x)$ is strictly increasing in x. Then from Theorem 4.1 we have

$$J^V_{F(X),F(Y)}(u,v) = J^V_{X,Y}(F^{-1}(u), F^{-1}(v)).$$

R e m a r k 4.2. Let $\varphi(x) = \overline{F}(x)$, where $\overline{F}(x)$ is the survival function of X. Here, assume that $\overline{F}(x)$ is strictly decreasing in x. From Theorem 4.1 we obtain

$$J^V_{\overline{F}(X),\overline{F}(Y)}(u,v) = J^V_{X,Y}(\overline{F}^{-1}(v), \overline{F}^{-1}(u)).$$

R e m a r k 4.3. For two nonnegative random variables X and Y

$$J^V_{aX,aY}(u,v) = J^V_{X,Y}\Big(\frac{u}{a}, \frac{v}{a}\Big) \quad \text{and} \quad J^V_{X+b,Y+b}(u,v) = J^V_{X,Y}(u-b, v-b),$$

where $a > 0$ and $0 < b < \min(u,v)$.

As an application of Theorem 4.1, we consider the following example.

E x a m p l e 4.1. Consider two nonnegative random variables X and Y with pdf's

$$f(x) = \begin{cases} \exp(-x), & \text{if } 0 < x < \infty, \\ 0, & \text{otherwise,} \end{cases} \tag{4.6}$$

and

$$g(x) = \begin{cases} 2\exp(-2x), & \text{if } 0 < x < \infty, \\ 0, & \text{otherwise,} \end{cases} \tag{4.7}$$

respectively. Consider a decreasing transformation $\varphi(x) = 1/x$ from $(0,\infty)$ to $(0,\infty)$. Then for $0 < u < v < \infty$, it can be shown that

$$\begin{aligned} J^V_{1/X,1/Y}(u,v) &= J^V_{X,Y}(1/u, 1/v) \\ &= \frac{1}{\alpha-\beta} \ln\Big[\frac{2^{1-\gamma}(\gamma-2)^{-1}(\exp((\gamma-2)/v) - \exp((\gamma-2)/u))}{(\exp(-1/u) - \exp(-1/v))^{\gamma}(\exp(-2/u) - \exp(-2/v))^{1-\gamma}}\Big]. \end{aligned} \tag{4.8}$$

Now consider an increasing transformation $\varphi(x) = x^{1/a}$, where $0 < x < \infty$ and $a > 0$. Under this transformation and for $0 < u < v < \infty$, we obtain

$$\begin{aligned} J^V_{X^{1/a},Y^{1/b}}(u,v) &= J^V_{X,Y}(u^a, v^a) \\ &= \frac{1}{\alpha-\beta} \ln\Big[\frac{2^{1-\gamma}(\gamma-2)^{-1}(\exp((\gamma-2)v^a) - \exp((\gamma-2)u^a))}{(\exp(-u^a) - \exp(-v^a))^{\gamma}(\exp(-2u^a) - \exp(-2v^a))^{1-\gamma}}\Big]. \end{aligned} \tag{4.9}$$

5. The simulation study

In this section, to present estimates of the doubly truncated generalized discrimination measure between two exponential populations with means $1/\lambda_1$ and $1/\lambda_2$, where $\lambda_1, \lambda_2 > 0$, a simulation study is carried out. Denote $X \sim \mathrm{Exp}(\lambda_1)$ and $Y \sim \mathrm{Exp}(\lambda_2)$. Then the doubly truncated generalized discrimination measure between X and Y is given by

$$J^V_{X,Y}(u,v) = \frac{1}{\alpha-\beta}[\ln H_1(\lambda_1,\lambda_2) - \gamma \ln H_2(\lambda_1,\lambda_2) - (1-\gamma)\ln H_3(\lambda_1,\lambda_2)], \tag{5.1}$$

