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STRATIFIED MODULES OVER AN EXTENSION ALGEBRA

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Abstract. Let A be a standard Koszul standardly stratified algebra and X an A-module. The paper investigates conditions which imply that the module $\operatorname{Ext}_A^*(X)$ over the Yoneda extension algebra A^* is filtered by standard modules. In particular, we prove that the Yoneda extension algebra of A is also standardly stratified. This is a generalization of similar results on quasi-hereditary and on graded standardly stratified algebras.

Keywords: standardly stratified algebra; homological dual; standard Koszul algebra

MSC 2010: 16E30, 16E05, 16S37

1. Introduction

In [2] and [4] Ágoston, Dlab and Lukács were looking for conditions which would imply that the Yoneda extension algebra of a quasi-hereditary algebra is again quasi-hereditary. They proved in [4] that a quasi-hereditary algebra which is standard Koszul, that is, its right and left standard modules have top projective resolutions, satisfies this property. They also showed that this homological duality respects the stratifying structure, i.e., the functor Ext_A^* maps standard A-modules to standard modules over the extension algebra. Later, the same authors investigated the analogous question for Koszul standardly stratified algebras under the additional assumption that the initial algebra was graded. They generalized the standard Koszul property for this class of algebras in [5], and achieved similar results for this case, using Poincaré and Hilbert matrices.

The present paper examines the more general case of (not necessarily graded) standardly stratified algebras. Our main goal is to find modules over a standard Koszul standardly stratified algebra, whose images under the natural functor Ext_A^*

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are filtered by standard modules. Notably, we extend former results about quasi-hereditary and graded Koszul standardly stratified algebras by showing that the homological dual of a standard Koszul standardly stratified (but not necessarily graded) algebra is standardly stratified. The lack of left-right symmetry in standardly stratified algebras, however, makes it necessary to deal separately with left and right modules.

We show in Section 3 that for certain A-modules, the functors $\operatorname{Hom}(\varepsilon_i A, -)\colon \operatorname{mod}\text{-}A \to \operatorname{mod}\text{-}\varepsilon_i A \varepsilon_i$ and the trace filtration (corresponding to the projective left A^*-modules) of the Ext^*_A-images of these modules are closely related when A or A° is standard Koszul and standardly stratified. After a short preparatory section, the refinement of this filtration is handled separately for the two cases in Sections 5 and 6. In both cases we define sufficiently large classes of modules (which contain simple and standard or proper standard modules, and are closed under top extensions), whose elements are mapped by Ext^*_A to $\overline{\Delta}^\circ$- or Δ-filtered A^*-modules. In particular, ${}_{A^*}A^*$ and $A^*_{A^*}$ prove to be $\overline{\Delta}^\circ$- and Δ-filtered, respectively. Finally, we present some examples and counterexamples in Section 7.

2. Preliminaries

Throughout the paper, A is a basic finite dimensional algebra over a field K. Modules are finitely generated and they are usually right modules. The category of finitely generated left or right A-modules will be denoted by $A\text{-}\operatorname{mod}$ and $\operatorname{mod}\text{-}A$, respectively.

For the algebra A we fix a complete ordered set of primitive orthogonal idempotents $\mathbf{e} = (e_1, \ldots, e_n)$. In the canonical decomposition $A_A = e_1 A \oplus \ldots \oplus e_n A$ of the regular module, the ith indecomposable projective module $e_i A$ will be denoted by $P(i)$ and its simple top $P(i)/\operatorname{rad} P(i)$ by $S(i)$. Besides, $\widehat{S}$ stands for the semisimple top of A_A, so $\widehat{S} = \bigoplus_{i=1}^{n} S(i)$. The corresponding left modules are denoted by $P^\circ(i), S^\circ(i)$ and $\widehat{S}^\circ$, respectively.

If $1 \leqslant i \leqslant n$, set $\varepsilon_i = e_i + \ldots + e_n$ and $\varepsilon_{n+1} = 0$. The centralizer algebras $\varepsilon_i A \varepsilon_i$ of A will be denoted by C_i, where the idempotents and their order are naturally inherited from A. The ith standard and proper standard A-modules are $\Delta(i) = e_i A / e_i A \varepsilon_{i+1} A$ and $\overline{\Delta}(i) = e_i A / e_i (\operatorname{rad} A) \varepsilon_i A$, respectively. That is, the ith standard module is the largest factor module of $P(i)$ which has no composition factor isomorphic to $S(j)$ if $j > i$, while the ith proper standard module is the largest factor module of $P(i)$ whose radical has no composition factor isomorphic to $S(j)$ if $j \geqslant i$. The left standard and the proper standard modules are defined analogously. The ith costandard module is

$\nabla(i) = D(\Delta^\circ(i))$ and the ith proper costandard module is $\overline{\nabla}(i) = D(\overline{\Delta}^\circ(i))$, where D stands for the usual K-duality functor $\mathrm{Hom}_K(-, K)$ of finitely generated modules.

Let $\mathcal{X}$ be a class of modules. We say that a module X is filtered by $\mathcal{X}$ if there is a sequence of submodules $X = X^0 \supseteq X^1 \supseteq \ldots$ such that $\bigcap_{i \geqslant 0} X^i = 0$, and all the factor modules X^i/X^{i+1} are isomorphic to some modules of $\mathcal{X}$. In this case, we write $X \in \mathcal{F}(\mathcal{X})$.

Given the ordered set $(e_1, \ldots, e_n)$ we can form the trace filtration of a module X with respect to the projective modules $P(i)$

$$X = X\varepsilon_1 A \supseteq X\varepsilon_2 A \supseteq \ldots \supseteq X\varepsilon_n A \supseteq 0.$$

We will refer to this filtration as the trace filtration of X. Following the terminology of [10], we call an algebra A (with a fixed complete ordered set $\mathbf{e}$ of primitive orthogonal idempotents) standardly stratified if the regular module $A_A \in \mathcal{F}(\Delta)$ (or equivalently, the left regular module ${}_AA \in \mathcal{F}(\overline{\Delta}^\circ)$, cf. [8]), where $\overline{\Delta}^\circ$ consists of the proper standard modules, while Δ consists of the left standard modules. We shall use later the fact that $\mathrm{Ext}^h_A(\Delta(i), S(j)) = 0$ for all $h \geqslant 0$ and $i \geqslant j$ when $A_A \in \mathcal{F}(\Delta)$ (cf. [7]), and similarly, $\mathrm{Ext}^h_A(\overline{\Delta}(i), S(j)) = 0$ for all $h \geqslant 0$ and $i > j$ when $A_A \in \mathcal{F}(\overline{\Delta})$.

A submodule $X \leqslant Y$ is a top submodule ($X \overset{t}{\leqslant} Y$), whenever $X \cap \operatorname{rad} Y = \operatorname{rad} X$. This is equivalent to the condition that the natural embedding of X into Y induces an embedding of $X/\operatorname{rad} X$ into $Y/\operatorname{rad} Y$ (such embeddings will be called top embeddings), or in other words, the induced map $\mathrm{Hom}_A(Y, \widehat{S}) \to \mathrm{Hom}_A(X, \widehat{S})$ is surjective. (See [1] for the origin of this concept.) Let

$$P_\bullet(X)\colon \qquad \ldots \to P_h(X) \to \ldots \to P_1(X) \to P_0(X) \to X \to 0$$

be a minimal projective resolution of X with the hth syzygy Ω_h. Using the concept of top submodules, we introduce the classes $\mathcal{C}^i_A$. The module X belongs to $\mathcal{C}^i_A$ if Ω_h is a top submodule of $\operatorname{rad} P_{h-1}$ for all $h \leqslant i$. We say that X has a top projective resolution, or X is Koszul if $X \in \mathcal{C}_A := \bigcap_{i=1}^{\infty} \mathcal{C}^i_A$. The algebra A is a Koszul algebra if $\widehat{S}$ (or equivalently $\widehat{S}^\circ$) has a top projective resolution (cf. [9]). Observe that the concept of top projective resolution generalizes the notion of a linear projective resolution for the non-graded setting.

