Yinzhen Mei; Yubin Gao; Yan Ling Shao; Peng Wang
A new family of spectrally arbitrary ray patterns

Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 4, 1049–1058

Persistent URL: `http://dml.cz/dmlcz/145917`

Czechoslovak Mathematical Journal, 66 (141) (2016), 1049–1058

A NEW FAMILY OF SPECTRALLY ARBITRARY RAY PATTERNS

Yinzhen Mei, Yubin Gao, Yanling Shao, Peng Wang, Taiyuan

(Received December 7, 2014)

Abstract. An $n \times n$ ray pattern $\mathcal{A}$ is called a spectrally arbitrary ray pattern if the complex matrices in $Q(\mathcal{A})$ give rise to all possible complex polynomials of degree n.

In a paper of Mei, Gao, Shao, and Wang (2014) was proved that the minimum number of nonzeros in an $n\times n$ irreducible spectrally arbitrary ray pattern is $3n-1$. In this paper, we introduce a new family of spectrally arbitrary ray patterns of order n with exactly $3n-1$ nonzeros.

Keywords: ray pattern; potentially nilpotent; spectrally arbitrary ray pattern

MSC 2010: 15A18, 15A29

1. Introduction

An $n \times n$ *ray pattern* $\mathcal{A} = (a_{ij})$ is an $n \times n$ matrix with entries $a_{ij} \in \{e^{i\theta}\colon 0 \leqslant \theta < 2\pi\} \cup \{0\}$. Its ray pattern class is (see [3])

$$Q(\mathcal{A}) = \{B = (b_{ij}) \in M_n(\mathbb{C})\colon\ b_{ij} = r_{ij}a_{ij},\ r_{ij} \in \mathbb{R}^+,\ 1 \leqslant i,j \leqslant n\}.$$

A ray pattern $\mathcal{A}$ is called potentially nilpotent if there is a complex matrix $B \in Q(\mathcal{A})$ and a positive integer k such that $B^k = 0$.

An $n\times n$ ray pattern $\mathcal{A}$ is called a spectrally arbitrary ray pattern if the complex matrices in the class give rise to all possible complex polynomials of degree n. It is clear that any spectrally arbitrary ray pattern must be potentially nilpotent. If $\mathcal{A}$ is a spectrally arbitrary ray pattern and no proper subpattern of $\mathcal{A}$ is spectrally arbitrary, then $\mathcal{A}$ is a minimal spectrally arbitrary ray pattern.

The concept of spectrally arbitrary sign pattern was introduced in [1] and the nilpotent Jacobi method involving the Implicit Function Theorem was used to prove

The research has been supported by NNSF of China (No. 11071227) and Shanxi Scholarship Council of China (No. 2012-070).

some sign patterns are spectrally arbitrary. In 2008, McDonald and Stuart extended the nilpotent Jacobi method to ray patterns. They provided a family of spectrally arbitrary ray patterns that have exactly $3n$ nonzeros and demonstrated that every $n\times n$ irreducible spectrally arbitrary ray pattern must have at least $3n-1$ nonzeros (see [3]). Next, Gao and Shao showed that the $n\times n$ ray pattern that they called $\mathcal{A}_{n,m}$ is a spectrally arbitrary ray pattern ([2]). In [4], we find a family of spectrally arbitrary ray patterns of order n with exactly $3n-1$ nonzeros, and so the minimum number of nonzeros in an $n\times n$ irreducible spectrally arbitrary ray pattern is $3n-1$. In this paper, we provide a new family of spectrally arbitrary ray patterns of order n with exactly $3n-1$ nonzeros.

