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(m,r)-CENTRAL RIORDAN ARRAYS AND THEIR APPLICATIONS

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Abstract. For integers $m > r \geqslant 0$, Brietzke (2008) defined the (m,r)-central coefficients of an infinite lower triangular matrix $G = (d,h) = (d_{n,k})_{n,k\in\mathbb{N}}$ as $d_{mn+r,(m-1)n+r}$, with $n = 0,1,2,\ldots$, and the (m,r)-central coefficient triangle of G as

$$G^{(m,r)} = (d_{mn+r,(m-1)n+k+r})_{n,k\in\mathbb{N}}.$$

It is known that the (m,r)-central coefficient triangles of any Riordan array are also Riordan arrays. In this paper, for a Riordan array $G = (d,h)$ with $h(0) = 0$ and $d(0), h'(0) \neq 0$, we obtain the generating function of its (m,r)-central coefficients and give an explicit representation for the (m,r)-central Riordan array $G^{(m,r)}$ in terms of the Riordan array G. Meanwhile, the algebraic structures of the (m,r)-central Riordan arrays are also investigated, such as their decompositions, their inverses, and their recessive expressions in terms of m and r. As applications, we determine the (m,r)-central Riordan arrays of the Pascal matrix and other Riordan arrays, from which numerous identities are constructed by a uniform approach.

Keywords: Riordan array; central coefficient; central Riordan array; generating function; Fuss-Catalan number; Pascal matrix; Catalan matrix

MSC 2010: 05A05, 05A10, 05A19, 15A09

1. Introduction

The concept of a (proper) Riordan array was introduced to generalize the properties of the Pascal triangle and Catalan triangle [10], [17]–[20]. An infinite lower triangular matrix $D = (d_{n,k})_{n,k\in\mathbb{N}}$ is called a Riordan array if its column k has the generating function $d(t)h(t)^k$, where $d(t) = \sum\limits_{n=0}^{\infty} d_n t^n$ and $h(t) = \sum\limits_{n=1}^{\infty} h_n t^n$ are formal

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power series with $d_0 \neq 0$ and $h_1 \neq 0$. The Riordan array corresponding to the pair $d(t)$ and $h(t)$ is denoted by $(d(t), h(t))$, and its generic entry is $d_{n,k} = [t^n]d(t)h(t)^k$, where $[t^n]$ denotes the coefficient operator.

The set of all Riordan arrays forms a group under ordinary row-by-column product with the multiplication identity $(1, t)$. If $(b_n)_{b\in\mathbb{N}}$ is any sequence having $b(t) = \sum_{n=0}^{\infty} b_n t^n$ as its generating function, then for every Riordan array $(d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$ we have

$$\sum_{k=0}^{n} d_{n,k} b_k = [t^n]d(t)b(h(t)). \tag{1}$$

This is called the fundamental theorem of Riordan arrays and it can be rewritten as

$$(d(t), h(t))b(t) = d(t)b(h(t)). \tag{2}$$

From the fundamental theorem of Riordan arrays, the multiplication rule of Riordan arrays is

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))). \tag{3}$$

The inverse of $(d(t), h(t))$ is

$$(d(t), h(t))^{-1} = \Big(\frac{1}{d(\bar{h}(t))}, \bar{h}(t)\Big), \tag{4}$$

where $\bar{h}(t)$ is the compositional inverse of $h(t)$, i.e., $h(\bar{h}(t)) = \bar{h}(h(t)) = t$.

A Riordan array $G = (d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$ can be characterized (see [12], [11], [14], [20]) by two sequences, the A-sequence, $A = (a_n)_{n\in\mathbb{N}}$ and the Z-sequence, $Z = (z_n)_{n\in\mathbb{N}}$ such that

$$d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \ldots,$$
$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \ldots$$

for all $n, k \geqslant 0$. If $A(t)$ and Z(t) are the generating functions for the corresponding A- and Z-sequences, respectively, then it follows that

$$d(t) = \frac{1}{1 - tZ(h(t))} \quad \text{and} \quad h(t) = tA(h(t)). \tag{5}$$

Hence, a Riordan array $G = (d(t), h(t))$ is completely characterized by the functions $A(t)$ and $Z(t)$. Furthermore, if the inverse of (d, h) is $(d, h)^{-1} = (g, f)$, then we have

$$A(t) = \frac{t}{f(t)} \quad \text{and} \quad Z(t) = \frac{1 - g(t)}{f(t)}, \tag{6}$$

where $f = \bar{h}$ and $g = 1/d(\bar{h})$. For example, the Pascal matrix $P = (\binom{n}{k})_{n,k\geqslant 0}$ is the element $(1/(1-t), t/(1-t))$ of the Riordan group.

It is natural to ask, for a given Riordan array, is its submatrix also a Riordan array? There are many related results that can be found from the references [3], [4], [5], [6]. Particularly, a kind of submatrices, called the (m, r)-central coefficient triangles, or (m, r)-central Riordan arrays for $m > r \geqslant 0$, of a Riordan array were introduced by Brietzke in [5]. More precisely, we have

Definition 1.1. Let $G = (d_{n,k})_{n,k\in\mathbb{N}}$ be an infinite lower triangular matrix. For integers $m > r \geqslant 0$, we define the (m, r)-central coefficients of G to be $d_{mn+r,(m-1)n+r}$, with $n = 0, 1, 2, \ldots$. The (m, r)-central coefficient triangle of G is defined as the infinite lower triangular matrix $G^{(m,r)} = (d_{mn+r,(m-1)n+k+r})_{n,k\in\mathbb{N}}$. If G is a Riordan array, then $G^{(m,r)}$ is called the (m, r)-central array.