A standardly stratified algebra A is said to be standard Koszul if $\Delta(i) \in \mathcal{C}_A$ and $\overline{\Delta}^\circ(i) \in \mathcal{C}_{A^\circ}$ for all i. In this case $\varepsilon_i(\operatorname{rad} A)^2\varepsilon_i = \varepsilon_i(\operatorname{rad} A)\varepsilon_i(\operatorname{rad} A)\varepsilon_i$ holds for all i (see Corollary 1.2 of [10]). Let us also state here some earlier results about these algebras, which we shall later use freely. The next theorem summarizes the statements of Lemma 2.1 and Theorem 2.9 of [10].

Theorem 2.1. *If A is a standard Koszul standardly stratified algebra, then A is Koszul. Furthermore, the centralizer algebras C_i are also standard Koszul and standardly stratified algebras, moreover, $\Delta_{C_i}(j) \cong \Delta_A(j)\varepsilon_i$ and $\overset{\circ}{\overline{\Delta}}_{C_i}(j) \cong \varepsilon_i \overset{\circ}{\overline{\Delta}}_A(j)$ for all $j \geqslant i$.*

The extension algebra (or homological dual) of A is the positively graded algebra A^* whose underlying vector space is $\bigoplus_{h \geqslant 0} (A^*)_h = \bigoplus_{h \geqslant 0} \operatorname{Ext}_A^h(\widehat{S}, \widehat{S})$, and the multiplication is given by the Yoneda composition of the extensions. A graded (left) A^*-module $X = \bigoplus_{h \in \mathbb{Z}} X_h$ is an A^*-module for which $(A^*)_h X_k \subseteq X_{h+k}$, and by an A^*-module homomorphism $f\colon X \to Y$ we mean a graded A^*-module homomorphism f having any degree $d \in \mathbb{Z}$. In this sense, we say that two graded A^*-modules X and Y are isomorphic if there exists a bijective A^*-homomorphism $f\colon X \to Y$ (not necessarily of degree 0). The ith graded shift of the graded A^*-module X is denoted by $X[i]$, which is a graded module such that $X[i]_h = X_{h-i}$. For graded modules we shall also use the notation $X_{\geqslant i} = \bigoplus_{h \geqslant i} X_h$.

The functor $\operatorname{Ext}_A^*\colon \operatorname{mod} A \to A^*\text{-}\operatorname{grmod}$ is defined as the direct sum of the functors $\operatorname{Ext}_A^h(-, \widehat{S})$. Namely, if $X \in \operatorname{mod}\text{-}A$, then $\operatorname{Ext}_A^*(X)$ is the graded left module $\bigoplus_{h \geqslant 0} \operatorname{Ext}_A^h(X, \widehat{S})$. For simplicity, we denote $\operatorname{Ext}_A^*(X)$ by X^*, while for its homogeneous part of degree h we write $(X^*)_h$. We use the notation $\varphi^* = \operatorname{Ext}_A^*(\varphi, \widehat{S})\colon \operatorname{Ext}_A^*(Y, \widehat{S}) \to \operatorname{Ext}_A^*(X, \widehat{S})$, where $\varphi\colon X \to Y$ is a module homomorphism, and we denote by E_X^h the canonical isomorphism between the spaces $\operatorname{Hom}_A(\Omega_h(X), \widehat{S})$ and $\operatorname{Ext}_A^h(X, \widehat{S})$. Thus, we have the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(\Omega_h(Y), \widehat{S}) \arrow[r, "(\widetilde{\varphi}_{h-1})^*"] \arrow[d, "E_Y^h"] & \operatorname{Hom}_A(\Omega_h(X), \widehat{S}) \arrow[d, "E_X^h"] \\
\operatorname{Ext}_A^h(Y, \widehat{S}) \arrow[r, "\varphi^*"] & \operatorname{Ext}_A^h(X, \widehat{S})
\end{tikzcd}$$

of left $(A^*)_0$-modules, where $\varphi_\bullet\colon P_\bullet(X) \to P_\bullet(Y)$ is a lifting of φ, while $\widetilde{\varphi}_{h-1}$ is the restriction of φ_{h-1} to the submodule $\Omega_h(X) \subseteq P_{h-1}(X)$.

The module X has a top projective resolution if and only if X^* is generated in degree 0, that is, $(X^*)_h = \operatorname{Ext}_A^h(X, \widehat{S}) = (A^*)_h \cdot (X^*)_0$ for $h \geqslant 0$. In particular, if A is Koszul (for example, when A is standard Koszul and standardly stratified), then A^* is *tightly* graded, i.e., $\operatorname{Ext}_A^h(\widehat{S}, \widehat{S}) = (\operatorname{Ext}_A^1(\widehat{S}, \widehat{S}))^h$ for $h \geqslant 1$ (cf. [9]).

The notion of S-Koszul modules for semisimple S generalizes the concept of Koszul modules. The module X is S-Koszul if satisfies $\operatorname{Ext}_A^h(X, S) = \operatorname{Ext}^1(\widehat{S}, S) \operatorname{Ext}_A^{h-1}(X, \widehat{S})$ for all $h \geqslant 0$. In this sense, a module has a top projective resolution if and only if it is S-Koszul for all simple modules S.

Let $(e_1, \ldots, e_n)$ be a complete ordered set of primitive orthogonal idempotents of A. The set $\{f_i = \mathrm{id}_{S(i)}\colon 1 \leqslant i \leqslant n\}$ defines a complete set of primitive orthogonal idempotents in A^*. We will always consider this set with the opposite order $(f_n, \ldots, f_1)$. In this way, the ith standard A^*-module $\Delta_{A^*}(i)$ is defined as $\Delta_{A^*}(i) = f_i A^*/f_i A^*(f_1 + \ldots + f_{i-1})A^*$, while the ith proper standard module is given by $\overline{\Delta}_{A^*}(i) = f_i A^*/f_i(A^*)_{\geqslant 1}(f_1 + \ldots + f_i)A^*$. The definitions of left standard and proper standard modules are analogous. The algebra A^* is standardly stratified if $A^*_{A^*}$ is filtered by right standard A^*-modules. In view of Theorem 1 of [5], if A^* is tightly graded, then this is equivalent to the condition that ${}_{A^*}A^*$ is filtered by left proper standard A^*-modules.