We consider the $n\times n$ $(n\geqslant 6)$ ray patterns

$$\mathcal{A}_n=\begin{bmatrix} -1 & 1 & 0 & \dots & \dots & \dots & \dots & 0 & 0\\ 1 & \mathrm{e}^{\mathrm{i}\theta} & 1 & 0 & \dots & \dots & \dots & 0 & 0\\ 1 & 0 & 0 & 1 & 0 & \dots & \dots & 0 & 0\\ 1 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & \dots & \dots & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & \dots & \dots & \dots & 0 & 1\\ \mathrm{e}^{\mathrm{i}\beta} & -\mathrm{i} & -\mathrm{i} & -\mathrm{i} & \dots & \dots & \dots & -\mathrm{i} & -\mathrm{i} \end{bmatrix}, \tag{1.1}$$

where $\theta\in(\pi/4,\pi/2)$ and $\beta\in(3\pi/2,2\pi)$. We shall prove that for any $\theta\in(\pi/4,\pi/2)$, there exist infinitely many choices for $\beta\in(3\pi/2,2\pi)$ such that $\mathcal{A}_n$ is a minimal spectrally arbitrary ray pattern.

Lemma 1.1 ([3], Extended nilpotent Jacobi method)**.**

(1) *Find a nilpotent matrix in the given ray pattern class.*

(2) *Change $2n$ of the positive coefficients (denoted $r_1,r_2,\dots,r_{2n}$) of the $\mathrm{e}^{\mathrm{i}\theta_{ij}}$ in this nilpotent matrix to variables $t_1,t_2,\dots,t_{2n}$.*

(3) *Express the characteristic polynomial of the resulting matrix as*

$$x^n+(f_1(t_1,t_2,\dots,t_{2n})+\mathrm{i}g_1(t_1,t_2,\dots,t_{2n}))x^{n-1}+\dots+(f_{n-1}(t_1,t_2,\dots,t_{2n})\\ +\mathrm{i}g_{n-1}(t_1,t_2,\dots,t_{2n}))x+(f_n(t_1,t_2,\dots,t_{2n})+\mathrm{i}g_n(t_1,t_2,\dots,t_{2n})).$$

(4) *Find the Jacobi matrix*

$$J=\frac{\partial(f_1,g_1,\dots,f_n,g_n)}{\partial(t_1,t_2,\dots,t_{2n})}.$$

If the determinant of J evaluated at $(t_1,t_2,\dots,t_{2n})=(r_1,r_2,\dots,r_{2n})$ is nonzero, then the given ray pattern and all of its superpatterns are spectrally arbitrary.

2. Main results

In this section we shall show that the $\mathcal{A}_n$ are minimal spectrally arbitrary ray patterns based upon the extended nilpotent Jacobi method. Using the method, finding an appropriate nilpotent matrix is a key step.

For convenience, we consider the $n \times n$ complex matrix

$$(2.1)\qquad B_n = \begin{bmatrix} -a_1 & 1 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\ a_2 & re^{i\theta} & 1 & 0 & \dots & \dots & \dots & 0 & 0 \\ a_3 & 0 & 0 & 1 & 0 & \dots & \dots & 0 & 0 \\ a_4 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ a_{n-3} & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 \\ 0 & -a_{n-2} & 0 & 0 & \dots & \dots & 0 & 1 & 0 \\ 0 & a_{n-1} & 0 & 0 & \dots & \dots & \dots & 0 & 1 \\ r_1e^{i\beta} & -ib_{n-1} & -ib_{n-2} & -ib_{n-3} & \dots & \dots & \dots & -ib_2 & -ib_1 \end{bmatrix},$$

where $r_1e^{i\beta} = a_n - ib_n$ with $r_1 > 0$, $a_n > 0$, $b_n > 0$, $r > 0$, $a_i > 0$ for $i = 1, 2, \ldots, n-1$, and $b_j > 0$ for $j = 1, 2, \ldots, n-1$. Then $B_n \in Q(\mathcal{A}_n)$.

Note that the coefficient of λ^{n-j} in the characteristic polynomial for B_n consists of the sum of signed weighted products of disjoint cycles whose total length is j. Then we have the following characteristic polynomial of B_n in Table 1.

