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DUALITY FOR HILBERT ALGEBRAS WITH SUPREMUM: AN APPLICATION

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Abstract. We modify slightly the definition of H-partial functions given by Celani and Montangie (2012); these partial functions are the morphisms in the category of $H^{\vee}$-space and this category is the dual category of the category with objects the Hilbert algebras with supremum and morphisms, the algebraic homomorphisms. As an application we show that finite pure Hilbert algebras with supremum are determined by the monoid of their endomorphisms.

Keywords: Hilbert algebra; duality; monoid of endomorphisms; BCK-algebra

MSC 2010: 06A12, 03G25

1. INTRODUCTION

A Hilbert algebra (also called a positive implication algebra) is a structure $\mathbf{H} = \langle H; \to, 1\rangle$ of type (2,0) that satisfies for all $a, b, c \in H$ the following:

$$a \to (b \to a) = 1; \tag{1}$$

$$(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1; \tag{2}$$

$$a \to b = 1 \text{ and } b \to a = 1 \text{ imply } a = b. \tag{3}$$

Here is an important example of Hilbert algebras (see [2]): For a poset $\langle X, \leqslant\rangle$, the set of its increasing subsets is denoted by $\mathcal{P}_i(X)$. Then $\mathcal{P}_i(\mathbf{X}) := \langle \mathcal{P}_i(X); \Rightarrow, X\rangle$ with the operation $\Rightarrow$ defined by the prescription

$$U \Rightarrow V := (U \cap V^{\complement}]^{\complement} = \{x\colon\ [x) \cap U \subseteq V\} \tag{4}$$

is a Hilbert algebra. Hilbert algebras represent the algebraic counterparts of the implicative fragment of Intuitionistic Propositional Logics. Diego in [6] proves that

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the class of Hilbert algebras forms a variety. The binary relation $\leqslant$ defined on H by $a \leqslant b$ if and only if $a \to b = 1$ is a partial order on H with last element 1. If the underlying set H of the Hilbert algebra $\mathbf{H}$ is a join-semilattice with respect to the order $\leqslant$, then it is possible to form a new algebra by adding the binary operation $\vee$ to the usual operations of $\mathbf{H}$. In this way, we arrive at the notion of Hilbert algebra with supremum. More precisely, a *Hilbert algebra with supremum* or $H^{\vee}$-algebra is an algebra $\langle H; \to, \vee, 1\rangle$ of type $(2,2,0)$ if $\langle H; \to, 1\rangle$ is a Hilbert algebra, $\langle H; \vee, 1\rangle$ is a join semilattice with last element 1 and $a \to b = 1$ if and only if $a \vee b = b$. Every Tarski algebra, for example, can be turned into a $H^{\vee}$-algebra since a Tarski algebra is a Hilbert algebra $\langle H; \to, 1\rangle$ such that $(a \to b) \to b = (b \to a) \to a$ for all $a, b \in H$ and it is known that H with the operation $\vee$ defined by $a \vee b = (a \to b) \to b$ is a join semilattice.

In [5] a duality for $H^{\vee}$-algebra is developed. It is the purpose of this paper to introduce and justify a slight modification to the condition (ii) of the definition of an H-partial function given therein. We think this modification makes the concept more precise. As an application of the mentioned duality we show that certain important family of $H^{\vee}$-algebras are determined by the monoid of their endomorphisms.

2. Preliminaries

It is known (see [10]) that Hilbert algebras and positive implicative BCK-algebras are dual isomorphic. For the basic facts about BCK-algebras in general and positive implicative BCK-algebras in particular we refer the reader to [9]. A BCK-algebra is an algebra $\mathbf{A} := \langle A; \to, 1\rangle$ of type $(2,0)$ that satisfies the following axioms:

$$(a \to b) \to ((a \to c) \to (b \to c)) = 1; \tag{5}$$

$$a \to ((a \to b) \to b) = 1; \tag{6}$$

$$a \to a = 1; \tag{7}$$

$$a \to 1 = 1; \tag{8}$$

$$a \to b = 1 \text{ and } b \to a = 1 \text{ imply } a = b. \tag{9}$$

This presentation of BCK-algebras given in [8] is dual to the original presentation given in [9] and we adopt such a presentation since it serves our purpose better. For Hilbert algebras as well as for BCK-algebras, the relation $\leqslant$ given by the prescription $a \leqslant b$ if and only if $a \to b = 1$, defines a partial order on the underlying set of the algebra. Let $\mathbf{A}$ be a BCK-algebra or a Hilbert algebra such that the underlying set A of the algebra is a join-semilattice with respect to the partial order defined above. Consider then the new algebra $\mathbf{A}^{\vee} := \langle A; \to, \vee, 1\rangle$ of type $(2,2,0)$ such that $\langle A; \vee, 1\rangle$

is a join semilattice with last element 1. Observe that by the condition imposed on the underlying set we have that $a \to b = 1$ if and only if $a \vee b = b$. If $\mathbf{A}$ is a BCK-algebra, the algebra $\mathbf{A}^{\vee}$ is called an upper BCK-semilattice. If $\mathbf{A}$ is a Hillbert algebra, $\mathbf{A}^{\vee}$ is called a Hilbert algebra with supremum.