3. Stratification of modules over A^*

Generalizing the concept of quasi-hereditary lean algebras (cf. [1]), we call an algebra A with a fixed ordered set $(e_1, \ldots, e_n)$ of primitive idempotents lean if $\varepsilon_i J^2 \varepsilon_i = \varepsilon_i J \varepsilon_i J \varepsilon_i$ for all i. In particular, A is lean if A or A° is standard Koszul, as it was shown in Corollary 1.2 of [10]. We should also note that the centralizer algebras $\varepsilon_i A \varepsilon_i$ of A are also lean if A is lean. In this section, we examine modules over the extension algebra of a lean algebra A. For induction purposes we define the classes

$$\mathcal{K}_2 = \{X \in \operatorname{mod} A\colon\ X\varepsilon_2 A \overset{t}{\leqslant} X,\ X\varepsilon_2 \in \mathcal{C}_{C_2}\} \quad \text{and} \quad \mathcal{K} = \mathcal{K}_2 \cap \mathcal{C}_A,$$

as they appeared in [10]. (We shall use the notation $\mathcal{K}_A$ when we need to specify the algebra.) We also introduce a recursive version $r\mathcal{K} \subset \mathcal{K}$ of $\mathcal{K}$ as

$$r\mathcal{K} = \{X \in \mathcal{K}\colon\ X\varepsilon_i \in \mathcal{K}_{C_i} \text{ for all } i\}.$$

Although $\mathcal{K}_2$ was originally defined for standard Koszul standardly stratified algebras, several useful features are preserved in this more general setting.

For an arbitrary module X we write $\widetilde{X} = X\varepsilon_2 A$ and $\overline{X} = X/\widetilde{X}$. Let the operator $\omega\colon$ mod-$A \to$ mod-A be defined by $\omega(X) = \Omega(\widetilde{X})$. If $h \geqslant 1$, then $\omega_h(X)$ stands for $\omega(\omega_{h-1}(X))$, while we denote the submodule $\omega_h(X)\varepsilon_2 A$ by $\widetilde{\omega}_h(X)$, and set $\omega_0(X) = X$.

Lemma 3.1. *Suppose that $X = X\varepsilon_2 A \in$ mod-A. Let $P_\bullet(X)$ denote a minimal projective resolution of X and let $P_\bullet(X\varepsilon_2)$ denote a minimal projective resolution of the C_2-module $X\varepsilon_2$. If $u_\bullet\colon P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$ is a lifting of $\mathrm{id}_{X\varepsilon_2}$, then $\widetilde{u}_0 = u_0\big|_{\Omega(X\varepsilon_2)}\colon \Omega(X\varepsilon_2) \to \Omega(X)\varepsilon_2$ is an isomorphism.*

P r o o f. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega(X\varepsilon_2) \arrow[r] \arrow[d, "\widetilde{u}_0"] & P(X\varepsilon_2) \arrow[r] \arrow[d, "u_0"] & X\varepsilon_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega(X)\varepsilon_2 \arrow[r] & P(X)\varepsilon_2 \arrow[r] & X\varepsilon_2 \arrow[r] & 0
\end{tikzcd}$$

with exact rows. As $X = X\varepsilon_2 A$, it follows that $P(X) = P(X)\varepsilon_2 A$, and so $P(X)\varepsilon_2$ is also a projective cover of $X\varepsilon_2$. Thus u_0 and $\widetilde{u}_0$ are isomorphisms. $\square$

Lemma 3.2. *Suppose that A is a lean algebra and $X \leqslant Y$ are A-modules such that $X\varepsilon_2 \in \mathcal{C}_{C_2}$ and the natural embedding $\varphi\colon \widetilde{X} \to Y$ is a top embedding. If $\varphi_\bullet\colon P_\bullet(\widetilde{X}) \to P_\bullet(Y)$ is a lifting of φ, then $\widetilde{\varphi}_0 = \varphi_0|_{\widetilde{\omega}(X)}\colon \widetilde{\omega}(X) \to \Omega(Y)$ is also a top embedding. Consequently, $\widetilde{\omega}(X) \overset{t}{\leqslant} \omega(X)$.*

P r o o f. By the horseshoe lemma we have the commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \omega(X) \arrow[r] \arrow[d] & \Omega(Y) \arrow[r] \arrow[d] & \Omega(Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(\widetilde{X}) \arrow[r, "\varphi_0"] \arrow[d] & P(Y) \arrow[r] \arrow[d] & P(Z) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{X} \arrow[r, "\varphi"] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the middle column is also a projective cover because φ is a top embedding. In view of Lemma 3.1, $\widetilde{\omega}(X)\varepsilon_2 \cong \Omega(X\varepsilon_2)$, so $X\varepsilon_2 \in \mathcal{C}_{C_2}$ implies that $\widetilde{\omega}(X)\varepsilon_2$ is a top submodule of $P(X\varepsilon_2)(\varepsilon_2 J\varepsilon_2) = P(\widetilde{X})J\varepsilon_2$, thus by Lemma 1.4 (2) of [10], $\widetilde{\omega}(X) \overset{t}{\leqslant} P(\widetilde{X})J$. On the other hand, φ_0 is a split monomorphism, so $P(\widetilde{X})J \overset{t}{\leqslant} P(Y)J$, giving $\varphi_0(\widetilde{\omega}(X)) \overset{t}{\leqslant} P(Y)J$. Since

$$\varphi_0(\widetilde{\omega}(X)) \subseteq \varphi_0(\omega(X)) \subseteq \Omega(Y) \subseteq P(Y)J,$$

we get $\widetilde{\varphi}_0(\widetilde{\omega}(X)) \overset{t}{\leqslant} \Omega(Y)$ and $\widetilde{\omega}(X) \overset{t}{\leqslant} \omega(X)$. $\square$

Corollary 3.3. *If A is lean and $X \in \mathcal{K}_2$, then $\omega(X) \in \mathcal{K}_2$.*

P r o o f. We apply Lemma 3.2 with $Y = X$, and Lemma 3.1. $\square$

Proposition 3.4. *If A is lean, then the classes $\mathcal{K}_2, \mathcal{K}$, and $r\mathcal{K}$ are closed under top extensions. That is, if*

$$0 \to X \overset{t}{\to} Y \to Z \to 0$$

is an exact sequence with top embedding, and both X and Z are in one of these classes, then Y is in the same class.

Proof. Since $\widetilde{X} \overset{t}{\leqslant} X \overset{t}{\leqslant} Y$ and $\widetilde{Z} \overset{t}{\leqslant} Z$, by Lemma 1.6 of [10], $\widetilde{Y} \overset{t}{\leqslant} Y$. Besides, $\widetilde{X} \overset{t}{\leqslant} Y$ also gives that $X\varepsilon_2 \overset{t}{\leqslant} Y\varepsilon_2$, so $Y\varepsilon_2$ is a top extension of the Koszul modules $X\varepsilon_2$ and $Z\varepsilon_2$, thus $Y\varepsilon_2 \in \mathcal{C}_{C_2}$ by Lemma 2.4 of [2]. Hence we get that the class $\mathcal{K}_2$ is closed under top extensions; and this also implies the same condition for $\mathcal{K} = \mathcal{K}_2 \cap \mathcal{C}_A$. To prove the statement for $r\mathcal{K}$, we can use the previous argument recursively for $X\varepsilon_i$ and $Z\varepsilon_i$. □

Proposition 3.5. *Suppose that $\varepsilon_2 J^2 \varepsilon_2 = \varepsilon_2 J \varepsilon_2 J \varepsilon_2$. If $X \in \mathcal{K}_2$, then for every $h \geqslant 0$ we have an exact sequence*

$$0 \to \widetilde{\omega}_h(X) \xrightarrow{\alpha_h} \Omega_h(X) \xrightarrow{\beta_h} Y_h(X) \to 0 \tag{1}$$

with α_h a top embedding.

