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GENERALIZED CONVEXITIES RELATED TO AGGREGATION OPERATORS OF FUZZY SETS

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We analyze the existence of fuzzy sets of a universe that are convex with respect to certain particular classes of fusion operators that merge two fuzzy sets. In addition, we study aggregation operators that preserve various classes of generalized convexity on fuzzy sets.

We focus our study on fuzzy subsets of the real line, so that given a mapping $F : [0,1] \times [0,1] \to [0,1]$, a fuzzy subset, say X, of the real line is said to be F-convex if for any $x, y, z \in \mathbb{R}$ such that $x \leq y \leq z$, it holds that $\mu_X(y) \geq F(\mu_X(x), \mu_X(z))$, where $\mu_X : \mathbb{R} \to [0,1]$ stands here for the membership function that defines the fuzzy set X.

We study the existence of such sets paying attention to different classes of aggregation operators (that is, the corresponding functions F, as above), and preserving F-convexity under aggregation of fuzzy sets. Among those typical classes, triangular norms T will be analyzed, giving rise to the concept of norm convexity or T-convexity, as a particular case of F-convexity.

Other different kinds of generalized convexities will also be discussed as a by-product.

Keywords: fuzzy sets, convexity and its generalizations, aggregation functions, fusion operators, triangular norms

Classification: 03E72, 26A51

1. INTRODUCTION

Convexity, as one of the most important notions in Geometry, has been studied thoroughly from different points of view and has been generalized in different ways. One of the most important generalizations is based on its crucial property, namely, convexity is preserved under set intersection. Based on that property, systems of subsets of a given set, that define an structure called a "generalized convexity" have been defined and studied in depth in [13] (see also [2, 3, 4, 8, 12, 17, 18]).

Our aim is to study convexity for fuzzy sets keeping in mind the classical geometrical interpretation of convex sets in an Euclidean space (e.g: the real plane, the real space, etc.), where for each pair of points of a convex set it holds true that the whole line segment that joins them also belongs to that set. However, for fuzzy sets we have to specify the notion of membership to a set. The unifying idea, in our considerations and approach in this manuscript, will be the fact that the grade of membership for the points

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on a line segment that joints two points depends on the grade of memberships of those given two points, both of them being, obviously, the endpoints of the line segment.

We will pay attention to the problem of characterizing the existence of such fuzzy sets, depending on bivariate functions given a priori. Furthermore, we will also analyze conditions under which suitable bivariate functions create a generalized convexity. We will formulate our main results not only for the intersection of fuzzy sets, but for an arbitrary aggregation of fuzzy sets as well.

2. PRELIMINARY CONCEPTS AND RESULTS CONCERNING DIFFERENT KINDS OF ABSTRACT CONVEXITIES ON FUZZY SETS

We start by recalling the standard definition of a fuzzy set.

Definition 2.1. (Zadeh [19]) Let X be a nonempty set, usually called the *universe*. A *fuzzy set* A in X is defined by means of a map $\mu_A : X \to [0,1]$. The map μ_A is said to be the *membership function* (or *indicator*) of A.

The *support* of A is the crisp set $Supp(A) = \{t \in X : \mu_A(t) \neq 0\} \subseteq X$, whereas the *core* of A is the crisp set $Cor(A) = \{t \in X : \mu_A(t) = 1\} \subseteq X$. The fuzzy set A is said to be *normal* provided that it has nonempty core.

Given $\alpha \in (0,1]$, the crisp subset of X defined by $A_\alpha = \{t \in X : \mu_A(t) \geq \alpha\}$ is said to be the *α-cut (level set)* of the fuzzy set A.

In the literature that deals with fuzzy sets, perhaps the most common definition for the concept of a convexity (as a matter of fact, usually called "quasi-convexity") has been introduced in [1] as follows:

Definition 2.2. Let X be a linear space. A fuzzy subset A of the universe X is said to be *quasi-convex* if for all $x, y \in X,\ \lambda \in [0,1]$ it holds true that $\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$, where μ_A stands here for the membership function of the fuzzy set A.