Brietzke in [5] presented that if a lower triangular matrix $G = (d_{n,k})$ is a Riordan array, then its (m, r)-central triangle $G^{(m,r)}$ is also a Riordan array. More precisely, we cite Theorem 2.1 of [5] below.

Theorem 1.2 ([5]). *For a given Riordan array $G = (d_{n,k})_{n,k\in\mathbb{N}}$, and for any integers $m \geqslant 2$ and $r \geqslant 0$,*

$$G^{(m,r)} = (\tilde{d}_{n,k})_{n,k\geqslant 0} = (d_{mn+r,(m-1)n+r+k})_{n,k\geqslant 0}$$

defines a new Riordan array, called the (m, r)-central Riordan array. Moreover, the generating function of the A-sequence of the new array is $(A(t))^m$, where $A(t)$ is the generating function of the A-sequence of the given Riordan array G.

The central coefficients of many Riordan arrays play an important role in combinatorics. It is interesting therefore to be able to give the generating functions of such central terms in a systematic way. In two recent papers [3], [13], it has been shown how to find the generating function of the central elements of the Bell subgroup of the Riordan group. Barry in [4] uses the Lagrange inversion theorem to characterize the generating function of the central coefficients of a Riordan array. In [5], the (m, r)-central Riordan arrays of some Riordan arrays are used to give a simple approach to constructing known or new identities.

2. The (m,r)-central coefficients and (m,r)-central Riordan arrays of a Riordan array

2.1. The (m,r)-central coefficients. In this subsection, we will give a characterization for the central coefficients of a Riordan array. Our proof will heavily depend on the Lagrange inversion formula (LIF), which we give in the following form and whose proof may be found in several papers, for example, [15], [21]:

Lemma 2.1 (LIF). *Suppose that a formal power series $w = w(t)$ is implicitly defined by a functional equation $w = t\varphi(w)$, where $\varphi(t)$ is a formal power series such that $\varphi(0) \neq 0$. Let $F(t)$ be any formal power series. Then we have*

$$[t^n]F(w(t)) = [t^n]F(t)\varphi(t)^{n-1}(\varphi(t) - t\varphi'(t)), \tag{7}$$

$$[t^n]w(t)^k = \frac{k}{n}[t^{n-k}]\varphi(t)^n. \tag{8}$$

Theorem 2.2. *Let $G = (d(t), h(t)) = (d_{n,k})_{n,k\geqslant 0}$ be a Riordan array, and let m and r be positive integers with $m > r \geqslant 0$. Suppose that $f(t)$ is the generating function defined by the functional equation $f(t) = t(h(f(t))/f(t))^{m-1}$, or equivalently, $f^m = t(h\circ f)^{m-1}$. Then the generating function for the (m,r)-central coefficients of G is given by*

$$c^{(m,r)}(t) = \frac{1}{m-r}(f^{m-r})'(d\circ f)(h\circ f)^{r-m+1}. \tag{9}$$

Proof. Since $f^m = t(h\circ f)^{m-1}$, we have

$$f' = \frac{(h\circ f)^{m-1}}{mf^{m-1} - t(m-1)(h\circ f)^{m-2}(h'\circ f)}.$$

Substituting into the above equation

$$t = \frac{f^m}{(h\circ f)^{m-1}},$$

we obtain

$$f' = \frac{(h\circ f)^m}{f^{m-1}(m(h\circ f) - f(m-1)(h'\circ f))}.$$

Hence, we have

$$[t^n]\frac{1}{m-r}(f^{m-r})'(d\circ f)(h\circ f)^{r-m+1} = [t^n](d\circ f)\frac{(h\circ f)^{r+1}}{f^r(m(h\circ f) - f(m-1)(h'\circ f))}.$$

Applying Lemma 2.1 (7) and substituting into it $\varphi = \varphi(t) = (h(t)/t)^{m-1}$ and

$$\varphi(t) - t\varphi'(t) = \frac{h^{m-2}}{t^{m-1}}(mh - t(m-1)h'),$$

we obtain

$$\begin{aligned}
&[t^n]d\frac{h^{r+1}}{t^r(mh - t(m-1)h')}\varphi^{(m-1)(n-1)}(\varphi - t\varphi') \\
&= [t^n]d\frac{h^{r+1}}{t^r}\frac{h^{(m-1)(n-1)}}{t^{(m-1)(n-1)}}\frac{h^{m-2}}{t^{m-1}} \\
&= [t^n]d\frac{h^{mn-n+r}}{t^{mn-n+r}} \\
&= [t^{mn+r}]dh^{(m-1)n+r},
\end{aligned}$$

which is the entry $d_{mn+r,(m-1)n+r}$ of the Riordan array $(d,h) = (d_{n,k})_{n,k\geqslant 0}$. This completes the proof. □

By virtue this theorem, the generating function for the central coefficients of $G = (p(t), tq(t))$, $p(0) \neq 0$, $q(0) \neq 0$, is given by $d(t) = tf'(t)p(f(t))/f(t)$, where $f(t)$ is the generating function defined by the functional equation $f(t) = tq(f(t))$. This is the main result of [4].

Example 2.3. Let $d(t) = 1/(1-t)$ and $h(t) = (t+t^2)/(1-t)$. Then $G = (d(t), h(t)) = (1/(1-t), (t+t^2)/(1-t))$ is the Delannoy matrix [2], [8], [22]. If $m = 2$, then $f(t) = t(1+f(t))/(1-f(t))$, which implies $f(t) = \frac{1}{2}(1 - t - \sqrt{1-6t+t^2})$. Therefore, $c^{(2,0)}(t) = f'(t)d(f(t))(h(f(t))/f)^{-1} = 1/\sqrt{1-6t+t^2}$, the generating function of central Delannoy numbers.