Proof. Fix an A-module $X \in \mathcal{K}_2$, and consider the embeddings $e^h\colon \widetilde{\omega}_h(X) \to \omega_h(X)$. For $h \geqslant 0$ let $e^h_\bullet\colon P_\bullet(\widetilde{\omega}_h(X)) \to P_\bullet(\omega_h(X))$ denote a lifting of e^h (and also its restriction to $\Omega_{\bullet+1}(\widetilde{\omega}_h(X)) \subseteq P_\bullet(\widetilde{\omega}_h(X))$). Using Lemma 3.2 and Corollary 3.3, an induction on h shows that α_h as the composition of morphisms

$$\begin{aligned}\widetilde{\omega}_h(X) &\xrightarrow{e^h} \omega_h(X) = \Omega_1(\widetilde{\omega}_{h-1}(X)) \xrightarrow{e_0^{h-1}} \Omega_1(\omega_{h-1}(X)) = \Omega_2(\widetilde{\omega}_{h-2}(X)) \\ &\xrightarrow{e_1^{h-2}} \ldots \xrightarrow{e_{h-2}^1} \Omega_{h-1}(\omega_1(X)) = \Omega_h(\widetilde{\omega}_0(X)) \xrightarrow{e_{h-1}^0} \Omega_h(X),\end{aligned} \tag{2}$$

is a top embedding. □

Corollary 3.6. *Let A be lean and $X \in \mathcal{K}_2$. Using the earlier notation, the degree k part $\operatorname{Ext}^k_A(\alpha_h, \widehat{S})\colon \operatorname{Ext}^k_A(\Omega_h(X), \widehat{S}) \to \operatorname{Ext}^k_A(\widetilde{\omega}_h(X), \widehat{S})$ of $\operatorname{Ext}^*_A(\alpha_h)$ can be written as*

$$\operatorname{Ext}^k_A(\alpha_h, \widehat{S}) = (\alpha_{h,k-1})^* = E^k_{\widetilde{\omega}_h(X)} \circ (e^h_{k-1})^* \circ \ldots \circ (e^0_{h+k-1})^* \circ (E^k_{\Omega_h(X)})^{-1},$$

where $\alpha_{h,\bullet}\colon P_\bullet(\widetilde{\omega}_h(X)) \to P_\bullet(\Omega_h(X))$ is a lifting of α_h, and $e^h_\bullet$ is the same as in the previous proof.

The functor $\mathrm{Hom}_A(\varepsilon_i A, -)$ maps exact sequences of mod-A to exact sequences of mod-C_i. For $i=2$, let us denote $\mathrm{Hom}_A(\varepsilon_2 A, -)$ by F. For an A-module X we define q_X to be the direct sum of linear maps

$$q_X = \bigoplus_{h\geqslant 0}(q_X)_h\colon\ \mathrm{Ext}_A^*(X) \to \mathrm{Ext}_{C_2}^*(X\varepsilon_2),$$

where $(q_X)_h$ sends every h-fold extension $0 \to \widehat{S} \to X_{h-1} \to \ldots \to X_0 \to X \to 0$ to an h-fold extension $0 \to \widehat{S}\varepsilon_2 \to X_{h-1}\varepsilon_2 \to \ldots \to X_0\varepsilon_2 \to X\varepsilon_2 \to 0$. The map q_X is well-defined because F preserves the equivalence of extensions. Since the functor F commutes with the Yoneda product of extensions, $q_{\widehat{S}}$ provides an algebra homomorphism from A^* to C_2^*. Consequently, q_X can be considered as a left graded A^*-module homomorphism having degree 0.

Lemma 3.7. *For $h\geqslant 1$, the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Ext}_A^h(X,\widehat{S}) \arrow[rr, "(q_X)_h"] \arrow[d, "(E_X^h)^{-1}"] & & \mathrm{Ext}_{C_2}^h(X\varepsilon_2,\widehat{S}\varepsilon_2) \\
\mathrm{Hom}_A(\Omega_h(X),\hat{S}) \arrow[r, "(q_{\Omega_h(X)})_0"] & \mathrm{Hom}_{C_2}(\Omega_h(X)\varepsilon_2,\hat{S}\varepsilon_2) \arrow[r, "(\widetilde{u}_{h-1})^*"] & \mathrm{Hom}_{C_2}(\Omega_h(X\varepsilon_2),\hat{S}\varepsilon_2) \arrow[u, "E_{X\varepsilon_2}^h"']
\end{tikzcd}$$

where $\widetilde{u}_{h-1}a\colon\ \Omega_h(X\varepsilon_2) \to \Omega_h(X)\varepsilon_2$ is the restriction of a lifting $u_\bullet\colon\ P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$ of $\mathrm{id}_{X\varepsilon_2}$. That is,

$$(q_X)_h = E_{X\varepsilon_2}^h \circ (\widetilde{u}_{h-1})^* \circ (q_{\Omega_h(X)})_0 \circ (E_X^h)^{-1}.$$

When $h=0$, the actions of $(q_X)_0$ and F coincide, i.e., $(q_X)_0(\xi)=F(\xi)$ for all $\xi\in\mathrm{Hom}_A(X,\widehat{S})$.

Proof. The statement for $h=0$ is an easy consequence of the construction of q. For $h\geqslant 1$, let $\xi\in\mathrm{Ext}_A^h(X,\widehat{S})$ and $\xi'=(E_X^h)^{-1}(\xi)\in\mathrm{Hom}_A(\Omega_h(X),\widehat{S})$. In the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_h(X\varepsilon_2) \arrow[r] \arrow[d, "\widetilde{u}_{h-1}"] & P_{h-1}(X\varepsilon_2) \arrow[r] \arrow[d] & \cdots \arrow[r] & X\varepsilon_2 \arrow[r] \arrow[d, "\mathrm{id}_{X\varepsilon_2}"] & 0 \\
0 \arrow[r] & \Omega_h(X)\varepsilon_2 \arrow[r] \arrow[d, "F(\xi')"] & P_h(X)\varepsilon_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & X\varepsilon_2 \arrow[r] \arrow[d, "\mathrm{id}_{X\varepsilon_2}"] & 0 \\
0 \arrow[r] & \widehat{S}\varepsilon_2 \arrow[r] & X_{h-1}\varepsilon_2 \arrow[r] & \cdots \arrow[r] & X\varepsilon_2 \arrow[r] & 0
\end{tikzcd}$$

the extensions $(q_X)_h(\xi)=((q_X)_h\circ E_X^h)(\xi')$ and $(E_{X\varepsilon_2}^h\circ(\widetilde{u}_{h-1})^*\circ F)(\xi')$ are both equivalent to the extension represented by the bottom row. □

Lemma 3.8. *The correspondence q_X is natural, that is, if $\varphi\colon X \to Y$ is an A-module homomorphism, then the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Ext}^*_A(Y) \arrow[r, "q_Y"] \arrow[d, "\varphi^*"] & \mathrm{Ext}^*_{C_2}(Y\varepsilon_2) \arrow[d, "F(\varphi)^*"] \\
\mathrm{Ext}^*_A(X) \arrow[r, "q_X"] & \mathrm{Ext}^*_{C_2}(X\varepsilon_2)
\end{tikzcd}$$