Notice that the literal transcription of the usual convexity condition that comes from the crisp case, namely $\mu_A(\lambda x+(1-\lambda)y) \geq \lambda\mu_A(x)+(1-\lambda)\mu_A(y)$ is not suitable for fuzzy sets for at least two reasons. On the one hand, the addition that appears on the right side of the inequality above could fail to make sense when working on lattice-valued fuzzy sets, where such operation is not defined (in general). On the other hand, the class of fuzzy sets that fulfill an inequality as the one given before, does not coincide, in general, with the class of the fuzzy sets whose α-cuts are always convex. It is straightforward to see that the aforementioned notion of quasi-convexity has none of these drawbacks.

In models arising in fuzzy logic, the minimum represents the classical conjunction. From this point of view Definition 2.2 can be read as the statement – if x **and** y are in the fuzzy set A then any point between them is also in A. However, if we use a different model for the conjunction, then the connective **and** is represented by some triangular norm. This leads to the notion of *T-convexity* or *convexity with respect to a triangular norm T*, discussed later.

Another similar concept may by inspired by ideas from [16]. Here the notion of a weakly convex fuzzy set has been defined in the following way:

Definition 2.3. A fuzzy subset A of a linear space X is said to be *weakly quasi-convex* if for all $x, y \in \operatorname{supp} \mu_A$ there exists $\lambda \in (0,1)$ such that $\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$.

The condition of weak quasi-convexity is mild, not too restrictive (see [11]). However, its underlying idea can be developed further. Roughly speaking, we bear in mind that the value of the membership function at an "inner point" may depend on the values that it takes at the "endpoints". This can be interpreted, defined and/or understood in a more general way than the (more restrictive) one introduced in Definition 2.2 and Definition 2.3.

As it has already been mentioned, we will deal with systems preserving convexity, thus we recall the definition of a generalized convexity from [13].

Definition 2.4. A system $\mathcal{C}$ of subsets of the universe X for which $\emptyset$ and X belong to $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersections is a generalized convexity on the given set X.

As we will work with fuzzy sets, the system of fuzzy subsets of the universe fulfilling the properties from Definition 2.4 we will also denote as a generalized convexity on X.

The notion of a generalized convexity is perhaps too wide. Observe, for instance, that any topology τ defined on the universe X immediately gives rise to a generalized convexity, after considering the class $\mathcal{C}$ of τ-closed subsets of X. Remember that given a topology τ on a set X, the intersection of any family of τ-closed subsets is also τ-closed, so that in this situation $\mathcal{C}$ is indeed closed under arbitrary intersections.

Obviously, not every generalized convexity $\mathcal{C}$ can be interpreted as the class of closed subsets of a topology τ. For this to happen we would need that the family $\mathcal{C}$ is not only stable under arbitrary intersections, but, in addition, it should also be stable under finite unions.

Example 2.5. Let $X = [0,1]$ be the unit interval of the real line. Let $\mathcal{C} = \{\emptyset, X\} \bigcup \{[0,a] : 0 \leq a \leq 1\}$. This family $\mathcal{C}$ is actually the class of τ-closed sets of the topology τ on $X = [0,1]$ defined as $\tau = \{\emptyset, [0,1]\} \bigcup \{(a,1] : 0 \leq a \leq 1\}$.

Now consider that class $\mathcal{C}' = \{\emptyset, X\} \bigcup \{\{a\} : 0 \leq a \leq 1\} \bigcup \{[a,b] : 0 \leq a < b \leq 1\}$. This second family $\mathcal{C}'$ also defines a generalized convexity on $X = [0,1]$. However, it fails to be the family of closed subsets of a topology on X, because, for instance $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ does not lie in $\mathcal{C}'$.

3. CONVEXITY WITH RESPECT TO TRIANGULAR NORMS AND AGGREGATION OPERATORS IN TWO VARIABLES

The notion of quasi-convexity has been widely studied and applied. However, it could still be too restrictive in several situations, especially in frameworks coming from fuzzy logic. In those contexts, it is typical to find models in which a triangular norm (t-norm, for short) other than the minimum is used. By this reason, the notion of convexity with respect to triangular norms (or T- convexity, for short) was launched in [9], as follows:

Definition 3.1. Let $\mathbb{R}$ be the set of all real numbers and let T be a t-norm. A fuzzy subset A of $\mathbb{R}$ is said to be *convex with respect to the t-norm T* (or *T-convex*, for short) if for all $x, y \in \mathbb{R},\ \lambda \in [0,1]$ it holds that $\mu_A(\lambda x + (1-\lambda)y) \geq T(\mu_A(x), \mu_A(y))$.