Proposition 1 ([8]). *The class of upper* BCK*-semilattices is a variety. The equations*

$$a \to (a \vee b) = 1; \tag{10}$$

$$a \vee ((a \to b) \to b) = (a \to b) \to b; \tag{11}$$

$$a \vee a = a; \tag{12}$$

$$a \vee b = b \vee a; \tag{13}$$

$$(a \vee b) \vee c = a \vee (b \vee c); \tag{14}$$

$$1 \to a = a, \tag{15}$$

together with (5) *and* (8) *constitute an equational basis of this variety.*

We recall here that a positive implicative BCK-algebra is a BCK-algebra satisfying the additional identity

$$(a \to b) \to (a \to c) = a \to (b \to c). \tag{16}$$

Having in mind Proposition 1 and the fact that Hilbert algebras are dual isomorphic to positive implicative BCK-algebras, it can be shown that the class of Hilbert algebras with supremum forms a variety defined by the identities that define Hilbert algebras, the identities that define join semi-lattices, (10) above and

$$(a \to b) \to ((a \vee b) \to b) = 1. \tag{17}$$

Proposition 2 ([5]). $\mathbf{A} = \langle A; \to \vee, 1\rangle$ *of type* $(2, 2, 0)$ *is a Hilbert algebra with supremum* ($H^{\vee}$*-algebra*) *if and only if* $\langle A; \to, 1\rangle$ *is a Hilbert algebra* (H*-algebra*), $\langle A; \vee, 1\rangle$ *is a join semilattice with last element* 1 *and* A *satisfies the identities* (10) *and* (17).

3. Duality for $H^{\vee}$-algebras

In this section we collect the basic facts about the simplified topological representation of Hilbert algebras with supremum given in [5] and [3] and we introduce a slight modification to the definition of the H-partial function that we think makes it more precise.

First we recall some topological concepts. Let $\mathbf{X} = \langle X, \tau \rangle$ be a topological space. For a set $Y \subseteq X$, $\mathrm{cl}(Y)$ will denote the closure of Y. The *specialization order* on X is defined by $x \preceq y$ if and only if $x \in \mathrm{cl}(y) = \mathrm{cl}(\{y\})$. It is easy to see that the relation $\preceq$ is reflexive and transitive and it is a partial order if $\mathbf{X}$ is T_0. The dual relation of $\preceq$ will be denoted by $\leqslant$ and defined by $x \leqslant y$ if and only if $y \in \mathrm{cl}(x)$. Notice that $\mathrm{cl}(x) = \{y \in X \colon x \leqslant y\} = [x)$ and that an open subset of X is decreasing whereas a closed one is increasing with respect to $\leqslant$, the dual relation of the specialization order $\preceq$. An arbitrary set $Y \subseteq X$ is said to be *irreducible* if $Y \subseteq Z \cup W$ for closed subsets Z and W of $\mathbf{X}$ implies $Y \subseteq Z$ or $Y \subseteq W$. The space $\mathbf{X}$ is said to be *sober* if for every closed irreducible subset Y of $\mathbf{X}$ there exists a unique $x \in X$ such that $Y = \mathrm{cl}(x)$. A sober space is obviously T_0. A *saturated set* is an intersection of open sets which is an equivalent to saying that it is decreasing. The smallest saturated set containing a given subset Y of X will be denoted by $\mathrm{sat}(Y)$. Observe that $\mathrm{sat}(Y) = (Y]$.

A *Hilbert space* or *H-space* for short is a sober topological space $\mathbf{X} := \langle X, \tau_{\mathcal{K}} \rangle$ such that

(i) $\mathcal{K}$ is a base of compact-open subsets of X for a topology $\tau_{\mathcal{K}}$ on X,
(ii) for every $A, B \in \mathcal{K}$, $(A \cap B^{\mathsf{C}}] \in \mathcal{K}$.

If additionally $\mathbf{X}$ satisfies

(iii) $U \cap V \in \mathcal{K}$ for all $U, V \in \mathcal{K}$,

$\mathbf{X}$ is called an $H^{\vee}$*-space*, i.e. an $H^{\vee}$-space is an H-space for which (iii) holds.

A nonempty subset D of an H-algebra $\mathbf{A}$ is called a *deductive system* if

(i) $1 \in D$, and
(ii) a, $a \to b \in D$ imply $b \in D$.

We denote the set of deductive systems of an H-algebra $\mathbf{A}$ by $\mathcal{D}_s(\mathbf{A})$. A deductive system D is said to be *irreducible* or *prime* if from $D = D_1 \cap D_2$ with $D_1, D_2 \in \mathcal{D}_s(\mathbf{A})$ it always follows that $D_1 = D$ or $D_2 = D$. In [3], Theorem 5, it is shown that D (deductive system) is irreducible if and only if for $a, b \notin D$ there exists $c \notin D$ such that $a \leqslant c$, $b \leqslant c$. The set of all irreducible deductive systems of $\mathbf{A}$ is denoted by $X(A)$. It is easy to prove that $D \in \mathcal{D}_s(\mathbf{A})$ is irreducible if and only if for all $a, b \in A$ such that $a \vee b \in D$, $a \in D$ or $b \in D$. It can be shown that

$$\mathcal{P}_i(X(\mathbf{A})) := \langle \mathcal{P}_i(X(A)), \Rightarrow, \cup, X \rangle$$

is an $H^{\vee}$-algebra and if $\mathbf{A}$ is an $H^{\vee}$-algebra, then the mapping $\varphi\colon A \to \mathcal{P}_i(X(A))$ given by

$$\varphi(a) = \{P \in X(A)\colon a \in P\} \tag{18}$$

is an injective homomorphism of $H^{\vee}$-algebras ([5], Lemma 5.1). Moreover,

$$\mathcal{K}_A := \{\varphi(a)\colon a \in A\} \tag{19}$$

is a basis for a topology $\tau_{\mathcal{K}_A}$ on $X(A)$ and $\mathbf{X}(A) := \langle X(A), \tau_{\mathcal{K}_A}\rangle$ is an $H^{\vee}$-space ([5], Theorem 5.6). Observe that the dual of the specialization order given by this topology on $X(A)$ is the set-theoretical inclusion.