Let $|\lambda I - B_n| = \lambda^n + \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + \ldots + \alpha_k\lambda^{n-k} + \ldots + \alpha_{n-1}\lambda + \alpha_n$, and $\alpha_k = f_k + ig_k$, $k = 1, 2, \ldots, n$.

Lemma 2.1. *For any $\theta \in (\pi/4, \pi/2)$, there exist infinitely many choices for $\beta \in (3\pi/2, 2\pi)$ such that the ray patterns $\mathcal{A}_n$ allow nilpotence.*

Proof. Let $B_n \in Q(\mathcal{A}_n)$ have the form (2.1), and suppose that B_n is nilpotent. By Table 1, we have that

$$(2.2)\qquad \begin{cases} a_1 = r\cos\theta, \\ a_2 = r(b_1\sin\theta - a_1\cos\theta), \\ a_3 = r\sin\theta(a_1b_1 + b_2), \\ a_4 = r\sin\theta(a_1b_2 + b_3), \\ a_j = r\sin\theta(a_1b_{j-2} + b_{j-1}),\ 5 \leqslant j \leqslant n-4, \\ a_{n-3} = r\sin\theta(a_1b_{n-5} + b_{n-4}) + a_{n-2}, \\ a_1a_{n-2} = a_{n-1} - r\sin\theta(a_1b_{n-4} + b_{n-3}), \\ a_1a_{n-1} = r\sin\theta(a_1b_{n-3} + b_{n-2}), \\ a_n = a_1b_{n-2}r\sin\theta, \end{cases}$$

Term	Coefficient
λ^{n-1}	$a_1 - r\cos\theta + \mathrm{i}(b_1 - r\sin\theta)$
λ^{n-2}	$-a_2 - a_1 r\cos\theta + b_1 r\sin\theta + \mathrm{i}[b_2 + (a_1 - r\cos\theta)b_1 - a_1 r\sin\theta]$
λ^{n-3}	$-a_3 + a_1 b_1 r\sin\theta + b_2 r\sin\theta$ $+\mathrm{i}[b_3 + (a_1 - r\cos\theta)b_2 - a_2 b_1 - a_1 b_1 r\cos\theta]$
λ^{n-4}	$-a_4 + a_1 b_2 r\sin\theta + b_3 r\sin\theta$ $+\mathrm{i}[b_4 + (a_1 - r\cos\theta)b_3 - a_2 b_2 - a_3 b_1 - a_1 b_2 r\cos\theta]$
λ^{n-j} $(5 \leqslant j \leqslant n-3)$	$-a_j + a_1 b_{j-2} r\sin\theta + b_{j-1} r\sin\theta$ $+\mathrm{i}\Big[b_j + (a_1 - r\cos\theta)b_{j-1} - \sum_{k=2}^{j-1} a_k b_{j-k} - a_1 b_{j-2} r\cos\theta\Big]$
λ^3	$-a_{n-3} + a_1 b_{n-5} r\sin\theta + b_{n-4} r\sin\theta + a_{n-2}$ $+\mathrm{i}\Big[b_{n-3} + (a_1 - r\cos\theta)b_{n-4} - \sum_{k=2}^{n-4} a_k b_{n-k-3} - a_1 b_{n-5} r\cos\theta\Big]$
λ^2	$a_1 a_{n-2} + a_1 b_{n-4} r\sin\theta + b_{n-3} r\sin\theta - a_{n-1}$ $+\mathrm{i}\Big[b_{n-2} + (a_1 - r\cos\theta)b_{n-3}$ $- \sum_{k=2}^{n-3} a_k b_{n-k-2} + a_{n-2} b_1 - a_1 b_{n-4} r\cos\theta\Big]$
λ	$-a_1 a_{n-1} + a_1 b_{n-3} r\sin\theta + b_{n-2} r\sin\theta$ $+\mathrm{i}\Big[b_{n-1} + (a_1 - r\cos\theta)b_{n-2} - \sum_{k=2}^{n-3} a_k b_{n-k-1}$ $+a_{n-2} b_2 - a_{n-1} b_1 + a_1 a_{n-2} b_1 - a_1 b_{n-3} r\cos\theta\Big]$
λ^0	$-a_n + a_1 b_{n-2} r\sin\theta$ $+\mathrm{i}\Big[b_n - \sum_{k=2}^{n-3} a_k b_{n-k} + a_1(a_{n-2} b_2 - a_{n-1} b_1 + b_{n-1} - b_{n-2} r\cos\theta)\Big]$