Notice that here, the triangular norm T has been assumed to play the role of the minimum. In this Definition 3.1, T could be any triangular norm. Nevertheless, both concepts, namely quasi-convexity and T-convexity are close and deeply related. In fact, they coincide in the case of normal fuzzy sets (see Definition 2.1 above) as it is proven in the next proposition.

Proposition 3.2. Let $\mathbb{R}$ be the set of all real numbers, let A be its fuzzy subset. If A is quasi-convex, then it is T-convex for any t-norm T. Moreover, if A is normal, the converse also holds true.

Proof. Since the minimum t-norm is the biggest triangular norm, we have that $T \leq \min$ holds true for any t-norm T. Therefore, quasi-convexity trivially implies T-convexity, for any triangular norm T. Conversely, if A is normal, then there exists at least one element $a \in \mathbb{R}$ such that $\mu_A(a) = 1$. Thus, for any pair $x, y \in \mathbb{R}, x < y$ and any its linear combination, that is, $\lambda x + (1-\lambda)y$, it follows that $x \leq \lambda x + (1-\lambda)y \leq a$ or, alternatively, $a \leq \lambda x + (1-\lambda)y \leq y$. In the former case we have $\mu_A(\lambda x + (1-\lambda)y) \geq T(\mu_A(x), \mu_A(a)) = \mu_A(x)$, in the latter one $\mu_A(\lambda x + (1-\lambda)y) \geq T(\mu_A(a), \mu_A(y)) = \mu_A(y)$. So, we conclude that $\mu_A(\lambda x + (1-\lambda)y) \geq \min\{\mu_A(x), \mu_A(y)\}$. □

While T-convexity may reflect the use of a particular t-norm T playing the role of the conjunction in a certain fuzzy logic model, notice that none of the special properties of a triangular norm has been mentioned in Definition 3.1. Indeed, only one of such classical properties has been used in the proof of Proposition 3.2. This suggests the study of convexity in an even more general form, in which we will introduce a further generalization of the concept of convexity as regards a t-norm T (also known as T-convexity).

The following results are stated for fuzzy subsets of the real line. However, they could be easily generalized for fuzzy subsets where the universe is the n-dimensional real space $\mathbb{R}^n$.

Definition 3.3. Let $F : [0,1]^2 \to [0,1]$ be an arbitrary bivariate mapping. A fuzzy subset A of the real line is said to be *convex with respect to the bivariate map F* (or *F-convex*, for short) if for each $x, y, z \in \mathbb{R}$ such that $x \leq y \leq z$ it holds true that $\mu_A(y) \geq F(\mu_A(x), \mu_A(z))$, where μ_A is the membership function of the fuzzy set A.

Similarly to Definition 2.3 we can also here take a linear combination of the points x, z in place of y.

Clearly for $F(\alpha, \beta) = \min\{\alpha, \beta\}$ we have the usual quasi-convexity, while by replacing F by a triangular norm T we obtain the notion of T-convexity introduced above.

Putting $y = x$ and $y = z$ in Definition 3.3 we obtain that F should fulfill the inequalities $F(\alpha, \beta) \leq \alpha$ and $F(\alpha, \beta) \leq \beta$, hence $F \leq \min$. This provides the following lemma:

Lemma 3.4. If there is at least one fuzzy set convex with respect to F, then $F \leq \min$.

Thus, in the rest of the paper we will consider only mappings F bounded from above by the minimum.

As a comment we may add that even in case that we require the convexity condition $\mu_A(y) \geq F(\mu_A(x), \mu_A(z))$ only for $x < y < z$, we do not obtain any more convex mappings for $F > \min$, as the inequality $\mu_A(y) > \min\{\mu_A(x), \mu_A(z)\}$ leads to discontinuity at one of the points x, z. Hence the mapping fulfilling such inequality should have more than countably many discontinuity points, what is a contradiction to Froda's theorem (see [6]).

Example 3.5. Consider the fuzzy set A whose membership function μ_A is given as follows:

$$\mu_A(x) = \begin{cases} \frac{1}{2}(x-1)^2 + \frac{1}{2}, & x \in [0,2], \\ 0, & x \in \mathbb{R} \setminus [0,2]. \end{cases}$$

This is an F-convex fuzzy subset of the real line, where $F(\alpha, \beta) = \frac{1}{2}\min\{\alpha, \beta\}$. Clearly the fuzzy set A is neither quasi-convex nor T-convex for any t-norm T, since $\mu_A(1) = \frac{1}{2} \not\geq T(\mu_A(0), \mu_A(1)) = T(1,1) = 1$ holds true for any t-norm T.