If $\mathbf{X} := \langle X, \tau_{\mathcal{K}}\rangle$ is an $H^{\vee}$-space, then $D(\mathbf{X}) := \langle D(X); \Rightarrow, \cup, X\rangle$, where

$$D(X) := \{U^{\complement}\colon U \in \mathcal{K}\}$$

and the operation $\Rightarrow$ given by the formula (4) is an $H^{\vee}$-algebra, see [5], Proposition 5.3. The image of the mapping φ given by formula (18) is $D(X(A))$, so

$$\varphi\colon \mathbf{A} \cong D(X(\mathbf{A})).$$

Let $\mathbf{X}$ be an $H^{\vee}$-space. Then the mapping $\varepsilon_X\colon \mathbf{X} \to X(D(\mathbf{X}))$ given by

$$\varepsilon_X(x) := \{U \in D(X)\colon x \in U\} \tag{20}$$

is a homeomorphism between topological spaces ([5], Proposition 5.7).

Let $\mathbf{X}_1 := \langle X, \tau_{\mathcal{K}_1}\rangle$ and $\mathbf{X}_2 := \langle X, \tau_{\mathcal{K}_2}\rangle$ be two H-spaces. A relation $R \subseteq X_1 \times X_2$ is said to be an *H-relation* provided that

(i) $R^{-1}(U) \in \mathcal{K}_1$ for every $U \in \mathcal{K}_2$,

(ii) $R(x)$ is a closed subset of $\mathbf{X}_2$ for all $x \in X_1$.

If additionally

(iii) $(x, y) \in R$ implies $[y) = R(z)$ for some $z \geqslant x$,

R is said to be an *H-functional relation* (remember that here $\leqslant$ is the dual of the specialization order given by the topology). The relation R is said to be *irreducible* if for $x \in X_1$ with $R(x) \neq \emptyset$, $R(x)$ is a closed irreducible subset of $\mathbf{X}_2$.

It is proved (see [4] and [5]) that $H^{\vee}$-spaces as objects and irreducible H-functional relations as arrows form a category denoted by $\mathcal{SF}^{\vee}$.

If $\mathbf{X}_1$ and $\mathbf{X}_2$ are $H^\vee$-spaces and $R \subseteq X_1 \times X_2$ is an irreducible H-functional relation, then the mapping $h_R\colon D(X_2) \to D(X_1)$ given by

$$h_R(U) = \{x \in X_1\colon R(x) \subseteq U\} \tag{21}$$

is a homomorphism of $H^\vee$-algebras (Theorem 5.12 and Corollary 5.13 of [5]).

Let $\mathbf{A}$ and $\mathbf{B}$ be two $H^\vee$-algebras and $h\colon \mathbf{A} \to \mathbf{B}$ a homomorphism. Consider the relation $R_h \subseteq X(B) \times X(A)$ given by the prescription

$$(P, Q) \in R_h \text{ if and only if } h^{-1}(P) \subseteq Q. \tag{22}$$

Then R_h is an irreducible H-functional relation.

Proposition 3. *R_h is irreducible if and only if for all $P \in X(B)$, $h^{-1}(P) \in X(A)$ or $h^{-1}(P) = A$.*

P r o o f. See Theorem 5.10 of [5]. □

Let $\mathcal{HH}^\vee$ be the category of $H^\vee$-algebras with morphisms the algebraic homomorphisms. It follows from the above results that the categories $\mathcal{HH}^\vee$ and $\mathcal{SF}^\vee$ are dually equivalent (see [5], page 248).

When $X(\mathbf{B})$ is a sober space, it happens that, since the relation $R_h \subseteq X(B) \times X(A)$ (corresponding to the map h, see (21)), is irreducible for each $P \in X(B)$ with $R_h(P) \neq \emptyset$, there is a unique $Q \in X(B)$ such that $R_h(P) = [Q)$. Then a partial function f_h may be defined from the $H^\vee$-space $X(\mathbf{B})$ to the $H^\vee$-space $X(\mathbf{A})$, having the domain $\{P \in X(B)\colon R_h(P) \neq \emptyset\}$ and the prescription $P \mapsto Q$. By the definition of R_h, see (22), $R_h(P) = \{Q \in X(A)\colon h^{-1}(P) \subseteq Q\}$. By Proposition 3, $h^{-1}(P) \in X(A)$ or $h^{-1}(P) = A$. So, if P is in the domain of the mentioned partial function, then the mapping should be $P \mapsto h^{-1}(P)$. Next, observe that if $P \notin \mathrm{dom}(f_h)$, i.e. if $h^{-1}(P) = A$, then $[P) \cap \mathrm{dom}(f_h) = \emptyset$, whereas if $P \in \mathrm{dom}(f_h)$, then $(P] \subseteq \mathrm{dom}(f_h)$. We think this discussion justifies the following modification of Definition 6.1 in [5].

Let $\mathbf{X}_1 := \langle X, \tau_{\mathcal{K}_1}\rangle$ and $\mathbf{X}_2 := \langle X, \tau_{\mathcal{K}_2}\rangle$ be two $H^\vee$-spaces. Let $f\colon X_1 \to X_2$ be a partial function with the domain denoted by $\mathrm{dom}(f)$. Then f is said to be an *H-partial function* if the following conditions are satisfied:

(i) $[f(x)) = f([x))$ for each $x \in \mathrm{dom}(f)$;
(ii) $[x) \cap \mathrm{dom}(f) = \emptyset$ for each $x \notin \mathrm{dom}(f)$ and $(x] \subseteq \mathrm{dom}(f)$ if $x \in \mathrm{dom}(f)$;
(iii) $(f^{-1}(U)] \in \mathcal{K}_1$ for each $U \in \mathcal{K}_2$.

A bijective correspondence between irreducible $H^\vee$-functional relations and H-partial functions can now be established as follows: for a given $H^\vee$-functional relation R consider the map $f_R\colon X_1 \to X_2$ with the domain $\{t \in X_1\colon R(t) \neq \emptyset\}$

and defined by $f_R(x) = y$, where $y \in X_2$ is the unique element of X_2 such that $R(x) = [y)$. Since the spaces we are considering are sober spaces, f is well defined and it can be proved that f is an H-partial function. Conversely, if $f\colon X_1 \to X_2$ is an H-partial function, then $R_f := \{(x, y)\colon x \in \mathrm{dom}(f) \text{ and } f(x) \leqslant y\} \subseteq X_1 \times X_2$ is an irreducible H-functional relation.