where $\gamma = \alpha+\beta-1$, $0 \leqslant \beta - 1 < \alpha < \beta$ and

$$\begin{aligned} H_1(\lambda_1,\lambda_2) &= [\exp(-(\lambda_1\gamma+\lambda_2(1-\gamma))u) - \exp(-(\lambda_1\gamma+\lambda_2(1-\gamma))v)] \\ &\quad\times \frac{\lambda_1^{\gamma}\lambda_2^{1-\gamma}}{\lambda_1\gamma+\lambda_2(1-\gamma)}, \\ H_2(\lambda_1,\lambda_2) &= \exp(-\lambda_1 u) - \exp(-\lambda_1 v), \\ H_3(\lambda_1,\lambda_2) &= \exp(-\lambda_2 u) - \exp(-\lambda_2 v). \end{aligned}$$

To estimate $J^V_{X,Y}(u,v)$ given by (5.1), we use the method of maximum likelihood. First we estimate the unknown parameters λ_1 and λ_2 based on the doubly truncated exponential data. Then we plug these in (5.1) to get the maximum likelihood estimator (MLE) of $J^V_{X,Y}(u,v)$. Here $[u,v]$ is the truncated interval, where u is the left truncation point and v is the right truncation point. Suppose that $x_1, x_2, \ldots, x_n$ are the realizations of identically, independently distributed random observations from exponential population with mean $1/\lambda_1$ subject to the constraints $x_i \in [u,v]$, $i = 1,2,\ldots,n$. Then the truncated density of x_i subject to $x_i \in [u,v]$ is

$$f_T(x_i \mid \lambda_1) = \begin{cases} \dfrac{\lambda_1 \exp(-\lambda_1 x_i)}{\exp(-\lambda_1 u) - \exp(-\lambda_1 v)}, & \text{if } x_i \in [u,v], \\ 0, & \text{if } x_i \notin [u,v]. \end{cases} \tag{5.2}$$

Thus, the log-likelihood function for the data $(x_1, x_2, \ldots, x_n)$ is

$$l(\lambda_1 \mid x_1, x_2, \ldots, x_n) = \ln\biggl\{\prod_{i=1}^{n} \frac{\lambda_1 \exp(-\lambda_1 x_i)}{\exp(-\lambda_1 u) - \exp(-\lambda_1 v)}\biggr\}. \tag{5.3}$$

Hence, the MLE of λ_1 can be obtained after solving the equation $\mathrm{d}l/\mathrm{d}\lambda_1 = 0$ in λ_1, which is given by

$$\frac{n}{\lambda_1} - \sum_{i=1}^{n} x_i - \frac{n(v\exp(-\lambda_1 v) - u\exp(-\lambda_1 u))}{\exp(-\lambda_1 u) - \exp(-\lambda_1 v)} = 0. \tag{5.4}$$

As the solution of (5.4) cannot be obtained explicitly, we use the Newton-Raphson method. In analogy to the MLE of λ_1, the MLE of λ_2 can be obtained. Let $y_1, y_2, \ldots, y_n$ be the realizations of identically, independently distributed random observations from exponential population with mean $1/\lambda_2$ subject to the constraints $y_i \in [u, v]$, $i = 1, 2, \ldots, n$. Then MLE of λ_2 can be obtained as a solution of

$$\frac{n}{\lambda_2} - \sum_{i=1}^{n} y_i - \frac{n(v\exp(-\lambda_2 v) - u\exp(-\lambda_2 u))}{\exp(-\lambda_2 u) - \exp(-\lambda_2 v)} = 0. \tag{5.5}$$