Table 1. The characteristic polynomial of B_n.

and

$$
\begin{cases}
b_1 = r\sin\theta, \\
b_2 = r(b_1\cos\theta + a_1\sin\theta) - a_1 b_1, \\
b_3 = (r\cos\theta - a_1)b_2 + a_2 b_1 + a_1 b_1 r\cos\theta, \\
b_j = (r\cos\theta - a_1)b_{j-1} + \sum_{k=2}^{j-1} a_k b_{j-k} + a_1 b_{j-2} r\cos\theta, 4 \leqslant j \leqslant n-3, \\
b_{n-2} = (r\cos\theta - a_1)b_{n-3} + \sum_{k=2}^{n-3} a_k b_{n-k-2} - a_{n-2} b_1 + a_1 b_{n-4} r\cos\theta, \\
b_{n-1} = (r\cos\theta - a_1)b_{n-2} + \sum_{k=2}^{n-3} a_k b_{n-k-1} - a_{n-2}(b_2 + a_1 b_1) + a_{n-1} b_1 \\
\qquad +a_1 b_{n-3} r\cos\theta, \\
b_n = \sum_{k=2}^{n-3} a_k b_{n-k} - a_1(a_{n-2} b_2 - a_{n-1} b_1 + b_{n-1} - b_{n-2} r\cos\theta).
\end{cases} \tag{2.3}
$$

For convenience, let $r=1$, $\cos\theta=p$, $\sin\theta=q=\sqrt{1-p^2}$. Then for $\theta\in(\pi/4,\pi/2)$ we have $0<p<\sqrt{2}/2$.

First, by (2.2) and (2.3), we can obtain that

$$\begin{cases} a_1=p,\\ b_1=q,\\ a_2=q^2-p^2=1-2p^2,\\ b_2=pq,\\ a_3=2pq^2,\\ b_3=q^3,\\ a_4=q^2(p^2+q^2). \end{cases} \tag{2.4}$$

Via the substitution (2.4), we can obtain that

$$\begin{cases} a_j=q(b_{j-1}+pb_{j-2}), & 5\leqslant j\leqslant n-4,\\ b_j=q^2b_{j-2}+\sum\limits_{k=3}^{j-1}a_kb_{j-k}, & 5\leqslant j\leqslant n-3, \end{cases}$$

and

$$\begin{cases} a_{n-3}=a_{n-2}+q(b_{n-4}+pb_{n-5}),\\ pa_{n-2}=a_{n-1}-q(b_{n-3}+pb_{n-4}),\\ b_{n-2}=q^2b_{n-4}-qa_{n-2}+\sum\limits_{k=3}^{n-3}a_kb_{n-2-k},\\ pa_{n-1}=q(b_{n-2}+pb_{n-3}),\\ b_{n-1}=q^2b_{n-3}-2pqa_{n-2}+qa_{n-1}+\sum\limits_{k=3}^{n-3}a_kb_{n-k-1},\\ a_n=pqb_{n-2},\\ b_n=q^2b_{n-2}+\sum\limits_{k=3}^{n-3}a_kb_{n-k}-p(pqa_{n-2}-qa_{n-1}+b_{n-1}). \end{cases}$$