The above comments allow us to consider the category $\mathcal{SF}^{\vee}$ with morphisms, the H-partial functions instead of $H^{\vee}$-functional relations and the equivalence between the categories $\mathcal{HH}^{\vee}$ and $\mathcal{SF}^{\vee}$ is described now as follows: Let $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ be a homomorphism of $H^{\vee}$-algebras. Then $h_X\colon \mathbf{X}(A_2) \to \mathbf{X}(A_1)$ given by the formula

$$h_X(P) = h^{-1}(P)$$

is an H-partial function with the domain $\{P \in X(A_2)\colon h^{-1}(P) \in X(A_1)\}$.

Let $f\colon \mathbf{X}_1 \to \mathbf{X}_2$ be an H-partial function. Then $f_D\colon D(X_2) \to D(X_1)$ given by the formula

$$f_D(U) = (f^{-1}(U^{\complement})]^{\complement}$$

is a homomrphism of $H^{\vee}$-algebras.

More precisely, the correspondence X from the category $\mathcal{HH}^{\vee}$ of $H^{\vee}$-algebras with morphisms the algebraic homomorphisms to the category $\mathcal{SF}^{\vee}$ of $H^{\vee}$-spaces now with morphisms the H-partial functions given by the diagram

$$\begin{tikzcd}
\mathbf{A}_1 \arrow[r, "X"] \arrow[d, "h"'] & X(\mathbf{A}_1) \\
\mathbf{A}_2 \arrow[r, "X"'] & X(\mathbf{A}_2) \arrow[u, "h_X"']
\end{tikzcd}$$

defines a contra-variant functor from the category $\mathcal{HH}^{\vee}$ to the category $\mathcal{SF}^{\vee}$. Likewise, the correspondence D from the category $\mathcal{SF}^{\vee}$ to the category $\mathcal{HH}^{\vee}$ given by the diagram

$$\begin{tikzcd}
\mathbf{X_1} \arrow[r, "D"] \arrow[d, "f"'] & D(\mathbf{X_1}) \\
\mathbf{X_2} \arrow[r, "D"'] & D(\mathbf{X_2}) \arrow[u, "f_D"']
\end{tikzcd}$$

defines a contra-variant functor from the category $\mathcal{SF}^{\vee}$ to the category $\mathcal{HH}^{\vee}$. Moreover, XD is the identity (up to homeomorphisms) in the category $\mathcal{SF}^{\vee}$ and DX is the identity in the category $\mathcal{HH}^{\vee}$.

We summarize the above in the following theorem:

Theorem 4. *There exists a dual equivalence between the category of $H^{\vee}$-algebras with homomorphisms and the category of $H^{\vee}$-spaces with H-partial functions.*

An *H-partial endomorphism* of an $H^{\vee}$-space $\mathbf{X}$ is an H-partial function from $\mathbf{X}$ to itself. Denote the set of H-partial endomorphisms of an $H^{\vee}$-space $\mathbf{X}$ by $\mathrm{pEnd}(\mathbf{X})$. This set with a composition of H-partial functions is a monoid. Likewise, the set $\mathrm{End}(\mathbf{A})$ of endomorphisms of an $H^{\vee}$-algebra with a composition of functions is a monoid and, as an obvious consequence of the equivalence in Theorem 4, we have the following:

Corollary 5. *Let* $\mathbf{A}$ *be an* $H^{\vee}$*-algebra and* $\mathbf{X}$ *an* $H^{\vee}$*-space. Then* $\mathrm{End}(\mathbf{A})$ *is anti-isomorphic to* $\mathrm{pEnd}(X(\mathbf{A}))$*. Likewise,* $\mathrm{pEnd}(\mathbf{X})$ *is anti-isomorphic to* $\mathrm{End}(D(\mathbf{X}))$*.*

4. Dual space of a pure $H^{\vee}$-algebra

Let $\mathbf{A}$ be a Hilbert algebra. An element $p \in A \setminus \{1\}$ is called *irreducible* if for all $a \in A$, $a \to p = 1$ or $a \to p = p$. It follows that $\{p \in A\colon\ p \text{ is irreducible }\} \cup \{1\}$ is a sub-universe of $\mathbf{A}$; A Hilbert algebra such that all of its elements are irreducible is said to be given by the order. This kind of Hilbert algebras is named in [1] *pure Hilbert algebras.* In [3], Lemma 13 it is proved that $p \in A$ is irreducible if and only if $(p]^{\complement} \in X(\mathbf{A})$. In fact, it is proved in [7] that if A is finite, then $D \in X(\mathbf{A}))$ if and only if $D = (p]^{\complement}$ for some irreducible element $p \in A$. Another important fact we will use is that if $\mathbf{X} := \langle X, \tau_{\mathcal{K}}\rangle$ is a finite H-space, then $(x] \in \mathcal{K}_X$ for all $x \in X$, see [4], Lemma 4.1.