In the sequel, for a fixed mapping $F : [0,1]^2 \to [0,1]$ let us denote by $\mathcal{C}_F$ the system of all F-convex fuzzy sets. Clearly for $F \leq \min$ at least the fuzzy set $\mu_A = 0$ belongs to $\mathcal{C}_F$, because, in such case $F(0,0) = 0$.

In order to study generalized convexities from now on we will work only with bivariate mappings F with the property $F(\alpha, \beta) \leq \min\{\alpha, \beta\} \quad (0 \leq \alpha, \beta \leq 1)$. Considering F as an aggregation function, we restrict ourselves to those maps that are conjunctive (see [7]).

From the Definition 3.3 it is also clear that given two bivariate maps F, G such that or all $\alpha, \beta \in [0,1]$ it holds true that $F(\alpha, \beta) \leq G(\alpha, \beta)$, then $\mathcal{C}_G \subseteq \mathcal{C}_F$. Furthermore, we can see that the extreme cases are obtained whenever $F(\alpha, \beta) = 1$ for all $\alpha, \beta \in [0,1]$, so that $\mathcal{C}_F = \{0_{\mathbb{R}}, 1_{\mathbb{R}}\}$, as well as for $F(\alpha, \beta) = 0$ for all $\alpha, \beta \in [0,1]$, so that $\mathcal{C}_F = \mathcal{F}(X)$ (the system of all fuzzy subsets of X). Observe also that, for $F(\alpha, \beta) = \min\{\alpha, \beta\}$ the set $\mathcal{C}_F$ is exactly the system of all quasi-convex fuzzy subsets of $\mathbb{R}$.

Perhaps the most important property of classical convex sets is that they create a generalized convexity system, or, in other words, convexity is preserved under intersections. Bearing this in mind, we are also interested in conditions under which an intersection of F-convex fuzzy sets is again an F-convex fuzzy set. We will analyze this problem not only for intersections (represented by triangular norms), but also (and more generally) for arbitrary aggregations of fuzzy sets.

As usually, by a (binary) aggregation function on $[0,1]$ we will understand a mapping $A : [0,1]^2 \to [0,1]$ such that $A(0,0) = 0, A(1,1) = 1$ and, in addition, A is monotone in both variables. By an aggregation of fuzzy sets A_1, A_2 whose membership functions are, respectively, μ_{A_1} and μ_{A_2} we understand the fuzzy set B whose membership function is $\mu_B(x) = A(\mu_{A_1}(x), \mu_{A_2}(x))$. We usually denote it as follows: $\mu_B = A(\mu_{A_1}, \mu_{A_2})$.

To formulate our following result we recall the notion of domination for real valued mappings of two variables. For more details on both domination and aggregation see [14, 15].

Definition 3.6. Let $F, G : [0,1]^2 \to [0,1]$ denote two arbitrary bivariate mappings. Then we say that F *dominates* G ($F \gg G$) if for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ it holds true that

$$F(G(\alpha_1, \beta_1), G(\alpha_2, \beta_2)) \geq G(F(\alpha_1, \alpha_2), F(\beta_1, \beta_2)).$$

The next proposition provides a sufficient and necessary condition for the preservation of convexity as regards a bivariate mapping F. The idea of the proof is based on similar consideration used in [10].

Proposition 3.7. Let $F : [0,1]^2 \to [0,1]$ be an arbitrary mapping, let A be a binary aggregation function on $[0,1]$. Then the following are equivalent:

1. $A(\mu_{A_1}, \mu_{A_2})$ is F-convex for any F-convex fuzzy subsets A_1, A_2 of the real line,
2. A dominates F.

Proof. Let $F : [0,1]^2 \to [0,1]$ be a mapping, let A be an arbitrary binary aggregation function on the unit interval. Since throughout this proof we will work with F-convex subsets of the real line, due to the result already stated in Lemma 3.4 we will assume that $F(\alpha, \beta) \leq \min\{\alpha, \beta\}$ $(0 \leq \alpha, \beta \leq 1)$.