It is easy to verify that $a_j>0$ for $1\leqslant j\leqslant n-4$, $b_j>0$ for $1\leqslant j\leqslant n-3$, and

$$b_{n-2}>q^2b_{n-4}>p^2b_{n-4}.$$

Next, we consider the remaining seven equations.

$$a_{n-3}=a_{n-2}+q(b_{n-4}+pb_{n-5}), \tag{2.5 a}$$

$$b_{n-2}=q^2b_{n-4}-qa_{n-2}+\sum_{k=3}^{n-3}a_kb_{n-k-2}, \tag{2.5 b}$$

$$pa_{n-2}=a_{n-1}-q(b_{n-3}+pb_{n-4}), \tag{2.5 c}$$

$$(2.5\text{ d})\qquad pa_{n-1}=q(b_{n-2}+pb_{n-3}),$$

$$(2.5\text{ e})\qquad b_{n-1}=q^2b_{n-3}-2pqa_{n-2}+qa_{n-1}+\sum_{k=3}^{n-3}a_kb_{n-k-1},$$

$$(2.5\text{ f})\qquad a_n=pqb_{n-2},$$

$$(2.5\text{ g})\qquad b_n=q^2b_{n-2}-p(pqa_{n-2}-qa_{n-1}+b_{n-1})+\sum_{k=3}^{n-3}a_kb_{n-k}.$$

Firstly, substituting (2.5 a) into (2.5 b), we can obtain that

$$b_{n-2}=2q^2b_{n-4}+3pq^2b_{n-5}+\sum_{k=4}^{n-4}a_kb_{n-2-k},\quad\text{and}\quad b_{n-2}>2q^2b_{n-4}>p^2b_{n-4}.$$

Thus by (2.5 d), we can obtain that there is a positive solution a_{n-1}. By (2.5 f), we can obtain that there is a positive solution a_n.

Secondly, we can eliminate by (2.5 c) and (2.5 d). Then

$$p^2a_{n-2}=q(b_{n-2}-p^2b_{n-4}).$$

So $a_{n-2}>0$, and it is obviously that $a_{n-3}>0$ by (2.5 a).

Next, substituting (2.5 e) and (2.5 a) into (2.5 g) and sorting out, we can obtain that

$$\begin{aligned}b_n&=q^2b_{n-2}-p^2q^2(b_{n-4}+pb_{n-5})+pq^2b_{n-3}+\sum_{k=4}^{n-3}(a_k-pa_{k-1})b_{n-k}\\&=q^2(b_{n-2}-p^2b_{n-4})+pq^2(b_{n-3}-p^2b_{n-5})+q\sum_{k=4}^{n-3}(b_{k-1}-p^2b_{k-3})b_{n-k}>0.\end{aligned}$$

Finally, we shall prove that there is a positive solution b_{n-1}.

Substituting (2.5 a) and (2.5 c) into (2.5 e) and sorting out, we can obtain that

$$b_{n-1}=2q^2b_{n-3}+2pq^2b_{n-4}+p^2q^2b_{n-5}+\sum_{k=3}^{n-4}a_kb_{n-k-1}>0.$$

In summary, we obtain that there are some $\beta\in(3\pi/2,2\pi)$, such that B_n is nilpotent, while r, r_1 and $a_1,a_2,\dots,a_{n-1},a_n,b_1,b_2,\dots,b_{n-1},b_n$ are positive. □

Let B_n be nilpotent evaluated at

$$P=(t_1,t_2,\dots,t_{2n})=(r,a_1,b_1,\dots,a_k,b_k,\dots,a_{n-1},b_{n-1},r_1).$$

We can prove the following result.