P r o o f. Let $u_\bullet\colon P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$ denote a lifting of $\mathrm{id}_{X\varepsilon_2}$, and similarly let $v_\bullet\colon P_\bullet(Y\varepsilon_2) \to P_\bullet(Y)\varepsilon_2$ denote a lifting of $\mathrm{id}_{Y\varepsilon_2}$. In the diagram

$$\begin{tikzcd}
 & P_\bullet(X\varepsilon_2) \arrow[dl, "u_\bullet"'] \arrow[dr, "F(\varphi)_\bullet"] & \\
P_\bullet(X)\varepsilon_2 \arrow[dr, "F(\varphi_\bullet)"'] & & P_\bullet(Y\varepsilon_2) \arrow[dl, "v_\bullet"] \\
 & P_\bullet(Y)\varepsilon_2 &
\end{tikzcd}$$

the chain maps $F(\varphi_\bullet)\circ u_\bullet$ and $v_\bullet \circ F(\varphi)_\bullet$ are homotopic, since they are both liftings of the map $F(\varphi)\circ \mathrm{id}_{X\varepsilon_2} = \mathrm{id}_{Y\varepsilon_2}\circ F(\varphi)$. Let $\xi \in \mathrm{Ext}^h_A(Y, \widehat{S})$ for which $\xi' = (E^h_Y)^{-1}(\xi)$. Then we have

$$\begin{tikzcd}
E^h_Y(\xi') \arrow[r, mapsto, "q_Y"] \arrow[d, equal] & (E^h_{Y\varepsilon_2}\circ(\widetilde{v}_{h-1})^*_0\circ F)(\xi') \arrow[r, mapsto, "F(\varphi)^*"] & (E^h_{X\varepsilon_2}\circ(F(\varphi)_{h-1})^*_0\circ(\widetilde{v}_{h-1})^*_0\circ F)(\xi') \arrow[d, equal] \\
E^h_Y(\xi') \arrow[r, mapsto, "\varphi^*"] & (E^h_X\circ(\varphi_{h-1})^*)(\xi') \arrow[r, mapsto, "q_X"] & (E^h_{X\varepsilon_2}\circ(\widetilde{u}_{h-1})^*_0\circ F(\varphi_{h-1})^*_0)(F(\xi')).
\end{tikzcd}$$

□

Remark 3.9. We should point out that for any A-module X, the kernel of q_X contains $A^*f_1X^*$ because any extension $\xi \in \mathrm{Ext}^k_A(X, \widehat{S}) \cap A^*f_1X^*$ can be written as a Yoneda-composite of

$$0 \to \widehat{S} \to \ldots \to \bigoplus S(1) \to 0 \quad \text{and} \quad 0 \to \bigoplus S(1) \to \ldots \to X \to 0,$$

which has clearly a 0 image with respect to q_X.

Lemma 3.10. *Suppose that A is lean, $X \in \mathcal{K}_2$, and $P_\bullet(X)$ is a minimal projective resolution of X. Then there is a lifting*

$$u_\bullet \colon\ P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$$

of $\mathrm{id}_{X\varepsilon_2}$ *such that each* $\widetilde{u}_h \colon\ \Omega_{h+1}(X\varepsilon_2) \to \Omega_{h+1}(X)\varepsilon_2$ *is a top embedding, and*

$$\widetilde{u}_h(\Omega_{h+1}(X\varepsilon_2)) = F(\alpha_{h+1})(\widetilde{\omega}_{h+1}(X)\varepsilon_2) \cong \widetilde{\omega}_{h+1}(X)\varepsilon_2. \tag{3}$$

Proof. We use induction on h. The case $h = 0$ is proved by Lemma 3.1. Suppose that $h > 0$. We define the maps $\eta_h \colon\ P_h(X\varepsilon_2) \to P(\widetilde{\omega}_h(X))\varepsilon_2$ recursively as shown in the first two rows of the commutative diagram below.

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{h+1}(X\varepsilon_2) \arrow[r] \arrow[d, "\widetilde{\eta}_h"] & P_h(X\varepsilon_2) \arrow[r] \arrow[d, "\eta_h"] & \Omega_h(X\varepsilon_2) \arrow[r] \arrow[d, "\widetilde{\eta}_{h-1}", "\cong"'] & 0 \\
0 \arrow[r] & \omega_{h+1}(X)\varepsilon_2 \arrow[r] \arrow[d, tail, "F(\alpha_{h+1})"] & P(\widetilde{\omega}_h(X))\varepsilon_2 \arrow[r] \arrow[d, tail] & \widetilde{\omega}_h(X)\varepsilon_2 \arrow[r] \arrow[d, "F(\alpha_h)"] & 0 \\
0 \arrow[r] & \Omega_{h+1}(X)\varepsilon_2 \arrow[r] & P_h(X)\varepsilon_2 \arrow[r] & \Omega_h(X)\varepsilon_2 \arrow[r] & 0
\end{tikzcd}$$

We show by induction that η_h and $\widetilde{\eta}_h$ are isomorphisms for each h. If $\widetilde{\eta}_{h-1}$ is an isomorphism, then η_h is surjective because $P(\widetilde{\omega}_h(X))\varepsilon_2 \to \widetilde{\omega}_h(X)\varepsilon_2$ is a projective cover. As $P(\widetilde{\omega}_h(X))\varepsilon_2$ is projective, η_h splits. But $\ker \eta_h \subseteq \operatorname{rad} P_h(X\varepsilon_2)$, so η_h is also injective. Then, by the snake lemma, $\widetilde{\eta}_h$ is an isomorphism, too.

Finally, $\alpha_{h+1} \colon\ \widetilde{\omega}_{h+1}(X) \to \Omega_{h+1}(X)$ is a top embedding with $\widetilde{\omega}_{h+1}(X)$ generated by $\varepsilon_2 A$, so $F(\alpha_{h+1})$ and $\widetilde{u}_h := F(\alpha_{h+1}) \circ \widetilde{\eta}_h$ are also top embeddings. $\square$

For the remaining part of this section let us fix the notation of the previous lemma. That is, for a fixed arbitrary module $X \in \mathcal{K}_2$ let $u_\bullet$ denote a lifting $P_\bullet(X\varepsilon_2) \to P_\bullet(X)\varepsilon_2$ of $\mathrm{id}_{X\varepsilon_2}$ for which $\widetilde{u}_\bullet = F(\alpha_{\bullet+1}) \circ \widetilde{\eta}_\bullet$ and α_h—along with its cokernel β_h—is defined by the exact sequence (1).

Proposition 3.11. *Let A be lean and $X \in \mathcal{K}_2$. Then $q_X \colon\ X^* \to (X\varepsilon_2)^*$ is an epimorphism, whose kernel is* $\bigoplus\limits_{h \geqslant 0} E_X^h(\operatorname{im}(\beta_h)_0^*)$.

Proof. For an arbitrary index $h \geqslant 0$,

$$(q_X)_h \circ E_X^h = E_{X\varepsilon_2}^h \circ (\widetilde{\eta}_{h-1})_0^* \circ F(\alpha_h)_0^* \circ (q_{\Omega_h(X)})_0$$

by the definition of $\widetilde{u}_\bullet$ and Lemma 3.7. Both $E_{X\varepsilon_2}^h$ and E_X^h are isomorphisms, so we investigate $(\widetilde{\eta}_{h-1})_0^* \circ F(\alpha_h)_0^* \circ (q_{\Omega_h(X)})_0$. By Lemma 3.8,

$$(\eta_{h-1})_0^* \circ \big(F(\alpha_h)_0^* \circ (q_{\Omega_h(X)})_0\big) = (\eta_{h-1})_0^* \circ \big((q_{\widetilde{\omega}_h(X)})_0 \circ (\alpha_h)_0^*\big)\,.$$

As $(\eta_{h-1})_0^*$ and $(q_{\widetilde{\omega}_h(X)})_0$ are isomorphisms, $\ker((q_X)_h \circ E_X^h) = \ker(\alpha_h)_0^* = \mathrm{im}(\beta_h)_0^*$. Furthermore, the surjectivity of $(\alpha_h)_0^*$ follows from α_h being a top embedding. Hence $(q_X)_h$ is surjective with kernel $E_X^h(\mathrm{im}(\beta_h)_0^*)$. □

Proposition 3.12. *Suppose that A is lean and $X \in \mathcal{K}_2$. If $Y_h(X)$ is $\widehat{S}\varepsilon_2 A$-Koszul for all h, then* $\ker q_X = A^* f_1 X^*$.

