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NOTES ON LOCALLY INTERNAL UNINORM ON BOUNDED LATTICES

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In the study, we introduce the definition of a locally internal uninorm on an arbitrary bounded lattice L. We examine some properties of an idempotent and locally internal uninorm on an arbitrary bounded latice L, and investigate relationship between these operators. Moreover, some illustrative examples are added to show the connection between idempotent and locally internal uninorm.

Keywords: bounded lattice, uninorm, idempotent uninorm, locally internal

Classification: 03B52, 06B20, 03E72

1. INTRODUCTION

Uninorms, introduced by Yager and Rybalov [21] and studied by Fodor et al. [11], are special aggregation operators that have proven to be useful in many fields like fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modeling (see [19, 20]). The uninorms were also studied by many authors in other papers (see [3, 4, 7, 8, 9, 13, 14, 15, 18]). These operators generalize both t-norms and t-conorms (see [15, 16]). A uninorm is a binary operation $U : [0,1]^2 \to [0,1]$ that is commutative, associative, non-decreasing in both coordinates and has a neutral element $e \in]0,1[$. The class of uninorms with neutral element $e \in [0,1]$ coincides the class of t-norms if $e = 1$, and the class of t-conorms if $e = 0$.

Locally internal, monotonic operations on unit interval $[0,1]$ with fixed neutral element have been studied and descriptions of such functions have been emphasized in [17]. Furthormore, characterization of locally internal monotonic, associative operations with a neutral element was presented in the same work.

In [14], the existence of uninorms on any arbitrary bounded lattice L with the neutral element $e \in L\backslash\{0,1\}$ was proved by using the existence of t-norms and t-conorms and the smallest and greatest uninorms were revealed on this count. More specifically, the existence of idempotent uninorms was studied and the smallest and greatest idempotent uninorms on any arbitrary bounded lattice L playing role of a it's neutral element were obtained in [5].

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In [6], it is proposed that in any bounded lattice idempotent uninorms need not be internal extending definition of the term "internal" . Some properties of the idempotent uninorms were examined giving sufficient conditions for each idempotent function to be internal on bounded lattice.

In this paper, we give the definiton of locally internal uninorm on bounded lattices. It is shown that every locally internal uninorm on bounded lattices is idempotent uninorm. It is proved that every locally internal uninorm is either conjunctive or disjunctive uninorm. We give an example that would be different from the example given in [6, Example 1], and we showed in this example that each idempotent uninorm does not have to be locally internal uninorm. If U is an idempotent uninorm on a bounded lattice L with neutral element e such that the set of elements incomparable with e is nonempty, then we show that U may not be locally internal on L. It is proved that for arbitrary uninorm U on bounded lattice L with given neutral element $e \in L\backslash\{0,1\}$, $U(0,1)$ is zero element (null element) of U. Moreover, we study the structure of the bounded lattice L such that every uninorm on L with the neutral element e is idempotent uninorm (locally internal uninorm).

2. PRELIMINARIES

In this section, some definitions and results for binary operations on $[0,1]$ that are monotonic and satisfy the locally internal property and some preliminaries concerning bounded lattices and uninorms (t-norms, t-conorms) are recalled.

Definition 2.1. (Martin et al. [17]) A binary operation $F : [0,1]^2 \to [0,1]$ is called locally internal if it satisfies $F(x,y) \in \{x,y\}$ for all $x, y \in L$.

Lemma 2.2. (Martin et al. [17]) Let F be a locally internal operation. For any $a, b, c \in [0,1]$, we have $F(a, F(b,c)) = F(F(a,b),c)$ if and only if not all the values $F(a,b)$, $F(a,c)$ and $F(b,c)$ are different.

Lemma 2.3. (Martin et al. [17]) Let F be a locally internal, monotonic operation and $a, b, c \in [0,1]$ such that the restriction of F to $\{a,b,c\}$ is commutative. Then $F(a, F(b,c)) = F(F(a,b),c)$.

The following result is an immediate consequence of the Lemmas 2.2 and 2.3 given above and shows the relationship between commutativity and associativity for locally internal, monotonic operations.