Let $\langle X \leqslant\rangle$ be a finite poset such that the poset $\langle X \oplus \{1\}, \leqslant_d\rangle$, being $\leqslant_d$ the order dual of $\leqslant$, is a $\vee$-semilattice. Observe that in the poset $\langle X \leqslant\rangle$, for $x, y \in X$ either $x \wedge y = z \in X$ or $(x] \cap (y] = \emptyset$. Consider the Hilbert algebra with the universe $P := X \oplus \{1\}$ and $\to$ given by the order $\leqslant_d$. Clearly, $\mathbf{P}$ is a pure $H^{\vee}$-algebra with dual $H^{\vee}$-space $X(\mathbf{P}) := \langle X, \mathcal{K}_X\rangle$, where

$$\mathcal{K}_X := \{(x]\colon\ x \in X\}.$$

Notice that all the $H^{\vee}$-Hilbert algebras $\mathbf{A}$ having $\mathbf{P}$ as a subalgebra and being subalgebras of $\mathcal{P}_i(\mathbf{X})$ are such that the carrier of their dual $H^{\vee}$-space is X; in other words, $X(\mathbf{A}) = \langle X, \tau_{\mathcal{K}_A\}}\rangle$, where the base $\mathcal{K}_A$ depends on $\mathbf{A}$. The fact that the order on A given by the basic binary operation of $\mathbf{A}$ has a minimum means that $X \in \mathcal{K}_A$. For $\mathcal{P}_i(\mathbf{X})$ we have that

$$\mathcal{K}_{\mathcal{P}_i(X)} = \{Z \subseteq X\colon\ Z^{\complement} \text{ is increasing}\}.$$

Proposition 6. *The number of join irreducible elements of the* $H^{\vee}$*-Hilbert algebra* $\mathcal{P}_i(\mathbf{X})$ *(*$\mathbf{X}$ *is a finite poset) is the number of its irreducible elements (see the definition above) or, which is the same, it is* $|X|$*.*

Proof. Let $m \in X$. Clearly, $(m]^{\complement} \in \mathcal{P}_i(\mathbf{X})$. Since $U \in \mathcal{P}_i(\mathbf{X})$ is $\vee$-irreducible if and only if $U = [p)$ for some $p \in X$, it will be enough to show that $(m]^{\complement}$ is irreducible; let us prove that: we want to prove that for all $U \in \mathcal{P}_i(\mathbf{X})$,

$$U \Rightarrow (m]^{\complement} = (U \cap (m]]^{\complement} = \begin{cases} (m]^{\complement} & \text{or,} \\ X. \end{cases}$$

This is equivalent to proving that

$$(U \cap (m]] = \begin{cases} (m] & \text{or,} \\ \emptyset. \end{cases}$$

With such a purpose suppose that $(U \cap (m]] \neq \emptyset$, that is $U \cap (m] \neq \emptyset$. Clearly, $U \cap (m] \subseteq (m]$ and since $(m]$ is decreasing, $(U \cap (m]] \subseteq (m]$. Pick $x \in U \cap (m]$, i.e. $x \in A$ and $x \leqslant m$. Since $U \in \mathcal{P}_i(\mathbf{X})$, it is increasing, so $m \in A$. Let us see that $(m] \subseteq (U \cap (m]]$. Let $z \in (m]$. Then $z \leqslant m \in U \cap (m]$; that means, $z \in (U \cap (m]]$ as desired.

□

5. Special $H^{\vee}$-partial endomorphism

Throughout this section, $\langle X, \leqslant \rangle$ (the carrier set X with subindices if necessary) will represent a finite poset such that for $x, y \in X$ either $x \wedge y = z \in X$ or $(x] \cap (y] = \emptyset$; in other words, $\langle \{0\} \cup X, \leqslant_a \rangle$, where for $x, y \in X$, $x \leqslant_a y$ if and only if $x \leqslant y$ and $0 \leqslant_a x$ for all $x \in X$, is a meet-semilattice. So $\mathbf{X} := \langle X, \mathcal{K}_X \rangle$, where $\mathcal{K}_X = \{(x] \colon x \in X\}$ is the dual $H^{\vee}$-space of a pure $H^{\vee}$-algebra. We will call this kind of $H^{\vee}$-spaces *pure $H^{\vee}$-spaces*.

Denote by $\operatorname{Max}(X)$ the set of maximal elements of X and for each $x \in X$ let $M_x := \{m \in \operatorname{Max}(X) \colon x \leqslant m\}$. For each $x \in X$ consider a mapping $f_x \colon X \to X$ with the domain $([x)] = (M_x]$ such that

$$f_x(z) = z \vee x.$$

Indeed, if $z \in (M_x]$, let $m_1, m_2, \ldots, m_k \in M_x$ such that $z \leqslant m_i$, $1 \leqslant i \leqslant k$; then $m_1 \wedge m_2 \wedge \ldots \wedge m_k \geqslant f_x(z) = x \vee z$. It is easy to check that f_x is an H-partial function (or more precisely, an H-partial endomorphism of $\mathbf{X}$). Observe that $f_x = \mathrm{id}$ in $[x) = \operatorname{Im}(f_x)$ and $f_x \circ f_x = f_x$; further $x \leqslant y$ implies $f_x \circ f_y = f_y$. Moreover, if $x \in \operatorname{Max}(X)$, then $f_y \circ f_x = f_x$ (if $y \leqslant x$); otherwise, $f_y \circ f_x = \emptyset$. Further, in the poset $\langle X, \leqslant \rangle$, if $x \vee y = \sup\{x, y\}$ exists (that occurs if $M_x \cap M_y \neq \emptyset$), then $f_x \circ f_y = f_{x \vee y}$.

There is another important family of H-partial endomorphisms. Let us describe this family as follows: for each pair $(m, x) \in \mathrm{Max}(X) \times X$ define $f_{m,x}$ with the domain $(x]$ and just a single value m. It happens that this family is a submonoid of the monoid of all partial endomorphisms of $\mathbf{X}$. It is easy to check that

$$g \circ f_{m,x} = \begin{cases} f_{g(m),x} & \text{if } m \in \mathrm{dom}(g), \\ \emptyset & \text{otherwise,} \end{cases} \qquad f_{m,x} \circ g = \begin{cases} f_{m,y} & \text{if } (y] = g^{-1}((x]) \neq \emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

Observe that f_m coincides with $f_{m,m}$ for $m \in \mathrm{Max}(X)$.