First we suppose that for any F-convex fuzzy subsets A_1, A_2 of $\mathbb{R}$ their aggregation function of membership, namely $A(\mu_{A_1}, \mu_{A_2})$ is F-convex. Thus, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$. Consider a fixed interval $[a,b] \subseteq \mathbb{R}$, and the following fuzzy sets A_1, A_2 such that

$$\mu_{A_1}(x) = \begin{cases} \alpha_1, & x = a, \\ F(\alpha_1, \alpha_2), & x \in (a,b), \\ \alpha_2, & x = b, \end{cases} \qquad \mu_{A_2}(x) = \begin{cases} \beta_1, & x = a, \\ F(\beta_1, \beta_2), & x \in (a,b), \\ \beta_2 & x = b \end{cases}$$

and $\mu_{A_1}(x) = \mu_{A_2}(x) = 0$ for $x \in \mathbb{R} \setminus [a,b]$.

We observe that both μ_{A_1} and μ_{A_1} are F-convex (here we make use of the assumption $F \leq \min$). Hence the fuzzy set $A(\mu_{A_1}, \mu_{A_2})$ is also F-convex. In other words, for any $z \in (a,b)$ it follows that

$$A(\mu_{A_1}, \mu_{A_2})(z) \geq F(A(\mu_{A_1}, \mu_{A_2})(a), A(\mu_{A_1}, \mu_{A_2})(b))$$

which is equivalent to

$$A(\mu_{A_1}(z), \mu_{A_2}(z)) \geq F(A(\mu_{A_1}(a), \mu_{A_2}(a)), A(\mu_{A_1}(b), \mu_{A_2}(b)))$$

or

$$A(F(\alpha_1, \alpha_2), F(\beta_1, \beta_2)) \geq F(A(\alpha_1, \beta_1), A(\alpha_2, \beta_2))$$

and thus $A \gg F$.

To prove the converse assume that $A \gg F$. Let A_1, A_2 be arbitrary F-convex fuzzy subsets of the real line. Take $x, y, z \in \mathbb{R}$ such that $x < y < z$. Then we have that

$$\mu_{A_1}(y) \geq F(\mu_{A_1}(x), \mu_{A_1}(z)) \text{ and } \mu_{A_2}(y) \geq F(\mu_{A_2}(x), \mu_{A_2}(z))$$

and from the monotonicity of the map A we obtain

$$A(\mu_{A_1}, \mu_{A_2})(y) \geq A(F(\mu_{A_1}(x), \mu_{A_1}(z)), F(\mu_{A_2}(x), \mu_{A_2}(z)))$$
$$\geq F(A(\mu_{A_1}(x), \mu_{A_2}(x)), A(\mu_{A_1}(z), \mu_{A_2}(z)) = F(A(\mu_{A_1}, \mu_{A_2})(x), A(\mu_{A_1}, \mu_{A_2})(z)).$$

Therefore the aggregation of μ_{A_1} and μ_{A_2} is also F-convex. □

This Proposition 3.7 shows in fact that, for a bivariate mapping F accomplishing that $F \ll \min$, the family $\mathcal{C}_F$ is a generalized convexity. Moreover, it also shows that the intersection of a finite collection of T-convex fuzzy sets based on the t-norm T is again T-convex, as the minimum t-norm dominates any other t-norm (see [14]).

Another similar condition to achieve the same conclusion is the monotonicity of F, as shown in the next proposition.

Proposition 3.8. $\mathcal{C}_F$ is a generalized convexity if and only if $F : [0,1]^2 \to [0,1]$ is increasing.

Proof. In the case in which we assume that the bivariate mapping F is increasing the proof just consists in checking that the inequality

$$F(\min\{\alpha_1, \alpha_2\}, \min\{\beta_1, \beta_2\}) \leq \min\{F(\alpha_1, \beta_1), F(\alpha_2, \beta_2)\}$$

holds true for a small number of possible mutual positions of $\alpha_1, \alpha_2, \beta_1, \beta_2$ in the unit interval.