Proposition 2.4. (Martin et al. [17]) If a locally internal, monotonic operation on $[0,1]$ is commutative, then it is associative.

Proposition 2.5. (Martin et al. [17]) An idempotent, associative, monotonic operation on $[0,1]$ with a neutral element is locally internal.

Definition 2.6. (Birkhoff [2]) A lattice $(L, \leqslant)$ is a bounded lattice if L has the top and bottom elements, which are written as 1 and 0, respectively. That is, there exist two elements $1, 0 \in L$ such that $0 \leqslant x \leqslant 1$, for all $x \in L$.

Definition 2.7. (Birkhoff [2]) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, we use the notation $a \parallel b$.

Definition 2.8. (Birkhoff [2]) Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of L is defined as

$$[a, b] = \{x \in L | a \leq x \leq b\}.$$

Similarly, we define $(a, b] = \{x \in L \mid a < x \leq b\}, [a, b) = \{x \in L \mid a \leq x < b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L$. Let $A(e) = [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$, $I_e = \{x \in L \mid x \parallel e\}$ and $L_e = \{x \in L \mid x \leq e \text{ or } x \geq e\}$.

An element $a \in L$ is called a zero element (null element) of a function $U : L^2 \to L$ if $U(x, a) = a$ for all $x \in L$.

Definition 2.9. (Karaçal and Mesiar [14]) Let $(L, \leq, 0, 1)$ be a bounded lattice. Operation $U : L^2 \to L$ is called a uninorm on L (shortly a uninorm, if L is fixed) if it is commutative, associative, increasing with respect to both variables and there exists some element $e \in L$ called the neutral element such that $U(e, x) = x$ for all $x \in L$.

We denote by $\mathcal{U}(e)$ the set of all uninorms on L with the neutral element $e \in L$.

Definition 2.10. (Caylıet al. [5]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U be a uninorm on L with the neutral element e.

i) An element $x \in L$ is called an idempotent element if it satisfies $U(x, x) = x$.

ii) U is called an idempotent uninorm if it satisfies $U(x, x) = x$ for all $x \in L$.

Definition 2.11. (Caylıet al. [5]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U be a uninorm on L with the neutral element e. Then,

i) U is called a conjunctive uninorm if $U(0, 1) = 0$.

ii) U is called a disjunctive uninorm if $U(0, 1) = 1$.

Definition 2.12. (Aşıcı and Karaçal [1], Ertuğrul et al. [10], Kesicioğlu and Mesiar [12]) Operation $T : L^2 \to L$ $\big(S : L^2 \to L\big)$ is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e = 1$ $(e = 0)$.

Proposition 2.13. (Karaçal and Mesiar [14]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U be a uninorm on L with the neutral element e. Then

i) $T^* = U|_{[0,e]^2} : [0, e]^2 \to [0, e]$ is a t-norm on $[0, e]$.

ii) $S^* = U|_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ is a t-conorm on $[e, 1]$.

Proposition 2.14. (Karaçal and Mesiar [14]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U be a uninorm on L with the neutral element e. The following properties hold:

i) $x \wedge y \leq U(x,y) \leq x \vee y$ for all $(x,y) \in A(e)$.

ii) $U(x,y) \leq x$ for $(x,y) \in L \times [0,e]$.

iii) $U(x,y) \leq y$ for $(x,y) \in [0,e] \times L$.

iv) $x \leq U(x,y)$ for $(x,y) \in L \times [e,1]$.

v) $y \leq U(x,y)$ for $(x,y) \in [e,1] \times L$.

Corollary 2.15. (Caylıet al. [5]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0,1\}$ and U be an idempotent uninorm on L with the neutral element e.

i) If $(x,y) \in [e,1]^2$, then $U(x,y) = x \vee y$.

ii) If $(x,y) \in [0,e]^2$, then $U(x,y) = x \wedge y$.