To obtain λ_2, we use the Newton-Raphson method. We denote the MLEs of λ_1, λ_2 and $J^V_{X,Y}(u,v)$ as $\widehat{\lambda}_1$, $\widehat{\lambda}_2$ and $\widehat{J}^V_{X,Y}(u,v)$, respectively. The MLE of $J^V_{X,Y}(u,v)$ is given by

$$\widehat{J}^V_{X,Y}(u,v) = \frac{1}{\alpha-\beta}[\ln H_1(\widehat{\lambda}_1, \widehat{\lambda}_2) - \gamma \ln H_2(\widehat{\lambda}_1, \widehat{\lambda}_2) - (1-\gamma)\ln H_3(\widehat{\lambda}_1, \widehat{\lambda}_2)],$$

where $\gamma = \alpha + \beta - 1$, $0 \leqslant \beta - 1 < \alpha < \beta$. To get the MLE of λ_1 and λ_2, we generate the data from two exponential populations using Monte Carlo simulation. The estimated values are computed based on 1000 samples with sample size 100 for different truncation limits and parameter values. The standard deviations of the simulated estimates of $J^V_{X,Y}(u,v)$ have been added. In the following, we present a few of them. From the simulated data presented in the tables we observe the following points.

- ▷ For fixed values of α and β, the doubly truncated generalized discrimination measure is more when two distributions are less similar (here, in terms of the values of the parameters). It is true, since (1.6) measures the closeness between two distributions.
- ▷ The estimates are almost unbiased.

6. Concluding remarks

In this paper, we have proposed a generalized discrimination measure between two doubly truncated nonnegative random variables. Using the concepts of likelihood ratio order, we have obtained some bounds of the proposed measure. Some examples have been provided in support of our results. It is shown that the measure can never be equal to a nonzero constant, free from the left and right truncation points. We have studied the effect of the monotone transformation on the proposed measure. Finally, a simulation study is carried out for the purpose of estimation of the proposed discrimination measure. It is worthwhile to mention that the results obtained in

(λ_1, λ_2)	(u, v)	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$	$J^V_{X,Y}(u, v)$	$\widehat{J}^V_{X,Y}(u, v)$	$\widehat{J}^{VSD}_{X,Y}(u, v)$
(0.2,0.3)	(1, 5)	0.185716	0.305027	0.001908	0.002722	0.003672
	(1,10)	0.201858	0.303778	0.008088	0.008363	0.001409
	(2, 5)	0.205211	0.324040	0.001095	0.001542	0.002396
	(2,10)	0.204341	0.306743	0.006682	0.006953	0.012115
	(4, 7)	0.253295	0.314577	0.001095	0.000408	0.001468
	(4,10)	0.207764	0.311663	0.004056	0.004344	0.005015
(0.3,0.2)	(1, 5)	0.302579	0.194589	0.001898	0.002214	0.001242
	(1,10)	0.303595	0.200952	0.007901	0.008288	0.001433
	(2, 5)	0.320192	0.204296	0.001092	0.001462	0.001302
	(2,10)	0.306898	0.204585	0.006556	0.006803	0.001478
	(4, 7)	0.311436	0.257802	0.001092	0.000312	0.001246
	(4,10)	0.307800	0.208411	0.004011	0.003934	0.009219
(0.5,0.7)	(1, 5)	0.509791	0.719982	0.006198	0.006768	0.013489
	(1,10)	0.513021	0.721777	0.015015	0.015707	0.021671
	(2, 5)	0.508163	0.698431	0.003879	0.003504	0.003951
	(2,10)	0.516174	0.734642	0.013906	0.015753	0.018772
	(4, 7)	0.548922	0.937266	0.003879	0.013647	0.016294
	(4,10)	0.548683	0.874008	0.010655	0.024073	0.027963
(0.7,0.5)	(1, 5)	0.720064	0.512822	0.006065	0.006421	0.001138
	(1,10)	0.723613	0.513098	0.014068	0.014889	0.017683
	(2, 5)	0.719813	0.505005	0.003828	0.004386	0.009101
	(2,10)	0.667588	0.514261	0.013121	0.007906	0.009087
	(4, 7)	0.836676	0.552059	0.003828	0.012984	0.020374
	(4,10)	0.860928	0.545948	0.010229	0.021336	0.014429
(1.0,1.2)	(1, 5)	1.026330	1.261371	0.003808	0.005030	0.012377
	(1,10)	0.979469	1.256811	0.005074	0.009567	0.001489
	(2, 5)	1.087236	1.307370	0.002839	0.006547	0.007262
	(2,10)	1.098233	1.393130	0.005036	0.008701	0.007669
	(4, 7)	1.182961	1.302354	0.002839	0.002954	0.004119
	(4,10)	1.163243	1.325673	0.004774	0.002181	0.003237
(1.2,1.0)	(1, 5)	1.259091	1.028553	0.003700	0.004678	0.004633
	(1,10)	1.257250	1.034960	0.004838	0.005507	0.013748
	(2, 5)	1.338270	1.091170	0.002784	0.003886	0.003976
	(2,10)	1.301770	1.099900	0.004806	0.004135	0.005332
	(4, 7)	1.319360	1.213200	0.002784	0.000693	0.001998
	(4,10)	1.281960	1.471570	0.004580	0.002849	0.003564