Example 2.4. We consider the $(2,r)$-central coefficients of the Riordan array $G = (1/(1-(x+1)t), t/(1-t))$, where x is a parameter. In this case, $d(t) = 1/(1-(x+1)t)$ and $h(t) = t/(1-t)$. Let $f(t)^2 = th(f(t))$, that is, $f(t) = t/(1-f(t))$, then $f(t) = \frac{1}{2}(1-\sqrt{1-4t}) = tC(t)$. Thus, $c^{(2,r)}(t) = f'(t)d(f(t))(h(f(t))/f)^{r-1} = B(t)C(t)^{r-1}/(1-(1+x)tC(t))$, where $C(t) = \frac{1}{2}(1-\sqrt{1-4t})/t$ is the generating function for the Catalan numbers and $B(t) = 1/\sqrt{1-4t}$ is the generating function for the central coefficient numbers.

2.2. The (m,r)-central Riordan arrays. Brietzke in [5] described a recursive approach to obtaining the entries of the (m,r)-central Riordan arrays from a given Riordan array. In the next theorem, we give the expression of (m,r)-central Riordan arrays explicitly.

Theorem 2.5. *Let $G = (d(t), h(t)) = (d_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan array. If $f(t)$ is the generating function defined by the functional equation*

$$f(t) = t\Big(\frac{h\circ f}{f}\Big)^{m-1}, \quad or \quad f(t)^m = t(h\circ f)^{m-1},$$

then the (m,r)-central Riordan arrays for $m > r \geqslant 0$ of $G = (d(t), h(t))$ are given by

$$G^{(m,r)} = \Big(\frac{1}{m-r}(f^{m-r})'(d\circ f)(h\circ f)^{r-m+1}, h\circ f\Big). \tag{10}$$

Proof. From the proof of Theorem 2.2, we may evaluate the entries of $G^{(m,r)} = (d_{n,k}^{(m,r)})_{n,k\geqslant 0}$ for $m > r \geqslant 0$ as

$$d_{n,k}^{(m,r)} = [t^n]\frac{1}{m-r}(f^{m-r})'(d\circ f)(h\circ f)^{r-m+1}(h\circ f)^k = [t^{mn+r}]dh^{(m-1)n+r+k},$$

which is the entry $d_{mn+r,(m-1)n+r+k}$ of the Riordan array $(d,h) = (d_{n,k})_{n,k\geqslant 0}$. This completes the proof of Theorem 2.3. □

Theorem 2.5 gives an easy way how to find the generating function for the A-sequence of (m,r)-central Riordan arrays.

Theorem 2.6. *Let $G = (d(t), h(t))$ be a Riordan array, and let $A(t)$ be the generating function for the A-sequence of G. Then the generating function for the A-sequence of $G^{(m,r)}$ is $A^m(t)$.*

Proof. Since f satisfies $f^m = t(h\circ f)^{m-1}$, let $A(t)$ be the generating function of the A-sequence of G. Then $h = tA(h)$, which gives $h\circ f = fA(h\circ f)$. Hence, $(h\circ f)^m = f^m A(h\circ f)^m = t(h\circ f)^{m-1}A(h\circ f)^m$, and $h\circ f = tA(h\circ f)^m$ follows, which indicate that A^m is the generating function of the A-sequence of $G^{(m,r)}$. □

It is obvious that the key to constructing (m,r)-central Riordan arrays from a given Riordan array (d,h) is to find f from $f^m = t(h\circ f)^{m-1}$. Since the solutions of the equation depend on m, we denote $f \equiv f_m$. We have the following results concerning f, which will be used in the remaining part of the paper.

Theorem 2.7. *Let $f(t)$ and $h(t)$ be the functions satisfying $f^m = t(h\circ f)^{m-1}$. Then the compositional inverse of $f(t)$ is*

$$\overline{f} = t\Big(\frac{t}{h}\Big)^{m-1}. \tag{11}$$

Further, f satisfies

$$f(\overline{h\circ f}) = \bar{h}, \tag{12}$$

where $\overline{h\circ f}$ and $\bar{h}$ are the compositional inverses of $h\circ f$ and h, respectively.

P r o o f. Since h satisfies $h(0)=0$ and $h'(0)\neq 0$ we have $f(0)=0$ and $f'(0)\neq 0$. Hence f has its compositional inverse $\overline{f}$. In

$$f=t\Big(\frac{h\circ f}{f}\Big)^{m-1},$$

we substitute $t=\overline{f}$ and obtain $t=\overline{f}(t)(h(t)/t)^{m-1}$, which can be written as (11). From $(h\circ f\circ(\overline{h\circ f}))(t)=t=(h\circ\bar{h})(t)$, we have (12) immediately. □

Theorem 2.8. *Let $G=(d(t),h(t))$ be a Riordan array, and let $A\equiv A(t)$ be the generating function for the A-sequence of G. Then the inverse of the (m,r)-central coefficient triangle of G can be written as*

$$(G^{(m,r)})^{-1}=\Big(\frac{A(t)-mtA'(t)}{\bar{h}'(t)(d\circ\bar{h})(t)A(t)^{r+2}},\ \frac{t}{A(t)^m}\Big), \tag{13}$$

where $\bar{h}$ is the compositional inverse of h.