Theorem 2.16. (Karaçal and Mesiar [14]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L\backslash\{0,1\}$. If T_e is a t-norm on $[0,e]^2$ and S_e is a t-conorm on $[e,1]^2$, then the following functions $U_t : L^2 \to L$ and $U_s : L^2 \to L$ are uninorms with the neutral element e.

$$U_t(x,y) = \begin{cases} T(x,y), & \text{if } (x,y) \in [0,e]^2 \\ x \vee y, & \text{if } (x,y) \in A(e) \\ x, & \text{if } x \in [0,e] \text{ and } y \parallel e \\ y, & \text{if } y \in [0,e] \text{ and } x \parallel e \\ 1, & \text{otherwise,} \end{cases} \tag{1}$$

$$U_s(x,y) = \begin{cases} S(x,y), & \text{if } (x,y) \in [e,1]^2 \\ x \wedge y, & \text{if } (x,y) \in A(e) \\ x, & \text{if } x \in [e,1] \text{ and } y \parallel e \\ y, & \text{if } y \in [e,1] \text{ and } x \parallel e \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

3. LOCALLY INTERNAL UNINORMS

In this section, we give the definiton of a locally internal operation on an arbitrary bounded lattice L by extending definition of locally internal operation on unit interval $[0,1]$ introduced by Martin et al. in [17] and investigate some properties of these operators on bounded lattice L.

Definition 3.1. Let $(L, \leq, 0, 1)$ be a bounded lattice. The operation $U : L^2 \to L$ is called locally internal if it satisfies $U(x,y) \in \{x,\ x \wedge y,\ x \vee y,\ y\}$ for all $x, y \in L$.

Proposition 3.2. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0,1\}$ and U be a uninorm on L with the neutral element e. If U is locally internal, then U is an idempotent uninorm.

Proof. Consider a locally internal uninorm U on a bounded lattice L with the neutral element $e \in L\backslash\{0,1\}$. In this case, for all $x \in L$, we have

$$U(x,x) \in \{x,\ x \wedge x,\ x \vee x,\ x\} = \{x\}, i.e, U(x,x) = x.$$

Thus, U is an idempotent uninorm. □

Proposition 3.3. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0,1\}$ and U be a uninorm on L with the neutral element e. If U is a locally internal uninorm, U is either conjunctive or disjunctive uninorm.

Proof. Consider a locally internal uninorm U on a bounded lattice L with the neutral element $e \in L \backslash \{0,1\}$. Then,

$$U(0,1) \in \{0,\ 0 \wedge 1,\ 0 \vee 1,\ 1\} = \{0,1\}$$

Hence, we have that U is either a conjunctive uninorm (while $U(0,1) = 0$) or U is a disjunctive uninorm (while $U(0,1) = 1$). □

If the contrary of Proposition 3.2 is considered, a natural question occurs: Is U a locally internal uninorm when U is an idempotent uninorm on arbitrary bounded lattice L with the neutral element $e \in L \backslash \{0,1\}$? In the following example, this question is answered negatively.

Example 3.4. Given a bounded lattice $L = \{0, a, e, 1\}$ whose lattice diagram is displayed in Figure 1 and a mapping $U : L^2 \to L$ defined by Table 1. Then U is an idempotent uninorm on L with a neutral element e and $U(0,1) = a$. That is U is not a locally internal uninorm.

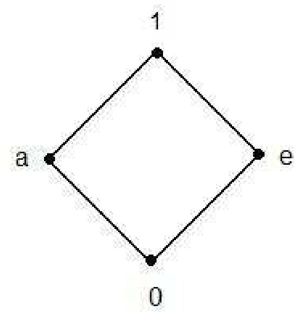

Fig. 1: The lattice L.

U	0	a	e	1
0	0	a	0	a
a	a	a	a	a
e	0	a	e	1
1	1	a	1	1

Tab. 1: The uninorm U on L.

Note that the uninorm in Example 3.4 is neither a conjunctive nor a disjunctive uninorm. In this case, another question occurs: Let U be an idempotent uninorm on arbitrary bounded lattice L with the neutral element $e \in L\backslash\{0,1\}$. If U is either conjunctive or disjunctive idempotent uninorm, is U a locally internal uninorm? It has been given a negative example to this question showing that neither a conjunctive uninorm nor an idempotent uninorm is locally internal in [6, Example 1]. Furthermore, in the following example we give also a negative example to this question.