Proposition 7. *For any H-partial idempotent endomorphism f ($f^2 = f \circ f = f$) we have that if $x \in \mathrm{Im}(f)$, then $\mathrm{dom}(f \circ f_x) = \mathrm{dom}(f_x) = ([x)]$ and $\mathrm{Im}(f \circ f_x) = [x)$; that is, $f \circ f_x = f_x$. If $x \notin \mathrm{dom}(f)$, then $f \circ f_x = \emptyset$.*

Proof. Straightforward. Just observe that since f is idempotent, $\mathrm{Im}(f) \subseteq \mathrm{dom}(f)$ and $f(x) = x$. $\square$

Proposition 8. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two finite pure $H^{\vee}$-spaces and $\Gamma : \mathrm{pEnd}(\mathbf{X}_1) \to \mathrm{pEnd}(\mathbf{X}_2)$ be a monoid isomorphism. Then for $m \in \mathrm{Max}(X_1)$, $\Gamma(f_m) = f_{m'}$ for some $m' \in \mathrm{Max}(X_2)$.*

Proof. Set $\sigma = \Gamma(f_m)$. As $f_m \neq \emptyset$, $\sigma \neq \emptyset$. Let $x \in \mathrm{dom}(\sigma)$, say, $\sigma(x) = y$. Since $\sigma([x)) = [y)$, we may choose $m' \in M_y$ and $x' \in [x)$ such that $\sigma(x') = m'$. Indeed, $f_m^2 = f_m$ implies $\sigma^2 = \sigma$ so $\sigma(m') = m'$. By the formulas given before Proposition 7, $f_{m'} \circ \sigma = f_{c',m'}$, where $(c'] = \sigma^{-1}((m'])$. It follows from this that $\Gamma^{-1}(f_{m'}) \circ f_m = \Gamma^{-1}(f_{m',c'})$. Setting $g := \Gamma^{-1}(f_{m'})$ and $g_1 := \Gamma^{-1}(f_{m',c'})$ we have that $g \circ f_m = g_1$. On the other hand, again using the formulas before Proposition 7, we have that $g \circ f_m = f_{g(m),m}$; so $g_1 = \Gamma^{-1}(f_{m',c'}) = f_{g(m),m}$. Now remember that $(c'] = \sigma^{-1}((m'])$. Then $m' \in (c']$ and since $m' \in \mathrm{Max}(X_2)$, $m' = c'$, so $f_{m',c'} = f'_m$. Finally, since $f_{m'}^2 = f_{m'}$, we have that $g(m) = m$ and $\sigma = f_{m'}$. $\square$

Corollary 9. *In the previous proposition, if $m \neq x \in X_1$, then $\Gamma(f_{m,x}) = f_{m',y}$ for some $y \in X_2$, $y \neq m'$.*

Proof. Set $\Gamma(f_{m,x}) := \sigma$. Clearly $f_m \circ f_{m,x} = f_{m,x}$. Then by the previous proposition, $f_{m'} \circ \sigma = \sigma$; so $m' \in \mathrm{Im}(\sigma)$. On the other hand, by the formulas before Proposition 7, $f_{m'} \circ \sigma = f_{m',y}$, where $(y] = \sigma^{-1}((m'])$. The desired result now follows. $\square$

Proposition 10. *With the hypothesis of the previous proposition, if m is an isolated point of $\mathbf{X}_1$ and $\Gamma(f_m) = f_{m'}$, then m' is an isolated point of $\mathbf{X}_2$.*

P r o o f. Observe that for all $\chi \in \mathrm{pEnd}(\mathbf{X}_1)$ we have

$$\begin{cases} \mathrm{dom}(\chi \circ f_m) = \{m\} \text{ and } \mathrm{Im}(\chi \circ f_m) = \{\chi(m)\} & \text{if } m \in \mathrm{dom}(\chi), \\ \emptyset & \text{otherwise.} \end{cases}$$

Likewise,

$$\begin{cases} \mathrm{dom}(f_m \circ \chi) = (s] \text{ and } \mathrm{Im}(\chi \circ f_m) = \{m\} & \text{if } m \in \mathrm{Im}(\chi), \\ \emptyset & \text{otherwise.} \end{cases}$$

(Notice that $(s] = \chi^{-1}((m]))$. If additionally χ is idempotent, then $f_m \circ \chi = f_m \Rightarrow \chi = f_m$ since in this case $\chi^2 = \chi$ implies $\mathrm{Im}(\chi) \subseteq \mathrm{dom}(\chi)$ and if $m \in \mathrm{Im}(\chi)$, $m \in \mathrm{dom}(\chi)$, that is $m \in (s]$.

Let $\Gamma(f_m) = f_{m'}$, where $m' \in \mathrm{Max}(\mathbf{X}_2)$; this we may write by virtue of Proposition 8. Since monoid isomorphisms preserve idempotent endomorphisms, $f_{m'}$ must satisfy the above condition, namely, $f_{m'} \circ \chi' = f_{m'} \Rightarrow \chi' = f_{m'}$, where $\chi' \in \mathrm{pEnd}(\mathbf{X}_2)$, with χ' idempotent. But this is not true if m' is not isolated. To see this just take a $t \in X_2$, an element in X_2 covered by m', i.e. $t < m'$ with nothing in between and observe that f_t is idempotent and $f_{m'} \circ f_t = f_{m'}$. $\square$

Proposition 11. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two finite pure H-spaces such that $\langle X_i \oplus \{1\}, \leqslant_{\mathrm{up}}\rangle$, $i = 1, 2$ is a $\vee$-semilattice, where $\leqslant_{\mathrm{up}}$ is the dual of the specialization order. This means that $\mathcal{K}_{X_i} = \{(x]\colon x \in X_i\}$, $i = 1, 2$. Let $\Gamma\colon \mathrm{pEnd}(\mathbf{X}_1) \to \mathrm{pEnd}(\mathbf{X}_2)$ be a monoid isomorphism. Then for $x \in X_1$, $\mathrm{Im}(\Gamma(f_x))$ has a minimum value.*