P r o o f. From Theorems 2.2 and 2.7 and noting that $f(\overline{h\circ f})=\bar{h}$ (see (12)) implies $f'(\overline{h\circ f})=\bar{h}'/(\overline{h\circ f})'$, we have

$$\begin{aligned}(G^{(m,r)})^{-1}&=\Big(\frac{1}{(f(\overline{h\circ f}))^{m-r-1}f'(\overline{h\circ f})(d\circ(f(\overline{h\circ f})))(h\circ(f(\overline{h\circ f})))^{r-m+1}},\overline{h\circ f}\Big)\\&=\Big(\frac{(\overline{h\circ f})'}{\bar{h}(t)^{m-r-1}\bar{h}'(t)(d\circ\bar{h})(t)t^{r-m+1}},\frac{t}{A(t)^m}\Big)\\&=\Big(\frac{(t/A^m(t))'}{\bar{h}'(t)(d\circ\bar{h})(t)(t/\bar{h})^{r-m+1}},\frac{t}{A(t)^m}\Big)\\&=\Big(\frac{A(t)-mtA'(t)}{\bar{h}'(t)(d\circ\bar{h})(t)A(t)^{r+2}},\frac{t}{A(t)^m}\Big),\end{aligned}$$

which implies (13). □

Remark. Note that $(G^{(m,r)})^{-1}\neq(G^{-1})^{(m,r)}$ in general because $\bar{h}\circ\tilde{f}\neq\overline{h\circ f}$, where $\tilde{f}$ satisfies $\tilde{f}^m=t(\bar{h}\circ\tilde{f})^{m-1}$.

Theorem 2.9. *If $G=(d(t),t)$ is an Appell Riordan array, then the corresponding $G^{(m,r)}$ is also an Appell Riordan array and equals G. Hence, the collection of all these $G^{(m,r)}$ forms a subgroup of the Riordan group. Furthermore, we have the following recurrence relationship of $G^{(m,r)}$ for a Riordan array $G=(d(t),h(t))$ in terms of the related Appell Riordan and Associated Riordan array:*

$$G^{(m+1,r)}=\Big(1,tA\Big(\overline{\frac{t}{A^{m+1}(t)}}\Big)\Big)G^{(m,r)}\Big(\frac{A(t)-t(m+1)A'(t)}{A(t)-tmA'(t)},t\Big), \tag{14}$$

where A is the generating function of the A-sequence of $G=(d,h)$ and $A(0)\neq 0$.

Proof. If $G = (d(t), h(t))$ with $h(t) = t$, then $f(t) = t$. Hence, $G^{(m,r)} = G = (d(t), h(t))$ with $h(t) = t$. To prove the second half of the theorem, we denote

$$g(t) = t\Big(\frac{t}{f}\Big)^{1/(m-1)},$$

where f satisfies $f^m = t(h \circ f)^{m-1}$. Hence, noting that $t/A^m = \overline{h \circ f}$ and $f(t/A^m) = f(\overline{h \circ f}) = \bar{h} = t/A$, we have

$$g\Big(\frac{t}{A^m}\Big) = \frac{t}{A^m}\Big(\frac{t/A^m}{t/A}\Big)^{1/(m-1)} = \frac{t}{A^{m+1}}.$$

Denote $w_m(t) = t/A^m(t)$. Then the above equation can be written as $g(w_m(t)) = w_{m+1}(t)$. It is obvious that g and w_m have the compositional inverses $\overline{g}$ and $\overline{w}_m$, respectively, because of $A(0) \neq 0$. Hence, $w_m = \overline{g}(w_{m+1})$, which implies

$$\begin{aligned}\overline{g} = w_m(\overline{w}_{m+1}) = \frac{\overline{w}_{m+1}}{A^m(\overline{w}_{m+1})} = A(\overline{w}_{m+1})\frac{\overline{w}_{m+1}}{A^{m+1}(\overline{w}_{m+1})}\\ = A(\overline{w}_{m+1})w_{m+1}(\overline{w}_{m+1}) = tA(\overline{w}_{m+1}).\end{aligned}$$

One may check that

$$\begin{aligned}\Big(\frac{A(t) - t(m+1)A'(t)}{A(t) - tmA'(t)}, t\Big)(G^{(m,r)})^{-1}(1, g(t)) = (A, t)\Big(\frac{A^{m-r-1}}{\bar{h}'(t)(d \circ \bar{h})(t)}, g(w_m)\Big)\\ = \Big(\frac{A(t) - t(m+1)A'(t)}{\bar{h}'(t)(d \circ \bar{h})(t)A^{r+2}}, w_{m+1}\Big) = (G^{(m+1,r)})^{-1},\end{aligned}$$

which implies (14) since $\overline{g} = tA(\overline{w}_{m+1})$. □

Example 2.10. We consider the $(2, r)$-central coefficients triangles of $G = (1/(1-(x+1)t), t/(1-t))$, where x is a parameter. By Example 2.4, the generating function for the $(2, r)$-central coefficients of $G = (1/(1-(x+1)t), t/(1-t))$ is given by $B(t)C(t)^{r-1}/(1-(x+1)tC(t))$, and the solution for the equation $f(t) = t/(1-f(t))$ is $f(t) = tC(t)$. Thus, by Theorem 2.5, we have

$$G^{(2,r)} = \Big(\frac{B(t)C(t)^{r-1}}{1-(x+1)tC(t)}, tC(t)^2\Big).$$

Setting $x = 1$, we obtain the following $(2, r)$-central coefficient triangles of $G = (1/(1-2t), t/(1-t))$:

$$G^{(2,0)} = \Big(\frac{B(t)^2}{C(t)}, tC(t)^2\Big), \quad G^{(2,1)} = (B(t)^2, tC(t)^2).$$

The general entries of $G=(d_{n,k})$ and $G^{(m,r)}=(d^{(m,r)}_{n,k})$ are: $d_{n,k}=\sum\limits_{i=0}^{n-k}2^i\binom{n-i-1}{k-1}$, $d^{(2,0)}_{n,k}=\sum\limits_{i=0}^{n-k}4^i((2k-1)/(2n-2i-1))\binom{2n-2i-1}{n-i-k}$, and $d^{(2,1)}_{n,k}=\sum\limits_{i=0}^{n-k}4^i(2k/(2n-2i))\times\binom{2n-2i}{n-i-k}$.