Assume, by contradiction, that F fails to be increasing. Then there are $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$ such that $\alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2$, but $F(\alpha_1, \alpha_2) > F(\beta_1, \beta_2)$. Take an arbitrary interval $[a, b] \in \mathbb{R}$ and the fuzzy sets A_1 and SA_2 whose membership functions are

$$\mu_{A_1}(x) = \begin{cases} \alpha_1, & x = a, \\ F(\alpha_1, \alpha_2), & x \in (a, b), \\ \alpha_2, & x = b, \end{cases} \qquad \mu_{A_2}(x) = \begin{cases} \beta_1, & x = a, \\ F(\beta_1, \beta_2), & x \in (a, b), \\ \beta_2 & x = b \end{cases}$$

and $\mu_{A_1}(x) = \mu_{A_2}(x) = 0$ for $x \in \mathbb{R} \setminus [a, b]$. We may notice that both A_1 and A_2 are F-convex fuzzy sets. However, their intersection is the fuzzy set defined by means of the membership function

$$(\mu_{A_1} \cap \mu_{A_2})(x) = \begin{cases} \alpha_1, & x = a, \\ F(\beta_1, \beta_2), & x \in (a, b), \\ \alpha_2, & x = b, \end{cases}$$

which is not F-convex. The reason is that, for any $y \in (a, b)$, we have that

$$(\mu_{A_1} \cap \mu_{A_2})(y) = F(\beta_1, \beta_2) \ngeq F(\alpha_1, \alpha_2) = F((\mu_{A_1} \cap \mu_{A_2})(a), (\mu_{A_1} \cap \mu_{A_2})(b)).$$

We conclude that $\mathcal{C}_F$ is not a generalized convexity. □

Finally, in the next proposition we explain the relationship between F-convex fuzzy sets and crisp convex sets. (Here we consider crisp sets as a special case of fuzzy sets).

Proposition 3.9. Let $\mathcal{C}$ denote the system of all crisp convex subsets of the real line $\mathbb{R}$ and let $\mathcal{F} = \{F : [0,1]^2 \to [0,1]; F(1,1) > 0\}$, F being a bivariate map. Then $\mathcal{C} = \cap_{F\in\mathcal{F}}(\mathcal{C}_F \cap 2^{\mathbb{R}})$.

Proof. Suppose $C \in \mathcal{C}$. Let $F \in \mathcal{F}$. We will show that C is F-convex. To do so, take $x, y, z \in \mathbb{R}, x \le y \le z$.

If $C(y) = 0$, then from its convexity at least one of the values $C(x), C(z)$ should be zero. Hence $F(C(x), C(z)) = 0$ too. If $C(y) = 1$ then the inequality $C(x) \ge F(C(x), C(z))$ is fulfilled for any $F \in \mathcal{F}$. Thus C is F-convex for any F.

Now let C be an F-convex crisp set for any $F \in \mathcal{F}$. Suppose C is not convex. Then there are $x, y, z \in \mathbb{R}$ with $x < y < z$, and such that $C(x) = C(z) = 1, C(y) = 0$. Consider a bivariate mapping $F \in \mathcal{F}$. From the F-convexity of C we have

$$0 = C(y) \ge F(C(x), C(z)) = F(1,1) > 0$$

which is a contradiction. This concludes the proof. □

4. FINAL COMMENTS, DISCUSSION AND SUGGESTIONS FOR FURTHER RESEARCH

A former suggestion for a further development of these ideas could be trying to avoid working with linear spaces, and defining generalized convexities (see [13]) on a nonempty set U, called universe, as suitable mappings $f : U \times U \times [0,1] \to U$ that accomplish certain conditions (e.g.: $f(x, y, \alpha) = f(y, x, 1-\alpha)$ for every $x, y \in U$ and $\alpha \in [0,1]$) so that $f(x, y, \alpha)$ could play the role of the point "$\alpha \cdot x + (1-\alpha) \cdot y$" that is typical in the case in which U is a linear space.

Another suggestion could be trying to work with some more general kinds of fuzzy sets, as, for instance, those in which the membership function takes values in a lattice, instead of in the unit interval $[0,1]$.

In a new complementary direction, already pointed out in Section 2, an appealing idea to be explored in further pieces of research could be the analysis of aggregation maps that preserve some kind of topology defined on the fuzzy sets of the real line. Notice that a topology can be defined (see e.g. [5] as well as the remark after Definition 2.4 in Section 2) by means of the family of closed sets. The intersection of an arbitrary family of closed sets should also be closed, whereas the union of two closed sets should also be a closed set. Here, the property relative to the intersection is similar to the one we have used in the previous section to first define and then deal with and analyze the concept of a generalized convexity. What we need in addition in a topological context is the additional restriction of the aggregation maps preserving also the fact that the union of two closed sets must also be closed. It seems a priori that this kind of studies could be tackled following similar steps to the ones considered in this manuscript when working with generalized convexities.