P r o o f. Let us proceed by induction on the height of x and the number of maximal elements that cover it. The first step of the induction is the case when x is maximal and the result in this case follows from Proposition 8. Suppose now that $x \in X_1$ is co-maximal and it is covered by just one maximal element of X_1, i.e. $[x) = \{x, m\}$, where $m \in \mathrm{Max}(X_1)$. Observe that $\mathrm{dom}(f_x) = (m]$ and $f_m \circ f_x = f_m = f_x \circ f_m$. Moreover, for every other maximal element z of X_1, $f_x \circ f_z = \emptyset = f_z \circ f_x$. Set $\sigma = \Gamma(f_x)$. By Proposition 8 we may write $\Gamma(f_m) = f_{m'}$, m' being a maximal element of X_2. Suppose that $\mathrm{Im}(\sigma)$ has two minimals t_1 and t_2. Then since $\sigma^2 = \sigma$,

$$f_{m'} \circ \sigma = \sigma \circ f_{m'} = f_{m'}, \quad \sigma \circ f_{t_i} = f_{t_i}, \quad i = 1, 2.$$

The last equality above is due to Proposition 7. Let $\delta_i := \Gamma^{-1}(f_{t_i})$, $i = 1, 2$ such that $f_x \circ \delta_i = \delta_i$. Clearly, $\mathrm{Im}(\delta_i) \subseteq [x)$, so $\mathrm{Im}(\delta_i) = \{m\}$ or $\mathrm{Im}(\delta_i) = \{x, m\}$.

Case I. $\mathrm{Im}(\delta_1) = \mathrm{Im}(\delta_2) = \{x, m\}$: in this case, $\delta_1 = \delta_2 = f_x$ (since necessarily by the definition of H-partial functions, $\mathrm{dom}(\delta_i) = (m]$), so $f_{t_1} = f_{t_2}$, a contradiction.

Case II. $\mathrm{Im}(\delta_1) = \{m\}$ and $\mathrm{Im}(\delta_2) = \{x, m\}$: this leads to $\delta_1 = f_m$ and $\delta_2 = f_x$, that is $f_{t_1} = f_{m'}$, again, a contradiction.

The other cases are treated in a similar way. So $\mathrm{Im}(\sigma)$ cannot have two minimals.

The next step of the induction is the case when x, co-maximal in X_1, is covered exactly by two maximals m_1 and m_2. Let $m_1', m_2' \in \mathrm{Max}(X_2)$ such that $\Gamma(f_{m_i}) = f_{m_i'}$, $i = 1, 2$. Let $\sigma = \Gamma(f_x)$ as above and suppose again that $\mathrm{Im}(\sigma)$ has two minimals t_1 and t_2 with $\delta_i := \Gamma^{-1}(f_{t_i})$, $i = 1, 2$. We have that $f_x \circ \delta_i = \delta_i$ because certainly $\sigma \circ f_{t_i} = f_{t_i}$, $i = 1, 2$. Then $\mathrm{Im}(\delta_i) \subseteq [x) = \{x, m_1, m_2\}$, $i = 1, 2$.

Case $\mathrm{Im}(\delta_i) = \{m_i\}$, $i = 1, 2$: in this case we have that $t_i = m_i'$, $i = 1, 2$ because $\Gamma(\delta_i) = f_{t_i}$, whence $\mathrm{Im}(\sigma) = \{m_1', m_2'\}$ (since t_1, t_2 are the minimal elements of $\mathrm{Im}(\sigma)$) and consequently, $f_{t_i} = f_{m_i'}$, $i = 1, 2$. Observe that $(m_1', m_2'] \subseteq \mathrm{Im}(\sigma) \subseteq \mathrm{dom}(\sigma)$. Indeed, $(m_1', m_2'] = \mathrm{dom}(\sigma)$ because if there existed $t \in \mathrm{dom}(\sigma)$ such that $M_t \cap \{m_i', m_2'\} = \emptyset$ and $\sigma(t) = m_1'$, for example, then as $\sigma^{-1}((m_1']) = (m_1']$ (by condition (iii) of the definition of an H-partial function), we have that $t \in (m_1']$, a contradiction. Then if $(m_1', m_2']$ has a minimum $z = m_1' \wedge m_2'$, σ cannot satisfy condition (i) in the definition of the H-partial function for z. So $(m_1'] \cap (m_2'] = \emptyset$. It could not happen $(m_i'] = \{m_i'\}$ because if that happened, then by Proposition 10 m_i would be isolated and certainly, it is not.

For $i \in \{1, 2\}$ let $z_i \in X_2$ such that m_i' is one of its covers. Clearly, $\sigma \circ f_{z_i} = f_{m_i'}$ (remember that $\mathrm{dom}(f_{z_i}) = ([z_i)]$, $\mathrm{Im}(f_{z_i}) = [z_i)$, $f_{z_i} = \mathrm{id}$ in $\mathrm{Im}(f_{z_i})$, $(m_1', m_2'] = \mathrm{dom}(\sigma)$ and $\sigma(z_i) = m_i'$ because $(m_1'] \cap (m_2'] = \emptyset$), so $f_x \circ \varrho_i = f_{m_i}$, where $\varrho_i = \Gamma^{-1}(f_{z_i})$. It follows from this that $\mathrm{dom}(\varrho_i) \supseteq (m_i]$, $\varrho_i(m_i) = m_i$. We have also that $f_{z_i} \circ \sigma = f_{m_i'}$, so $\varrho_i \circ f_x = f_{m_i}$ and it follows from this that $\varrho_i(x) = m_i$; now since $\varrho_i([x)) = [\varrho_i(x)) = [m_i)$, it follows that $\mathrm{Im}(\varrho_i) = \{m_i\}$ and this implies that $\varrho_i = f_{m_i}$ (because $\mathrm{dom}(\varrho_i)$ has to be $(m_i]$) and this is a contradiction.