Example 3.5. Given a bounded lattice $L = \{0, a, e, b, c, d, 1\}$ whose lattice diagram is displayed in Figure 2 and a mapping $U : L^2 \to L$ defined by Table 2. Then U is a conjunctive idempotent uninorm on L with a neutral element e. But U is not a locally internal uninorm.

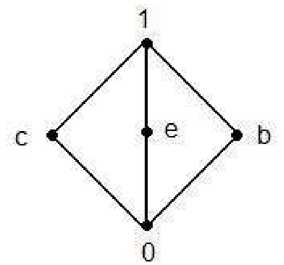

Fig. 2: The lattice L.

U	0	b	c	e	1
0	0	0	0	0	0
b	0	b	b	b	1
c	0	b	c	c	1
e	0	b	c	e	1
1	0	1	1	1	1

Tab. 2: The uninorm U on L.

Consider a bounded lattice $(L, \leq, 0, 1)$ and $e \in L\backslash\{0,1\}$. The following proposition shows that every idempotent uninorm U on L with the neutral element e is locally internal under the additional assumption that all elements $x \in L$ are comparable with e [6, Proposition 7].

Proposition 3.6. (Çaylı and Drygaś [6]) Let $(L, \leq, 0, 1)$ be a bounded lattice and U be an idempotent uninorm on L with the neutral element $e \in L\backslash\{0,1\}$ such that all $x \in L$ are comparable with e. Then, $U(x, y) \in \{x \wedge y, x \vee y\}$ for all $(x, y) \in L^2$.

That is, every idempotent uninorm U on bounded lattice L with the neutral element $e \in L\backslash\{0,1\}$ is locally internal under an assumption that all $x \in L$ are comparable with e.

Corollary 3.7. Let $(L, \leq, 0, 1)$ be a linearly ordered set and U be an idempotent uninorm on L with the neutral element $e \in L\backslash\{0,1\}$. Then, U is locally internal uninorm.

When considering Lemma 2.3, it is quite natural to come up with a problem like this: Let $(L, \leq, 0, 1)$ be an arbitrary bounded lattice. Is it valid Lemma 2.3 on every bounded lattice L? That is, if F is a locally internal, monotone operation and $a, b, c \in L$ such that the restriction of F to $\{a, b, c\}$ is commutative, is the equality $F(a, F(b, c)) = F(F(a, b), c)$ satisfied? The following example provides an example that Lemma 2.3 may not be provided in every bounded lattice L.

Example 3.8. Given a bounded lattice $L = \{0, a, b, c, 1\}$ with order in the Figure 3 and define a mapping $F : L^2 \to L$ by Table 3. Then F is a locally internal, monotone operation and $a, b, c \in L$ such that the restriction of F to $\{a, b, c\}$ is commutative. But we have that

$$F\left(a, F(b, c)\right) = F\left(a, c\right) = 0$$

and

$$F(F(a, b), c) = F(b, c) = c$$

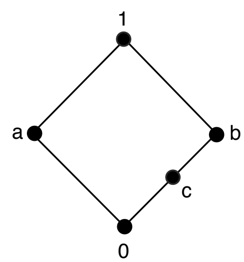

Fig. 3: The lattice L.

U	0	a	c	b	1
0	0	0	0	0	1
a	0	a	0	b	1
c	0	0	c	c	1
b	0	b	c	b	1
1	1	1	1	1	1

Tab. 3: The uninorm U on L.

Proposition 3.9. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0,1\}$ and U be a uninorm on L with the neutral element e. Then, we obtain that $U\left(x, U\left(0, 1\right)\right) = U\left(0, 1\right)$ for all $x \in L$.