Table 1. Estimates of $J^V_{X,Y}(u, v)$ when $\alpha = 0.5$ and $\beta = 1.2$.

(λ_1, λ_2)	(u, v)	$\widehat{\lambda}_1$	$\widehat{\lambda}_2$	$J^V_{X,Y}(u,v)$	$\widehat{J}^V_{X,Y}(u,v)$	$\widehat{J}^{VSD}_{X,Y}(u,v)$
(1.5,1.7)	(1, 5)	1.587580	1.824951	0.002217	0.002805	0.002743
	(1,10)	1.599251	1.839232	0.002388	0.002985	0.003067
	(2, 5)	1.463465	1.931831	0.001914	0.009857	0.013542
	(2,10)	1.629754	1.837385	0.002387	0.004584	0.011638
	(4, 7)	1.635623	1.877412	0.001914	0.002466	0.005189
	(4,10)	1.670945	1.838124	0.002372	0.001377	0.001496
(1.7,1.5)	(1, 5)	1.835670	1.597400	0.002156	0.002708	0.002758
	(1,10)	1.836380	1.600860	0.002309	0.002772	0.002637
	(2, 5)	1.901370	1.828440	0.001872	0.000201	0.000387
	(2,10)	1.824570	1.607360	0.002308	0.002367	0.005213
	(4, 7)	1.958580	1.749320	0.001872	0.001664	0.005011
	(4,10)	2.108600	1.950190	0.002295	0.000904	0.002469
(0.1,2.0)	(1, 5)	0.120810	2.200040	0.425101	0.469663	0.437601
	(1,10)	0.101252	2.206350	0.915016	0.994609	0.901134
	(2, 5)	0.127476	2.216070	0.287632	0.325704	0.424751
	(2,10)	0.098347	2.191900	0.843525	0.922928	0.878996
	(4, 7)	0.116580	1.939690	0.287632	0.267469	0.381946
	(4,10)	0.105630	1.882850	0.663418	0.611375	0.695472
(2.0,0.1)	(1, 5)	2.190770	0.101191	0.333439	0.367477	0.395110
	(1,10)	1.812910	0.098228	0.580998	0.544335	0.564458
	(2, 5)	2.085750	0.139836	0.243202	0.244011	0.278596
	(2,10)	2.134000	0.096641	0.548960	0.578928	0.535176
	(4, 7)	1.929890	0.120601	0.243202	0.225009	0.363851
	(4,10)	1.958780	0.094239	0.463813	0.460499	0.570549

Table 2. Estimates of $J^V_{X,Y}(u,v)$ when $\alpha = 0.5$ and $\beta = 1.2$.

the above sections are general in the sense that they reduce to some of the results for doubly truncated discrimination information measure obtained by Misagh and Yari [12] and Al-Rahman and Kittaneh [1], when $\beta = 1$ and α tends to 1. The results in the present paper also hold for Renyi's doubly truncated discrimination information measure when $\beta = 1$. It is also noted that some results obtained by Kundu [10] and Kayal [8] can be derived from the results of the present paper by taking $(u \to 0, v \to t)$ and $(u \to t, v \to \infty)$.

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