Case $\mathrm{Im}(\delta_i) = \{m_1, m_2\}$, $i \in \{1, 2\}$: since $\delta_i^2 = \delta_i$, $\mathrm{Im}(\delta_i) = \{m_1, m_2\} \subseteq \mathrm{dom}(\delta_i)$ and consequently $(m_1, m_2] \subseteq \mathrm{dom}(\delta_i)$, so $x \in \mathrm{dom}(\delta_i)$ with, say, $\delta_i(x) = m_i$. But then this violates the property $\delta_i([x)) = [\delta_i(x) = m_i)$ since necessarily $\delta_i(m_j) = m_j$.

The last case leads to $\delta_i = f_x$ which is a contradiction.

Suppose now that the result is true for the elements of X_1 above x. Looking for a contradiction, suppose that $\sigma = \Gamma(f_x)$ has more than one minimal element, say, t_i, $i = 1, 2, \ldots, k$, $k > 1$. Let $\varrho_i = \Gamma^{-1}(f_{t_i})$. Clearly $\sigma \circ f_{t_i} = f_{t_i} \circ \sigma = f_{t_i}$, so $f_x \circ \varrho_i = \varrho_i \circ f_x = \varrho_i$. This means that $\mathrm{Im}(\varrho_i) \subseteq [x)$. If $\mathrm{Im}(\varrho_i) = [x)$, then $\varrho_i = f_x$ which is not true. So $\mathrm{Im}(\varrho_i) \subsetneq [x)$. We assert that $\mathrm{Im}(\varrho_i)$ has a minimum and in fact, $\varrho_i = f_{d_i}$ for some $d_i > x$. For if $\mathrm{Im}(\varrho_i)$ has two minimals r_1 and r_2, as $x \leqslant r_1$, $x \leqslant r_2$ then $r_1 \wedge r_2$ exists and it must belong to $\mathrm{dom}(\varrho_i)$, which makes impossible condition (i) of the definition of an H-partial function to be satisfied. Notice that $(t_i] \cap (t_j] = \emptyset$ if $i \neq j$ since

$$\mathrm{dom}(\sigma) = (\{m_i' \in \mathrm{Max}(X_2)\colon\ f_{m_i'} = \Gamma^{-1}(f_{m_i}) \ \text{ with } \ m_i \in \mathrm{Max}(X_1),\ x \leqslant m_i\}]$$

and $\mathrm{Im}(\sigma) = [t_1, t_2, \dots, t_k) \subseteq \mathrm{dom}(\sigma)$. Observe now that for each $m \in \mathrm{Max}(X_1)$ there are exactly $|[x)|$ H-partial endomorphisms of the form $f_{m,y}$ such that $f_{m,y} \circ f_x = f_{m,y}$ (indeed, $f_{m,y} \circ f_x = f_{m,y}$ if and only if $x \leqslant y$). Now since the partial endomorphisms $f_{m,y}$ are preserved under monoid isomorphism (Corollary 9), we have that the same is true for σ. But there are exactly $|[t_1, t_2, \dots, t_k)|$ H-partial endomorphisms of the form $f_{m',t}$ such that $f_{m',t} \circ \sigma = f_{m',t}$ and it can be proved, using the induction hypothesis, that $|[t_1, t_2, \dots, t_k)| = |[d_1, d_2, \dots, d_k)| = |(x]| - 1$ (indeed, $d_1, \dots, d_k$ are all the covers of x in X_1). This ends the proof. □

Theorem 12. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ and Γ be as in the previous proposition. Then the mapping $\sigma\colon X_1 \to X_2$ given by the rule*

$$\sigma(x) = y \quad \text{if and only if} \quad y = \min(\Gamma(f_x))$$

is an H-function, so $\mathbf{X}_1 \cong \mathbf{X}_2$.

Proof. It is clear that σ is a well defined one to one and onto mapping. Since $\mathbf{X}_i$ is the dual space of a pure Hilbert algebra, $\mathcal{K}_{X_i} = \{(x]\colon x \in X_i\}$. Due to the properties of the H-partial functions of the form f_x (see the observations on the H-partial functions f_x given at the beginning of this section) the mapping σ satisfies properties (i) and (iii) of the definition of H-partial functions. Let us see, for example, that $\sigma((x]) = (\sigma(x)]$: Let $t \in \sigma((x])$. Then $t = \sigma(y)$ for some $y \geqslant x$. We observed before that this implies $f_y \circ f_x = f_x$ and from this it follows $\Gamma(f_y) \circ \Gamma(f_x) = \Gamma(f_x)$, i.e. $f_{\sigma(y)} \circ f_{\sigma(x)} = f_{\sigma(x)}$ and, as mentioned before, this means $t = \sigma(x) \geqslant \sigma(x)$; so $t \in [\sigma(x))$, as wanted. Conversely, if $t \in [\sigma(x))$, $t \geqslant \sigma(x)$. Observe that $\mathrm{dom}(\sigma) = X_1$ and $\mathrm{Im}(\sigma) = X_2$; so $t = \sigma(y)$ for some y and it is easy to see that $\sigma(y) \geqslant \sigma(x)$ which in turn implies $y \geqslant x$ and this means that $t \in \sigma([x))$. Let us now check that σ satisfies property (iii): For $y \in X_2$ there is a unique $x \in X_1$ such that $y = \sigma(x)$. It is routine to check that $\sigma^{-1}((y]) = (x]$.

Property (ii) is readily satisfied because σ is actually a total function, i.e. $\mathrm{dom}(\sigma) = X_1$. This concludes the proof. □

Corollary 13. *Two pure finite $H^{\vee}$-algebras share the same monoid of endomorphisms if and only if they are isomorphic.*

Proof. It follows from Corollary 5 and the previous theorem. □

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