Theorem 2.11. *Let $G=(d(t),h(t))$ be a Riordan array, and let $f(t)$ be the generating function determined by the functional equation*

$$f(t)=t\Big(\frac{h\circ f}{f}\Big)^{m-1},\quad \text{or}\quad f(t)^m=t(h\circ f)^{m-1}.$$

Then the (m,r)-central coefficient triangle of G can be factorized as

$$G^{(m,r)}=(1,f)\Big(\frac{dh^{r+1}}{t^r(mh-(m-1)th')},h\Big), \tag{15}$$

$$G^{(m,r)}=\Big(\frac{1}{m-r}(f^{m-r})'(h\circ f)^{r-m+1},f\Big)G, \tag{16}$$

$$G^{(m,r)}=\Big(\frac{h\circ f}{f},t\Big)G^{(m,r-1)}, \tag{17}$$

$$G^{(m,r)}=G^{(m,r-1)}(A(t),t). \tag{18}$$

Proof. The proofs of these formulas are straightforward from equation (3), Theorems 2.5 and 2.8. Hence, we omit them. □

Corollary 2.12. *Let $G=(d(t),h(t))=(d_{n,k})_{n,k\in\mathbb{N}}$ be a Riordan array, and denote the coefficient $[t^{n+s}]h(t)^s$ by $h^{(s)}_n$. Then we have*

$$d_{mn+r,mn+r-n+k}=\sum_{j=k}^{n}d_{j,k}h^{(nm-n+r)}_{n-j}. \tag{19}$$

Proof. By using Lemma 2.1 argument similar to the proof of Theorem 2.5, the (n,j) entry of $((m-r)^{-1}(f^{m-r})'(h\circ f)^{r-m+1},f)$ is

$$\begin{aligned}
&[t^n]\frac{1}{m-r}(f^{m-r})'(h\circ f)^{r-m+1}f^j\\
&=[t^n]\frac{(h\circ f)^{r+1}f^j}{f^r(m(h\circ f)-(m-1)f(h'\circ f)}\\
&=[t^n]\frac{t^jh^{r+1}}{t^r(mh-(m-1)th')}\frac{h^{(m-1)(n-1)}}{t^{(m-1)(n-1)}}\frac{h^{m-2}}{t^{m-1}}(mh-(m-1)th')\\
&=[t^n]\frac{t^jh^{mn-n+r}}{t^{mn-n+r}}\\
&=[t^{n-j+mn-n+r}]h^{mn-n+r}=h^{mn-n+r}_{n-j}.
\end{aligned}$$

Hence, the (n,k) entry of $G^{(m,r)}$ is

$$d_{n,k}^{(m,r)} = d_{mn+r,(m-1)n+r+k} = \sum_{j=k}^{n} d_{j,k}[t^n]\frac{1}{m-r}(f^{m-r})'(h\circ f)^{r-m+1}f^j$$
$$= \sum_{j=k}^{n} d_{j,k}h_{n-j}^{mn-n+r}.$$

□

In order to obtain the generating functions of the (m,r)-central coefficient triangles of the Pascal matrix, we need the concept of the m-Catalan numbers and their generating function. For any integer $m \geqslant 0$, the m-Catalan numbers or Fuss-Catalan numbers (see [7], [9]) are defined by the formula

$$C_n^{(m)} = \frac{1}{mn+1}\binom{mn+1}{n}, \quad n = 0,1,2,\ldots \tag{20}$$

Their generating function $\mathcal{B}_m(t) = \sum\limits_{n=0}^{\infty}(1/(mn+1))\binom{mn+1}{n}t^n$ satisfies the functional equation

$$\mathcal{B}_m(t) = 1 + t\mathcal{B}_m(t)^m. \tag{21}$$

It can be shown from [9], [16] that the following identities are valid:

$$\mathcal{B}_m(t)^s = \sum_{n=0}^{\infty}\frac{s}{mn+s}\binom{mn+s}{n}t^n, \tag{22}$$

$$\frac{\mathcal{B}_m(t)^{s+1}}{1-(m-1)t\mathcal{B}_m(t)^m} = \sum_{n=0}^{\infty}\binom{mn+s}{n}t^n, \tag{23}$$

$$\mathcal{B}_m'(t) = \frac{\mathcal{B}_m(t)^{m+1}}{1-(m-1)t\mathcal{B}_m(t)^m} = \sum_{n=0}^{\infty}\binom{mn+m}{n}t^n, \tag{24}$$

$$\mathcal{B}_{m-s}(t\mathcal{B}_m(t)^s) = \mathcal{B}_m(t). \tag{25}$$

Theorem 2.13. *The (m,r)-central coefficient triangle of the Pascal matrix $P = (1/(1-t), t/(1-t))$ is given by*

$$P^{(m,r)} = \Big(\frac{\mathcal{B}_m(t)^{r+1}}{1-(m-1)t\mathcal{B}_m(t)^m}, t\mathcal{B}_m(t)^m\Big),$$

and it can be factorized as

$$P^{(m,r)} = \Big(\frac{\mathcal{B}_m(t)^{r}}{1-(m-1)t\mathcal{B}_m(t)^m}, t\mathcal{B}_m(t)^{m-1}\Big)\Big(\frac{1}{1-t}, \frac{t}{1-t}\Big),$$

and

$$P^{(m,r)} = (\mathcal{B}_m(t), t)\Big(\frac{\mathcal{B}_m(t)^r}{1-(m-1)t\mathcal{B}_m(t)^m}, t\mathcal{B}_m(t)^m\Big).$$