P r o o f. In view of Proposition 3.11 and Remark 3.9, it is enough to show that $\bigoplus_{h\geqslant 0} E_X^h(\mathrm{im}(\beta_h)_0^*) \subseteq A^* f_1 X^*$, or equivalently,

$$(E_X^h \circ (\beta_h)_0^*)(\mathrm{Hom}_A(Y_h(X), \widehat{S})) \subseteq (A^* f_1 X^*)_h$$

for all h. We prove this by induction on h. If $h = 0$, then $Y_0(X) = \overline{X} \in \mathcal{F}(S(1))$, and that implies

$$E_X^0(\mathrm{im}(\beta_0)_0^*) = \mathrm{im}(\beta_0)_0^* = \mathrm{Hom}_A(X, S(1)) \subseteq (A^* f_1 X^*)_0.$$

It is clear that $(E_X^h \circ (\beta_h)_0^*)(\mathrm{Hom}_A(Y_h(X), S(1))) \subseteq A^* f_1 X^*$, so we only have to deal with the image of $\mathrm{Hom}_A(Y_h(X), \widehat{S}\varepsilon_2 A)$. Since α_h is a top embedding, we get, using the horseshoe lemma, the short exact sequence of the respective syzygies as the bottom row of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\omega}_{h+1}(X) \arrow[r, "\alpha_{h+1}"] \arrow[d] & \Omega_{h+1}(X) \arrow[r, "\beta_{h+1}"] \arrow[d, equal] & Y_{h+1}(X) \arrow[r] \arrow[d, "\theta_{h+1}"] & 0 \\
0 \arrow[r] & \omega_{h+1}(X) \arrow[r] & \Omega_{h+1}(X) \arrow[r, "\widetilde{\beta}_{h,0}"] & \Omega(Y_h(X)) \arrow[r] & 0.
\end{tikzcd} \tag{4}$$

Here the snake lemma yields the exact sequence

$$0 \to \overline{\omega}_{h+1}(X) \longrightarrow Y_{h+1}(X) \xrightarrow{\theta_{h+1}} \Omega(Y_h(X)) \to 0. \tag{5}$$

By (4), $(\beta_{h+1})^* \circ (\theta_{h+1})^* = (\widetilde{\beta}_{h,0})^*$. Besides, $\overline{\omega}_{h+1}(X) \in \mathcal{F}(S(1))$ gives the isomorphism $(\theta_{h+1})^*\colon \mathrm{Hom}_A(\Omega(Y_h(X)), \widehat{S}\varepsilon_2 A) \to \mathrm{Hom}_A(Y_{h+1}(X), \widehat{S}\varepsilon_2 A)$, so

$$(\beta_{h+1})_0^*\big(\mathrm{Hom}_A(Y_{h+1}(X), \widehat{S}\varepsilon_2 A)\big) = (\beta_{h,0})_0^*\big(\mathrm{Hom}_A(\Omega(Y_h(X)), \widehat{S}\varepsilon_2 A)\big).$$

Suppose that φ is an element of $\mathrm{Hom}_A(\Omega(Y), \widehat{S}\varepsilon_2 A)$. Then from the diagram

$$\begin{tikzcd}
\xi\colon 0 \arrow[r] & \Omega_{h+1}(X) \arrow[r] \arrow[d, "\beta_{h,0}"] & P_h(X) \arrow[r] \arrow[d] & \Omega_h(X) \arrow[r] \arrow[d, "\beta_h"] & 0 \\
0 \arrow[r] & \Omega(Y_{h+1}) \arrow[r] \arrow[d, "\varphi"] & P(Y_{h+1}) \arrow[r] & Y_{h+1} \arrow[r] & 0 \\
& \widehat{S}\varepsilon_2 A & & &
\end{tikzcd}$$

we get

$$
\begin{aligned}
(E_X^{h+1} \circ (\beta_{h,0})_0^*)(\varphi) &\subseteq \varphi * \beta_{h,0} * \xi * \operatorname{Ext}_A^h(X, \Omega_h(X)) \\
&\subseteq \varphi * \operatorname{Ext}_A^1(Y_{h+1}, \Omega(Y)) * \beta_h * \operatorname{Ext}_A^h(Y_{h+1}, \Omega_h(X)) \\
&\subseteq \operatorname{Ext}_A^1(Y_{h+1}, \widehat{S}\varepsilon_2 A) * \beta_h * \operatorname{Ext}_A^h(X, \Omega_h(X)),
\end{aligned}
$$

where $*$ stands for the Yoneda product of extensions of arbitrary modules, to emphasize that this product is not necessarily a product in A^*. It was assumed that Y_{h+1} is $\widehat{S}\varepsilon_2 A$-Koszul, so the latter is included in

$$
\begin{aligned}
&(A^*)_1 * \operatorname{Hom}_A(Y_{h+1}, \widehat{S}) * \beta_h * \operatorname{Ext}_A^h(X, \Omega_h(X)) \\
&\qquad\qquad \subseteq (A^*)_1 * E_X^h(\operatorname{im}(\beta_h)_0^*) \subseteq (A^* f_1 X^*)_{h+1}.
\end{aligned}
$$

□

4. $\overline{\Delta}$-filtration of modules over an infinite dimensional graded algebra

Suppose that $\Lambda = \bigoplus\limits_{h \geqslant 0} \Lambda_h$ is a tightly graded K-algebra, i.e., $\Lambda_h \cdot \Lambda_k = \Lambda_{h+k}$ for all $h, k \geqslant 0$. Let Λ-grfmod denote the category of left graded Λ-modules $X = \bigoplus\limits_{h \in \mathbb{Z}} X_h$ such that $\dim_{\mathrm{K}} X_h < \infty$ for every h, and there exists a $t \in \mathbb{Z}$ for which $X_h = 0$ whenever $h < t$. The homomorphisms and isomorphisms in Λ-grfmod will be graded, but not necessarily of degree 0. We assume that $f_1 \in \Lambda_0$ is an idempotent element and the proper standard module belonging to f_1 is defined as

$$\overline{\Delta}^{\circ}(1) = \Lambda f_1 / \Lambda f_1 \, (\Lambda_{\geqslant 1}) \, f_1.$$

Clearly, $\operatorname{Ext}_\Lambda^1(\overline{\Delta}^{\circ}(1), S) = 0$ for all simple modules with $f_1 S = 0$. We call a chain of submodules $X = X^0 \supseteq X^1 \supseteq \ldots$ a $\overline{\Delta}^{\circ}(1)$-filtration if $\bigcap\limits_{i=0}^{\infty} X^i = 0$ and $X^i / X^{i+1} \cong \overline{\Delta}^{\circ}(1)$ for each i.