Lemma 2.2.

$$|J|_P=\det\frac{\partial(f_1,g_1,f_2,g_2,\dots,f_k,g_k,\dots,f_{n-2},g_{n-2},f_{n-1},g_{n-1},f_n,g_n)}{\partial(r,a_1,b_1,\dots,a_{n-1},b_{n-1},r_1)}\Big|_P\neq 0.$$

P r o o f. The determinant $|J|_P$ has the following form

$$= \begin{vmatrix} A & B & \begin{matrix} O \\ C \end{matrix} \end{vmatrix}$$

where

$$A = \begin{bmatrix} -p & 1 \\ -q & 0 \\ a_2 & -p \\ -2b_2 & 0 \\ 2qb_2 & q^2 \\ -2pb_2 & 0 \\ a_4 & qb_2 \\ -p(b_3+pb_2) & b_3-pb_2 \\ \vdots & \vdots \\ a_k & qb_{k-2} \\ -p(b_{k-1}+pb_{k-2}) & b_{k-1}-pb_{k-2} \\ \vdots & \vdots \\ a_{n-4} & qb_{n-6} \\ -p(b_{n-5}+pb_{n-6}) & b_{n-5}-pb_{n-6} \\ a_{n-3}-a_{n-2} & qb_{n-5} \\ -p(b_{n-4}+pb_{n-5}) & b_{n-4}-pb_{n-5} \\ a_{n-1}-pa_{n-2} & qb_{n-4}+a_{n-2} \\ -p(b_{n-3}+pb_{n-4}) & b_{n-3}-pb_{n-4} \\ pa_{n-1} & qb_{n-3}-a_{n-1} \\ -p(b_{n-2}+pb_{n-3}) & b_{n-2}-pb_{n-3}+qa_{n-2} \\ pqb_{n-2} & qb_{n-2} \\ -p^2b_{n-2} & a_{n-2}b_2-pb_{n-2}-qa_{n-1}+b_{n-1} \end{bmatrix},$$

has two $2n$-columns, O is $(2n-8)\times 5$ zero matrix,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ p & 0 & -1 & 0 & 0 \\ q & 1 & 0 & 0 & 0 \\ 0 & q & -p & 0 & 0 \\ 2b_2 & 0 & -q & 1 & 0 \\ 0 & pq & 0 & 0 & \cos\beta \\ pb_2 & -q^2 & -pq & p & -\sin\beta \end{bmatrix},$$

and B has only zeros in the upper triangle, $B=$

$$\begin{bmatrix}
0 & 0 & \cdots & & & & & & & & \cdots & 0 \\
1 & 0 & & & & & & & & & & \vdots \\
q & -1 & \ddots & & & & & & & & & \\
0 & 0 & 1 & & & & & & & & & \\
pq & 0 & q & -1 & & & & & & & & \\
-q^2 & -q & 0 & 0 & \ddots & & & & & & & \\
0 & 0 & pq & 0 & \ddots & & & & & & & \\
-a_3 & -b_2 & -q^2 & -q & \cdots & -1 & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & 0 & 1 & & & & & \\
0 & 0 & \cdots & 0 & \cdots & 0 & q & \ddots & & & & \\
-a_{k-1} & -b_{k-2} & \cdots & -a_3 & \cdots & -q & 0 & \ddots & & & & \\
\vdots & \vdots & & \vdots & & \vdots & \vdots & & & & & \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & -1 & & & \\
-a_{n-5} & -b_{n-6} & -a_{n-6} & -b_{n-7} & \cdots & -b_{n-k-4} & -a_{n-k-4} & \cdots & 0 & 1 & 0 & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & q & -1 & 0 \\
-a_{n-4} & -b_{n-5} & -a_{n-5} & -b_{n-6} & \cdots & -b_{n-k-3} & -a_{n-k-3} & \cdots & -q & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & pq & 0 & q \\
a_{n-2}-a_{n-3} & -b_{n-4} & -a_{n-4} & -b_{n-5} & \cdots & -b_{n-k-2} & -a_{n-k-2} & \cdots & -b_2 & -q^2 & -q & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & pq \\
pa_{n-2}-a_{n-1} & -b_{n-3} & a_{n-2}-a_{n-3} & -b_{n-4} & \cdots & -b_{n-k-1} & -a_{n-k-1} & \cdots & -b_3 & -a_3 & -b_2 & -q^2 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-pa_{n-1} & -b_{n-2} & pa_{n-2} & -b_{n-3} & \cdots & -b_{n-k} & -a_{n-k} & \cdots & -b_4 & -a_4 & -b_3 & -a_3
\end{bmatrix}.$$