P r o o f. For the Riordan array $P = (1/(1-t), t/(1-t))$ with $h(t) = t/(1-t)$, $f(t)$ is determined by $f(t)^m = th(f(t))^{m-1}$. Then $f(t) = t/(1-f(t))^{m-1}$, which implies $1/(1-f(t)) = 1 + t/(1-f(t))^m$. By (21), $1/(1-f(t)) = \mathcal{B}_m(t)$. Hence, $f(t) = t\mathcal{B}_m(t)^{m-1}$. Let $P^{(m,r)} = (\tilde{d}(t), \tilde{h}(t))$. Then

$$\begin{aligned}\tilde{d}(t) &= \frac{1}{m-r}(f^{m-r})' d(f(t))\Big(\frac{f(t)}{1-f(t)}\Big)^{r-m+1}\\ &= \frac{\mathcal{B}_m(t)^{r+1}}{1-(m-1)t\mathcal{B}_m(t)^m}\\ &= \frac{\mathcal{B}_m(t)^r}{1-m+m\mathcal{B}_m(t)^{-1}},\end{aligned}$$

and $\tilde{h}(t) = f(t)/(1-f(t)) = t\mathcal{B}_m(t)^m$. Therefore, by Theorem 2.5, the (m,r)-central coefficient triangle of the Pascal matrix $G = (1/(1-t), t/(1-t))$ can be written as

$$P^{(m,r)} = (\tilde{d}(t), \tilde{h}(t)) = \Big(\frac{\mathcal{B}_m(t)^r}{1-m+m\mathcal{B}_m(t)^{-1}}, t\mathcal{B}_m(t)^m\Big).$$

By (23), the general element of R is

$$d_{n,k}^{(m,r)} = d_{mn+r,(m-1)n+r+k} = \binom{mn+r}{(m-1)n+r+k} = \binom{mn+r}{n-k}.$$

The factorizations are straightforward results of Theorem 2.9. □

Theorem 2.14. *We have the following recursive relation for the* (m,r)*-central binomial coefficients:*

$$\binom{mn+r}{n-k} = \sum_{j=0}^{n}\binom{m(n-j)}{n-j}\frac{mk+r}{mj+r}\binom{mj+r}{j-k}.$$

P r o o f. By (22), the generic term of $(\mathcal{B}_m(t)^r, t\mathcal{B}_m(t)^m)$ is

$$[t^n]\mathcal{B}_m(t)^r(t\mathcal{B}_m(t)^m)^k = [t^{n-k}]\mathcal{B}_m(t)^{mk+r} = \frac{mk+r}{mn+r}\binom{mn+r}{n-k}.$$

By considering

$$\Big(\frac{\mathcal{B}_m(t)^{r+1}}{1-(m-1)t\mathcal{B}_m(t)^m}, t\mathcal{B}_m(t)^m\Big) = \Big(\frac{\mathcal{B}_m(t)}{1-(m-1)t\mathcal{B}_m(t)^m}, t\Big)(\mathcal{B}_m(t)^r, t\mathcal{B}_m(t)^m),$$

we obtain the desired result. □

Theorem 2.15. *Let $P = (1/(1-t), t/(1-t))$. Then the inverse Riordan array of $P^{(m,r)}$ is*

$$(P^{(m,r)})^{-1} = \Big(\frac{1-(m-1)t}{(1+t)^{r+1}}, \frac{t}{(1+t)^m}\Big).$$

Proof. Let

$$P^{(m,r)} = (\tilde{d}(t), \tilde{h}(t)) = \Big(\frac{\mathcal{B}_m(t)^{r+1}}{1-(m-1)t\mathcal{B}_m(t)^m}, t\mathcal{B}_m(t)^m\Big),$$

then $\bar{\tilde{h}}(t) = t/(1+t)^m$, and $d(\bar{\tilde{h}}(t))^{-1} = (1-(m-1)t)/(1+t)^{r+1}$. Hence the result follows by (4). □

Since $\mathcal{B}_2(t) = C(t) = \sum\limits_{k=0}^{\infty}(1/(n+1))\binom{2n}{n}t^n$ is the generating function for the Catalan numbers, by Theorems 2.13 and 2.15, we have

$$P^{(2,0)} = \Big(\frac{1}{\sqrt{1-4t}}, tC(t)^2\Big), \quad P^{(2,1)} = \Big(\frac{C(t)}{\sqrt{1-4t}}, tC(t)^2\Big),$$
$$(P^{(2,0)})^{-1} = \Big(\frac{1-t}{1+t}, \frac{t}{(1+t)^2}\Big), \quad (P^{(2,1)})^{-1} = \Big(\frac{1-t}{(1+t)^2}, \frac{t}{(1+t)^2}\Big).$$

In the next example, we will apply the $(2,r)$-central arrays of the Pascal matrix and their inverse which were computed in the above to provide a simple proof of some identities of Andrews, see [1], [5].

Example 2.16. The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined by the generating function $\sum\limits_{n=0}^{\infty} F_n t^n = t/(1-t-t^2)$. It is well known that

$$\sum_{n=0}^{\infty} F_{2n}t^n = \frac{t}{1-3t+t^2}, \quad \sum_{n=0}^{\infty} F_{2n+1}t^n = \frac{1-t}{1-3t+t^2}.$$

Using (7), we can verify the following identities:

$$\Big(\frac{1-t}{(1+t)^2}, \frac{t}{(1+t)^2}\Big)\frac{1-t}{1-3t+t^2} = \frac{1-t-t^3+t^4}{1-t^5},$$
$$\Big(\frac{1-t}{1+t}, \frac{t}{(1+t)^2}\Big)\frac{t}{1-3t+t^2} = \frac{t(1-t-t^2+t^3)}{1-t^5},$$
$$\Big(\frac{1-t}{(1+t)^2}, \frac{t}{(1+t)^2}\Big)\frac{1}{1-3t+t^2} = \frac{1-2t^2+t^4}{1-t^5},$$
$$\Big(\frac{1-t}{1+t}, \frac{t}{(1+t)^2}\Big)\frac{1-t}{1-3t+t^2} = \frac{1-t^2-t^3+t^5}{1-t^5}.$$