Proof. By using monotonicity, commutativity and associativity of U, we have that

$$U(x, U(0,1)) = U(U(0,x), 1) \leq U(U(0,1), 1) = U(0, U(1,1)) = U(0,1)$$

and

$$U(0,1) = U(U(0,0), 1) \leq U(U(x,0), 1) = U(x, U(0,1))$$

for all $x \in L$. Thus, we obtain that $U(x, U(0,1)) \leq U(0,1) \leq U(x, U(0,1))$, that is $U(0,1) = U(x, U(0,1))$ for all $x \in L$. □

In other words, Proposition 3.9 states that for any uninorm U on a bounded lattice $L, U(0,1)$ is a zero element of U.

Remark 3.10. Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0,1\}$ and U be a uninorm on L with the neutral element e. If there exists a zero element of U, this zero element is unique.

Theorem 3.11. The structure of the bounded lattice $(L, \leq, 0, 1)$ such that every uninorm defined on L with neutral element $e \in L \backslash \{0,1\}$ is idempotent uninorm is as shown in Figure 4.

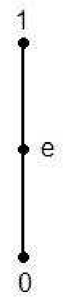

Fig. 4: The lattice L.

Proof. Let every uninorm on L with the neutral element $e \in L \backslash \{0,1\}$ be idempotent uninorm. The proof is split into all possible cases.

Case 1:

From Corollary 2.15 (i), we have that for all $(x,y) \in [e,1]^2$, $U(x,y) = x \vee y$. By considering the uninorm given by the formula (1), for all $(x,y) \in]e,1]^2$, $U(x,y) = 1$. Suppose that there is an element k such that $k \in]e,1[$. In this case, we have that $U(k,k) = k$ since U is idempotent uninorm and we obtain that $U(k,k) = 1$ from the formula (1). This is a contradiction. Therefore, there is not any element in interval $]e,1[$.

Case 2:

From Corollary 2.15 (ii), we have that for all $(x,y) \in [0,e]^2$, $U(x,y) = x \wedge y$. By considering the uninorm given by the formula (2), for all $(x,y) \in [0,e[^2$, $U(x,y) = 0$. Suppose that there is an element t such that $t \in]0,e[$. In this case, we have that $U(t,t) = t$ since U is idempotent uninorm and we obtain that $U(t,t) = 0$ from the formula (2). This is a contradiction. Thus, there is not any element in interval $]0,e[$.

Case 3:

In the uninorm given by the formula (1), it is $U(x,y) = 1$ for all $(x,y) \in I_e \times I_e$. Suppose that there is an element s such that $s \in I_e$. In this case, we have that $U(s,s) = s$ since U is idempotent uninorm and we obtain that $U(s,s) = 1$ from the formula (1). This is a contradiction. So, there is not any element in I_e.

Hence, structure of the bounded lattice L such that every uninorm on L with the neutral element e is idempotent uninorm is as the Figure 4. □

Theorem 3.12. Let $e \in L\backslash\{0,1\}$. Structure of the bounded lattice $(L, \leq, 0, 1)$ such that every uninorm on L with the neutral element e is locally internal uninorm is as the Figure 4.

Proof. By Proposition 3.2, every locally internal uninorm U on the bounded lattice L with the neutral element e is a idempotent uninorm. Thereofore, from Theorem 3.11, it can be seen that structure of the bounded lattice $(L, \leq, 0, 1)$ such that every uninorm on L with the neutral element e is a locally internal uninorm is as on Figure 4. □

Remark 3.13. Every uninorm on the bounded lattice L given by Figure 4 for indicated neutral element e is both an idempotent uninorm and a locally internal uninorm.

4. CONCLUSION

In this paper, for an arbitrary bounded lattice L, the definition of locally internal uninorm on L is given. Some properties of locally internal and idempotent uninorms on a bounded lattice L with the neutral element $e \in L \setminus \{0,1\}$ are investigated and some results on the relationship between locally internal and idempotent uninorm are given. In addition, the structure of the bounded lattice L such that every uninorm on L with the neutral element $e \in L \setminus \{0,1\}$ is an idempotent uninorm (a locally internal uninorm) is researched. In further studies, we deeply investigate the concept of locally internal uninorm on bounded lattices.

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