At this stage, another problem to be explored in next future is the following: even in case when some generalized convexities defined by suitable maps could actually give rise to a topology, it may happen that the topology is *discrete*. To see this, let X be a nonempty set, and let $F : [0,1]^2 \to [0,1]$ be an upper semicontinuous function $(\alpha_i \searrow \alpha_0, \beta_i \searrow \beta_0 \Rightarrow F(\alpha_i, \beta_i) \to F(\alpha_0, \beta_0))$ such that $F(0,x) = F(x,0) = 0$ for all $x \in X$ and let $N : X^2 \times [0,1] \to X$, such that $N(x,x,\lambda) = x$ for all $x \in X, \lambda \in [0,1]$.

Denote

$$\mathcal{C}_{FN} = \{f : X \to X;\ f(N(x,y,\lambda)) \geq F(f(x), f(y)) \ \text{ for all } x, y \in X,\ \lambda \in [0,1]\}.$$

For the case $N(x,y,\lambda) = \lambda x + (1-\lambda)y$ (in a linear space) and $F = \min$ we obtain the set of all convex fuzzy subsets of X. Some properties of such N and arbitrary F have been studied in the previous Section 3. Now we will consider arbitrary N and F and, by $\mathcal{C}^*_{FN}$ we denote the collection of all finite unions (using standard fuzzy union, i.e. the maximum) of fuzzy sets from $\mathcal{C}_{FN}$. We will show that $\mathcal{C}^*_{FN}$ can be considered as a base for the system of closed sets for some fuzzy topology on X.

By χ_A we will denote the characteristic function of A.

Proposition 4.1. There is a topology on X for which $\mathcal{C}^*_{FN}$ is the base for the system of all closed sets.

P r o o f. It is enough to show two properties of C^*_{FN}:

i) $\wedge_{f \in C^*_{FN}} = \emptyset$,

ii) for all $f, g \in C^*_{FN}$ the mapping $f \vee g$ is a meet of some subfamily from C^*_{FN}

To show property (i), take $x_0, y_0 \in X, x_0 \neq y_0$. It is easy to see that $f = \chi_{\{x\}} \in C_{FN}$ for any F and N: If $N(x,y,\lambda) = x_0$, then

$$f(N(x,y,\lambda)) = 1 \geq F(f(x), f(y))$$

for arbitrary F. If $N(x,y,\lambda) \neq x_0$, then at least one of the elements x, y is different from x_0 and on the right hand side of the inequality we have either $F(1,0)$ or $F(0,1)$ which is zero in both cases and the inequality is fulfilled. As the meet of these characteristic functions is the function identically equal to zero, the first property is proved.

The property (ii) is obvious, as for $f, g \in C^*_{FN}$ the mapping $f \vee g$ is an element of C^*_{FN}. $\square$

Let us denote by $\mathcal{T}_{FN}$ the topology on X, for which $\mathcal{C}^*_{FN}$ is a base of closed sets. In the following we will deal with separation axioms for the topological space $(X, \mathcal{T}_{FN})$.

Proposition 4.2. $(X, \mathcal{T}_{FN})$ is a T_1 space.

Proof. We have to show that singletons are closed. However, from the proof of Proposition 4.1 it follows, that singletons are member of the base for closed sets, and therefore closed. □

The problem with the topology introduced as above is the following: Take the fuzzy subsets of a real line and some $x \in \mathbb{R}$. Then for $F \leq \min$ and any mean N there is $\chi_{(-\infty,a)}, \chi_{(a,\infty)} \in C_{FN}$ and thus they are closed in $\mathcal{T}_{FN}$. So, their complements $\chi_{[a,\infty)}, \chi_{(-\infty,a]}$ are open and so is their meet. But this is a singleton and if singletons are open, we come to a *discrete topology*.

Thus, to conclude, we suggest as a new line for further research to analyze *smaller* families that also give rise to a generalized convexity and a topology defined by means of the class of closed sets, but such that the resulting topology is *not* the discrete one. In the previous discussion – namely in Proposition 4.1 and Proposition 4.2 above – the class is perhaps "too big", so that we might try to consider some suitable subclass.

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