Let

$$\begin{aligned}
l_1 &= p, \\
l_2 &= q, \\
l_3 &= a_2 - pl_1 + ql_2 = 2a_2, \\
l_4 &= -2b_2 + 0l_1 + 0l_2 + 0l_3 = -2b_2, \\
l_5 &= 2qb_2 + q^2 l_1 + pql_2 + 0l_3 + q(-l_4) = 3a_3, \\
l_6 &= -2pb_2 + 0l_1 - q^2 l_2 - ql_3 + 0(-l_4) + 0l_5 = -3b_3, \\
&\ \vdots
\end{aligned}$$

$$
\begin{aligned}
l_{2k-1} &= a_k + qb_{k-2}l_1 + pq(-l_{2k-4}) + b(-l_{2k-2}) = ka_k,\\
l_{2k} &= -p(b_{k-1} + pb_{k-2}) + (b_{k-1} - pb_{k-2})l_1 - a_{k-1}l_2 + q^2 l_{2k-4}\\
&\quad - \sum_{j=2}^{k-1} b_{k-j}l_{2j-1} + \sum_{j=2}^{k-3} a_{k-j}l_{2j} = -kb_k,\\
&\vdots\\
l_{2k-7} &= a_{n-3} - a_{n-2} + qb_{n-5}l_1 + pq(-l_{2n-10}) + b(-l_{2n-8}) = (n-3)(a_{n-3} - a_{n-2}),\\
l_{2n-6} &= -p(b_{n-4} + pb_{n-5}) + (b_{n-4} - pb_{n-5})l_1 - a_{n-4}l_2 + q^2 l_{2n-10}\\
&\quad - \sum_{j=2}^{n-4} b_{n-3-j}l_{2j-1} + \sum_{j=2}^{n-6} a_{n-3-j}l_{2j} = -(n-3)b_{n-3},\\
l_{2n-5} &= a_{n-1} - pa_{n-2} + p(qb_{n-4} + a_{n-2}) + pq(-l_{2n-8}) + q(-l_{2n-6})\\
&= (n-2)a_{n-1} - (n-3)pa_{n-2},\\
l_{2n-4} &= -p(b_{n-3} + pb_{n-4}) + p(b_{n-3} - pb_{n-4}) + q(a_{n-2} - a_{n-3}) + q^2 l_{2n-8}\\
&\quad - \sum_{j=2}^{n-3} b_{n-2-j}l_{2j-1} + \sum_{j=2}^{n-5} a_{n-2-j}l_{2j} = -(n-2)b_{n-2},\\
l_{2n-2} &= -p(b_{n-2} + pb_{n-3}) + p(b_{n-2} - pb_{n-3} + qa_{n-2}) + q(pa_{n-2} - a_{n-1}) + q^2 l_{2n-6}\\
&\quad + (a_{n-3} - a_{n-2})l_4 - \sum_{j=2}^{n-2} b_{n-1-j}l_{2j-1} + \sum_{j=3}^{n-4} a_{n-1-j}l_{2j} = -(n-1)b_{n-1}.
\end{aligned}
$$