Lemma 4.1 . *If $X \in \mathcal{F}(\overline{\Delta}^{\circ}(1))$, then X is generated by the projective module Λf_1, i.e. $X = \Lambda f_1 X$.*

Proof. If $X = X^0 \supseteq X^1 \supseteq \ldots$ is a $\overline{\Delta}^{\circ}(1)$-filtration, then $X^i = \Lambda u_i + X^{i+1}$ for some elements $u_i = f_1 u_i$. Then for any h the finiteness of the dimension of $(X)_{\leqslant h}$ and the condition $\bigcap X^i = 0$ implies that $(X^i)_h = 0$ for some i, thus,

$$X_h = \left(\sum_{j=0}^{i-1} \Lambda u_j \right)_h + (X^i)_h \leqslant \sum_{j=0}^{\infty} \Lambda u_j \leqslant \Lambda f_1 X.$$

□

Proposition 4.2. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence, where $Z \in \mathcal{F}(\overline{\Delta}^{\circ}(1))$, and $Y = \Lambda f_1 Y$. If S is a simple module such that $f_1 S = 0$, then*

$$\mathrm{Ext}^1_\Lambda(Z,S) \cong \mathrm{Hom}_\Lambda(X,S) = 0.$$

As a consequence, X is generated by Λf_1.

Proof. First suppose that $Z \cong \overline{\Delta}^{\circ}(1)$. Then $\mathrm{Ext}^1_\Lambda(Z,S) = 0$ and from the exact sequence

$$\mathrm{Hom}_\Lambda(Y,S) \to \mathrm{Hom}_\Lambda(X,S) \to \mathrm{Ext}^1_\Lambda(Z,S) \tag{6}$$

we get $\mathrm{Hom}_\Lambda(X,S) = 0$.

Now let $Z = Z^0 \supseteq Z^1 \supseteq \dots$ be a $\overline{\Delta}^{\circ}(1)$-filtration and assume that

$$\xi\colon\ 0 \to S \to W \to Z \to 0$$

is a short exact sequence. Let us denote by W^i the preimage of Z^i in W for each i. Then $\bigcap W^i = S$.

If $\Lambda f_1 W \neq W$, then the condition $Z = \Lambda f_1 Z$ (by Lemma 4.1) together with the simplicity of S implies that $W = S \oplus \Lambda f_1 W$, so the extension ξ is trivial.

If $\Lambda f_1 W = W$, then we may apply the first step of the proof to the sequences

$$0 \to W^{i+1} \to W^i \to W^i/W^{i+1} \to 0$$

to show by induction that $\mathrm{Hom}_\Lambda(W^i,S) = 0$ for all i.

On the other hand, the simple module S lies in W_h for some h. But $\bigcap\limits_{i=0}^{\infty} W^i = S$ yields that $\bigcap\limits_{i=0}^{\infty} (W^i)_k = 0$ for $k \neq h$, and $\bigcap\limits_{i=0}^{\infty} W^i = S$ for $k = h$. So $\dim_{\mathrm{K}} W_k < \infty$ implies that there is an i such that $(W^i)_k = 0$ for $k < h$ and S for $k = h$, which contradicts $\mathrm{Hom}_A(W^i,S) = 0$. We proved that $\mathrm{Ext}^1_\Lambda(Z,S) = 0$, thus (6) gives $\mathrm{Hom}_\Lambda(X,S) = 0$. $\square$

Proposition 4.3. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence, where $Y \in \mathcal{F}(\overline{\Delta}^{\circ}(1))$, and X is generated by Λf_1. Then both X and Z are $\overline{\Delta}^{\circ}(1)$-filtered.*

Proof. Let $Y = Y^0 \supseteq Y^1 \supseteq \dots$ be a $\overline{\Delta}^{\circ}(1)$-filtration. To prove that X is $\overline{\Delta}^{\circ}(1)$-filtered, we can show by induction that the terms in the chain of modules $X = X \cap Y^0 \supseteq X \cap Y^1 \supseteq \dots$ are generated by Λf_1 and the factors are isomorphic to $\overline{\Delta}^{\circ}(1)$. Indeed, if $X \cap Y^i$ is generated by Λf_1, then the factor module $(X \cap Y^i)/(X \cap Y^{i+1}) \cong (Y^{i+1} + (X \cap Y^i))/Y^{i+1}$, which is also generated by Λf_1, is embeddable into

$Y^i/Y^{i+1} \cong \overline{\Delta}^{\circ}(1)$, so it is either 0, or is isomorphic to $\overline{\Delta}^{\circ}(1)$. Then Proposition 4.2 implies that $X \cap Y^{i+1}$ is generated by Λf_1.

Next we show that the image of the chain $Y = X + Y^0 \supseteq X + Y^1 \supseteq \ldots$ gives a $\overline{\Delta}^{\circ}(1)$-filtration of Z. The modules $X + Y^i$ are Λf_1-generated, since X and Y^i are Λf_1-generated by Lemma 4.1. The factor $(X + Y^i)/(X + Y^{i+1}) \cong Y^i/(Y^i \cap (X + Y^{i+1}))$ is a homomorphic image of $Y^i/Y^{i+1} \cong \overline{\Delta}^{\circ}(1)$, where the kernel is $(Y^i \cap (X + Y^{i+1}))/Y^{i+1} = ((Y^i \cap X) + Y^{i+1})/Y^{i+1}$, and this is, by the first part of the proof, generated by Λf_1. So the kernel can only be 0 or Y^i/Y^{i+1}, consequently the factor is either isomorphic to $\overline{\Delta}^{\circ}(1)$ or 0.

It remains to be shown that $\bigcap(X + Y^i) = X$. Let x be an element of the intersection, which is in Y_h. Since the homogeneous parts of the graded module Y are finite dimensional, there is an i such that $Y^i \subseteq (Y)_{>h}$, hence $x \in X + Y^i$ implies that $x \in X$. □

Lemma 4.4. *A module $X \in \Lambda\text{-}\operatorname{grfmod}$ is $\overline{\Delta}^{\circ}(1)$-filtered if and only if the factors of the sequence*

$$X = \Lambda f_1(X)_{\geqslant t} \supseteq \Lambda f_1(X)_{\geqslant t+1} \supseteq \ldots \supseteq \Lambda f_1(X)_{\geqslant h} \supseteq \ldots \tag{7}$$

have finite $\overline{\Delta}^{\circ}(1)$-filtrations, or equivalently,

$$\Lambda f_1(X)_{\geqslant h}/\Lambda f_1(X)_{\geqslant h+1} \cong \bigoplus \overline{\Delta}^{\circ}(1) \quad \textit{for every } h.$$

P r o o f. If the factors have finite $\overline{\Delta}^{\circ}(1)$-filtrations, then the chain of modules in (7) can be refined to a $\overline{\Delta}^{\circ}(1)$-filtration of X.

On the other hand, if $X \in \mathcal{F}(\overline{\Delta}^{\circ}(1))$, then the factors of the sequence (7) are $\overline{\Delta}^{\circ}(1)$-filtered by Proposition 4.3, while $\dim_{\mathrm{K}} f_1\big(\Lambda f_1(X)_{\geqslant h}/\Lambda f_1(X)_{\geqslant h+1}\big) = \dim_{\mathrm{K}} f_1 X_h < \infty$ shows that they, in fact, have finite $\overline{\Delta}^{\circ}(1)$-filtrations.