Using the inverse matrices, we have

$$\Big(\frac{C(t)}{\sqrt{1-4t}}, tC(t)^2\Big)\frac{1-t-t^3+t^4}{1-t^5} = \frac{1-t}{1-3t+t^2},$$
$$\Big(\frac{1}{\sqrt{1-4t}}, tC(t)^2\Big)\frac{t(1-t-t^2+t^3)}{1-t^5} = \frac{t}{1-3t+t^2},$$
$$\Big(\frac{C(t)}{\sqrt{1-4t}}, tC(t)^2\Big)\frac{1-2t^2+t^4}{1-t^5} = \frac{1}{1-3t+t^2},$$
$$\Big(\frac{1}{\sqrt{1-4t}}, tC(t)^2\Big)\frac{1-t^2-t^3+t^5}{1-t^5} = \frac{1-t}{1-3t+t^2}.$$

We can represent these matrix equations by using the identities

$$\sum_{j\geqslant 0}\left(\binom{2n+1}{n+5j} - \binom{2n+1}{n+5j+1} - \binom{2n+1}{n+5j+3} + \binom{2n+1}{n+5j+4}\right) = F_{2n+1},$$
$$\sum_{j\geqslant 0}\left(\binom{2n}{n+5j+1} - \binom{2n}{n+5j+2} - \binom{2n}{n+5j+3} + \binom{2n}{n+5j+4}\right) = F_{2n},$$
$$\sum_{j\geqslant 0}\left(\binom{2n+1}{n+5j} - 2\binom{2n+1}{n+5j+2} + \binom{2n+1}{n+5j+4}\right) = F_{2n+2},$$
$$\sum_{j\geqslant 0}\left(\binom{2n}{n+5j} - \binom{2n}{n+5j+2} - \binom{2n}{n+5j+3} + \binom{2n}{n+5j+5}\right) = F_{2n+1},$$

which are equivalent to (19), (20), (21) and (22) of Brietzke in [5].

3. More examples

Example 3.1. Let $G = (1/(1-2t), t/(1-t))$. By Example 2.10, the $(2,r)$-central coefficient triangles of G are $G^{(2,0)} = (B(t)^2/C(t), tC(t)^2)$ and $G^{(2,1)} = (B(t)^2, tC(t)^2)$. In this example, we present some identities by using these matrices.

(i) Considering the Riordan array $G^{(2,0)} = (d(t), h(t)) = (B(t)^2/C(t), tC(t)^2)$, we have

$$d(t)f(h(t)) = \frac{B(t)^2}{C(t)}\frac{(1+tC^2)(1-tC^2)^2}{1-tC^2+t^2C^4} = \frac{1}{1-3t},$$

where

$$\begin{aligned} f(t) &= \frac{(1+t)(1-t)^2}{1-t+t^2} = t + \frac{1-t-2t^2}{1+t^3} \\ &= (1-2t^2) - t^3(1-t-2t^2) + t^6(1-t-2t^2) - t^9(1-t-2t^2) + \dots. \end{aligned}$$

Thus, we obtain the identity for the Riordan array entries $d_{n,k} = \sum_{i=n+k}^{2n} \binom{2n}{i}$, $k \leqslant n$:

$$d_{n,0} - 2d_{n,2} + \sum_{j=1}^{\infty}(-1)^j(d_{n,3j} - d_{n,3j+1} - 2d_{n,3j+2}) = 3^n.$$

Similarly, due to

$$d(t)f(h(t)) = \frac{B(t)^2}{C(t)} \frac{tC^2(1+tC^2)(1-tC^2)^3}{1-t^5C^{10}} = \frac{t}{1-3t+t^2},$$

where

$$\begin{aligned} f(t) &= \frac{t(1+t)(1-t)^3}{1-t^5} = \frac{t-2t^2+2t^4-t^5}{1-t^5} \\ &= t - 2t^2 + 2t^4 - t^5 + t^6 - 2t^7 + 2t^9 - t^{10} + \dots, \end{aligned}$$

we have the identity

$$\sum_{j=0}^{\infty}(d_{n,5j+1} - 2d_{n,5j+2} + 2d_{n,5j+4} - d_{n,5j+5}) = F_{2n}.$$

From the matrix identity

$$d(t)f(h(t)) = \frac{B(t)^2}{C(t)} \frac{(1-t^3C^6)(1+tC^2)(1-tC^2)^3}{1-t^5C^{10}} = \frac{1-t}{1-3t+t^2},$$

where

$$\begin{aligned} f(t) &= \frac{(1-t^3)(1+t)(1-t)^3}{1-t^5} = -1 + t + \frac{2-2t-t^2+t^4}{1-t^5} \\ &= 1 - t - t^2 + t^4 + 2t^5 - 2t^6 - t^7 + t^9 + 2t^{10} + \dots, \end{aligned}$$

we have the identities

$$-d_{n,0} + d_{n,1} + \sum_{j=0}^{\infty}(2d_{n,5j} - 2d_{n,5j+1} - d_{n,5j+2} + d_{n,5j+4}) = F_{2n+1},$$

$$\sum_{j=0}^{\infty}(2d_{n,5j} - 2d_{n,5j+1} - d_{n,5j+2} + d_{n,5j+4}) = F_{2n+1} + \binom{2n}{n}.$$

(ii) Considering the Riordan array $G^{(2,1)} = (d(t), h(t)) = (B(t)^2, tC(t)^2)$, we have

$$d(t)f(h(t)) = B(t)^2 \frac{(1-tC^2)^2}{1-tC^2+t^2C^4} = \frac{1}{1-3t},$$

where

$$f(t)=\frac{(1-t)^2}{1-t+t^2}=1-\frac{t+t^2}{1+t^3}=1-t-t^2+t^4+t^5-t^7-t^8-\dots.$$