First, add l_1 times the 2nd column and l_2 times the 3rd column to the first column. Secondly, add l_{2j-1} times the $(2j)$th column and $-l_{2j}$ times the $(2j+1)$th column to the first column, for $j = 2, \ldots, n-3$. Next, add l_{2n-5} times the $(2n-2)$th column, $(-l_{2n-4})$ times the $(2n-3)$th column and $-l_{2n-2}$ times the $(2n-1)$th column to the first column. Finally, expand the determinant along the first row from the top downwards. Then we have that

$$
|J|_P = (-1)^{n-5}\begin{vmatrix}
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & q & p & 0 & -1 & 0 & 0\\
0 & -q & 0 & q & 1 & 0 & 0 & 0\\
t & 0 & pq & 0 & q & -p & 0 & 0\\
0 & -pq & -q^2 & 2pq & 0 & -q & 1 & 0\\
s & 0 & 0 & 0 & pq & 0 & 0 & \cos\beta\\
l & -b_3 & -a_3 & p^2q & -q^2 & -pq & p & -\sin\beta
\end{vmatrix},
$$

where

$$
t = pa_{n-1} + p(qb_{n-3} - a_{n-1}) + pq(-l_{2n-6}) + q(-l_{2n-4}) - pl_{2n-5} = (n-3)p^2 a_{n-2},
$$

$$
\begin{aligned}
s &= pqb_{n-2} + p(qb_{n-2}) + pq(-l_{2n-4}) = npqb_{n-2},\\
l &= -p^2b_{n-2} + p(a_{n-2}b_2 - a_{n-1}q + b_{n-1} - pb_{n-2}) - pqa_{n-1} - pa_{n-2}l_4\\
&\quad - \sum_{k=2}^{n-3} b_{n-k}l_{2k-1} + \sum_{k=3}^{n-3} a_{n-k}l_{2k} + q^2l_{2n-4} - pql_{2n-5} - pl_{2n-2} = -nb_n.
\end{aligned}
$$

So

$$
\begin{aligned}
|J|_P &= (-1)^{n-3}(b_3t\cos\beta - sp^2\sin\beta - lp^2\cos\beta)\\
&= (-1)^{n-3}[(n-3)p^2q^3a_{n-2}\cos\beta + 2np^2b_n\cos\beta] \neq 0.
\end{aligned}
$$

□

By virtue of the extended nilpotent Jacobi method and Lemmas 2.1 and 2.2, the following theorem is immediate.

Theorem 2.1. *For any $\theta \in (\pi/4, \pi/2)$ there exist infinitely many choices for $\beta \in (3\pi/2, 2\pi)$ such that the ray patterns $\mathcal{A}_n$ and all of their superpatterns are spectrally arbitrary.*

Lemma 2.3 ([3]). *An $n\times n$ irreducible spectrally arbitrary ray pattern must have at least $3n-1$ nonzero entries.*

Note that there are $3n-1$ nonzeros in the ray pattern $\mathcal{A}_n$. So we have the following.

Theorem 2.2. *If $\mathcal{A}_n$ is spectrally arbitrary, then $\mathcal{A}_n$ is a minimal spectrally arbitrary ray pattern.*

Acknowledgment. The authors are very grateful to the referee for many valuable suggestions.

References

[1] *J. H. Drew, C. R. Johnson, D. D. Olesky, P. van den Driessche*: Spectrally arbitrary patterns. Linear Algebra Appl. *308* (2000), 121–137.

[2] *Y. Gao, Y. Shao*: New classes of spectrally arbitrary ray patterns. Linear Algebra Appl. *434* (2011), 2140–2148.

[3] *J. J. McDonald, J. Stuart*: Spectrally arbitrary ray patterns. Linear Algebra Appl. *429* (2008), 727–734.

[4] *Y. Mei, Y. Gao, Y. Shao, P. Wang*: The minimum number of nonzeros in a spectrally arbitrary ray pattern. Linear Algebra Appl. *453* (2014), 99–109.

Authors' address: Yinzhen Mei, Yubin Gao (corresponding author), Yanling Shao, Peng Wang, Department of Mathematics, North University of China, 3 Xueyuan Road, Taiyuan, Shanxi, 030051, P. R. China, e-mail: `myzmath@163.com`, `ybgao@nuc.edu.cn`, `ylshao@nuc.edu.cn`, `wpmath@163.com`.