Thus, we have the identities for the Riordan array entries $d_{n,k}=\sum_{i=n+k+1}^{2n+1}\binom{2n+1}{i}$, $k\leqslant n$:

$$d_{n,0}+\sum_{j=0}^{\infty}(-1)^{j+1}(d_{n,3j+1}+d_{n,3j+2})=3^n,$$
$$\sum_{j=0}^{\infty}(-1)^{j}(d_{n,3j+1}+d_{n,3j+2})=4^n-3^n.$$

Similarly, the matrix identity

$$d(t)f(h(t))=B(t)^2\frac{(1-tC^2)^2(1-t^3C^6)}{1-t^5C^{10}}=\frac{1-t}{1-3t+t^2},$$

where

$$f(t)=\frac{(1-t)^2(1-t^3)}{1-t^5}=1-\frac{2t-t^2+t^3-2t^4}{1-t^5}$$
$$=1-2t+t^2-t^3+2t^4-2t^6+t^7-t^8+2t^9-\dots,$$

yields the identities

$$d_{n,0}+\sum_{j=0}^{\infty}(-2d_{n,5j+1}+d_{n,5j+2}-d_{n,5j+3}+2d_{n,5j+4})=F_{2n+1},$$
$$\sum_{j=0}^{\infty}(-2d_{n,5j+1}+d_{n,5j+2}-d_{n,5j+3}+2d_{n,5j+4})=F_{2n+1}-4^n.$$

By using

$$d(t)f(h(t))=B(t)^2\frac{(1+tC^2)^3(1-tC^2)^3}{1-t^5C^{10}}=\frac{1-t}{1-3t+t^2},$$

where

$$f(t)=\frac{(1+t)^2(1-t)^3}{1-t^5}=1-\frac{t+2t^2-2t^3-t^4}{1-t^5}$$
$$=1-t-2t^2+2t^3+t^4-t^6-2t^7+2t^8+t^9-\dots,$$

we obtain the identities

$$d_{n,0}+\sum_{j=0}^{\infty}(-d_{n,5j+1}-2d_{n,5j+2}+d_{n,5j+3}+d_{n,5j+4})=F_{2n+2},$$

$$\sum_{j=0}^{\infty}(-d_{n,5j+1}-2d_{n,5j+2}+d_{n,5j+3}+d_{n,5j+4})=F_{2n+2}-4^n.$$

Example 3.2. We consider the Riordan array $G=((1-2t)/(1-t)^2,t/(1-t))$. Then, by Theorem 2.3, we obtain the $(2,r)$-central coefficient triangles

$$G^{(2,0)}=(C(t),tC(t)^2),\quad G^{(2,1)}=(C(t)^2,tC(t)^2).$$

(i) Denote $G^{(2,0)}=(C(t),tC(t)^2)=(d_{n,k})_{n,k\geqslant 0}$. Then $(G^{(2,0)})^{-1}=(1/(1+t), t/(1+t)^2)$. Since, $(C(t),tC(t)^2)(1/(1-3t+t^2))=1/(C(t)(1-5t))$, we have

$$\sum_{j=0}^{n}d_{n,j}F_{2j+2}=5^n-\sum_{j=1}^{n}5^{n-j}C_{j-1}.$$

Similarly, $(C(t),tC(t)^2)(1-t)/(1-3t+t^2)=\sqrt{1-4t}/(1-5t)$ yields

$$\sum_{j=0}^{n}d_{n,j}F_{2j+1}=B_n,$$

where $\{B_n\}$ is the integer sequence $A046748$.

(ii) Denote $G^{(2,1)}=(C(t)^2,tC(t)^2)=(B_{n,k})_{n,k\geqslant 0}$. Then $(G^{(2,1)})^{-1}=(1/(1+t)^2, t/(1+t)^2)$. Since $(1/(1+t)^2,t/(1+t)^2)(1/(1-3t+t^2))=(1-t^2)(1+t)/(1-t^5)$ and

$$\begin{aligned}\frac{(1-t^2)(1+t)}{1-t^5}&=\frac{1+t-t^2-t^3}{1-t^5}\\&=1+t-t^2-t^3+t^5+t^6-t^7-t^8+t^{10}+t^{11}-t^{12}-t^{13}+\dots,\end{aligned}$$

we have

$$\sum_{j=0}^{n}(B_{n,5j}+B_{n,5j+1}-B_{n,5j+2}-B_{n,5j+3})=F_{2n+2}.$$

Similarly, from $(1/(1+t)^2,t/(1+t)^2)(1-t)/(1-3t+t^2)=(1-t^3)/(1-t^5)$ and

$$\frac{1-t^3}{1-t^5}=1-t^3+t^5-t^8+t^{10}-t^{13}+\dots,$$

we obtain

$$\sum_{j=0}^{n}(B_{n,5j} - B_{n,5j+3}) = F_{2n+1},$$
$$\sum_{j=0}^{n}(B_{n,5j+1} - B_{n,5j+2}) = F_{2n}.$$

Furthermore, from $(C(t)^2, tC(t)^2)(1/(1-3t+t^2)) = 1/(1-5t)$ we have

$$\sum_{j=0}^{n} B_{n,j}F_{2j+2} = 5^n,$$

and from $(C(t)^2, tC(t)^2)(1-t)/(1-3t+t^2) = (2-C(t))/(1-5t)$ we obtain

$$\sum_{j=0}^{n} B_{n,j}F_{2j+1} = 5^n - \sum_{j=1}^{n} 5^{n-j}C_j.$$

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