For the second equivalence, let $0 \to \Omega \to P \to Z \to 0$ be the projective cover of a factor Z of the sequence (7). Then $Z = \Lambda f_1 Z$ gives $P = \bigoplus \Lambda f_1$, where $\Omega \subseteq (P)_{\geqslant 1}$ is generated by Λf_1 according to Proposition 4.2. So $\Omega \subseteq \Lambda f_1(P)_{\geqslant 1}$, while $\Lambda f_1(Z)_{\geqslant 1} = 0$ yields $\Lambda f_1(P)_{\geqslant 1} \subseteq \Omega$, thus $Z \cong P/\Lambda f_1(P)_{\geqslant 1} \cong \bigoplus \overline{\Delta}^{\circ}(1)$. □

Proposition 4.5. *If $0 \to X \to Y \to Z \to 0$ is a short exact sequence with X and Z both $\overline{\Delta}^{\circ}(1)$-filtered, then Y is also $\overline{\Delta}^{\circ}(1)$-filtered.*

P r o o f. We need to show that the factors of the chain of modules

$$Y = \Lambda f_1(Y)_{\geqslant t} \supseteq \Lambda f_1(Y)_{\geqslant t+1} \supseteq \ldots \supseteq \Lambda f_1(Y)_{\geqslant h} \supseteq \ldots.$$

have finite $\overline{\Delta}^{\circ}(1)$-filtrations.

For every index $h \geqslant 0$ we can form the short exact sequence

$$0 \to (X)_{\geqslant h} \cap \Lambda f_1(Y)_{\geqslant h} \to \Lambda f_1(Y)_{\geqslant h} \to \Lambda f_1(Z)_{\geqslant h} \to 0. \tag{8}$$

Since $\Lambda f_1(Z)_{\geqslant h} \in \mathcal{F}(\overline{\Delta}^{\circ}(1))$ and $\Lambda f_1(Y)_{\geqslant h}$ is generated by Λf_1, Proposition 4.2 gives that $(X)_{\geqslant h} \cap \Lambda f_1(Y)_{\geqslant h} = \Lambda f_1((X)_{\geqslant h} \cap \Lambda f_1(Y)_{\geqslant h}) = \Lambda f_1(X)_{\geqslant h}$. Therefore, we can rewrite (8) as

$$0 \to \Lambda f_1(X)_{\geqslant h} \to \Lambda f_1(Y)_{\geqslant h} \to \Lambda f_1(Z)_{\geqslant h} \to 0,$$

so we get the short exact sequences

$$\begin{aligned} 0 \to \Lambda f_1(X)_{\geqslant h}/\Lambda f_1(X)_{\geqslant h+1} &\to \Lambda f_1(Y)_{\geqslant h}/\Lambda f_1(Y)_{\geqslant h+1} \\ &\to \Lambda f_1(Z)_{\geqslant h}/\Lambda f_1(Z)_{\geqslant h+1} \to 0, \end{aligned}$$

where the first and third modules have finite $\overline{\Delta}^{\circ}(1)$-filtrations, providing finite $\overline{\Delta}^{\circ}(1)$-filtrations for the middle terms. By Lemma 4.4, this proves that Y is $\overline{\Delta}^{\circ}(1)$-filtered. $\square$

5. Δ-filtered algebras

In this section, we shall prove that the Ext_A^*-images of the modules of $r\mathcal{K}$ are filtered by left proper standard modules of A^*, when A is a standard Koszul standardly stratified algebra (s.K.s.s. algebra, for short).

For an easier reference, let us quote two lemmas from [2], which will be used repeatedly in the sequel.

Lemma 5.1. *Let $0 \to X \to Y \to Z \to 0$ be exact with the map $X \to Y$ a top embedding. If $X \in \mathcal{C}$, then the induced sequence of graded left A^*-modules $0 \to \mathrm{Ext}_A^*(Z) \to \mathrm{Ext}_A^*(Y) \to \mathrm{Ext}_A^*(X) \to 0$ is also exact with morphisms of degree* 0.

Lemma 5.2. *Let $0 \to X \to Y \to Z \to 0$ be exact with $X \subseteq \operatorname{rad} Y$. If $Y \in \mathcal{C}$, then the induced sequence of graded left A^*-modules $0 \to \mathrm{Ext}_A^*(X)[1] \to \mathrm{Ext}_A^*(Z) \to \mathrm{Ext}_A^*(Y) \to 0$ is also exact with morphisms of degree* 0.

Proposition 5.3. *If A is s.K.s.s. and $X \in \mathcal{K}_2$, then $X^*/A^*f_1X^* \cong (X\varepsilon_2)^*$.*

Proof. In view of Propositions 3.11 and 3.12, we only need to show that the modules $Y_h(X)$ defined in Proposition 3.5 by the short exact sequences

$$0 \to \widetilde{\omega}_h(X) \xrightarrow{\alpha_h} \Omega_h(X) \xrightarrow{\beta_h} Y_h(X) \to 0 \tag{9}$$

are in $\mathcal{K}_2$, since by Proposition 2.7 of [10] this will imply that $Y_h(X)$ is $\widehat{S}\varepsilon_2A$-Koszul.

Since $X \in \mathcal{K}_2$, its hth syzygy $\Omega_h(X)$ also lies in $\mathcal{K}_2$ by Proposition 2.6 of [10]. In particular, $\Omega_h(X)\varepsilon_2A$ is a top submodule of $\Omega_h(X)$. Hence, we can apply Lemma 1.6 of [10] to sequence (9) to get $Y_h(X)\varepsilon_2A \overset{t}{\leqslant} Y_h(X)$. Note that $Y_0(X) = \overline{X} \in \mathcal{F}(S(1)) \subseteq \mathcal{K}_2$, so it suffices to prove that $Y_h(X) \in \mathcal{K}_2$ implies $Y_{h+1}(X)\varepsilon_2 \in \mathcal{C}_{C_2}$.

In the short exact sequence (5) of Proposition 3.12:

$$0 \to \overline{\omega}_{h+1}(X) \to Y_{h+1}(X) \to \Omega(Y_h(X)) \to 0,$$

$\overline{\omega}_{h+1}(X) \in \mathcal{F}(S(1))$, so $Y_{h+1}(X)\varepsilon_2 \cong \Omega(Y_h(X))\varepsilon_2$. By the inductive hypothesis $Y_h(X) \in \mathcal{K}_2$, thus $\Omega(Y_h(X)) \in \mathcal{K}_2$ by Proposition 2.6 of [10], consequently $Y_{h+1}(X)\varepsilon_2 \cong \Omega(Y_h(X))\varepsilon_2 \in \mathcal{C}_{C_2}$. □

Applying Proposition 5.3 recursively, we immediately get the trace filtration of X^* for modules X of $r\mathcal{K}$.

Theorem 5.4. *If A is s.K.s.s. and $X \in r\mathcal{K}$, then $X^*/A^*(f_1 + \ldots + f_{i-1})X^* \cong (X\varepsilon_i)^*$ for all $i \geqslant 1$.*

Lemma 5.5. *If A is s.K.s.s., then $A^*/A^*f_1A^* \cong C_2^*$ as algebras.*

Proof. By Theorem 2.1, the module $\widehat{S}$ belongs to $\mathcal{K}_2$, so we can apply Proposition 5.3 to this module to get the isomorphism ${}_{A^*}A^*/A^*f_1{}_{A^*}A^* \cong {}_{A^*}C_2^*$ of (left) A^*-modules, which implies the required isomorphism of algebras. □

Lemma 5.6. *If A is s.K.s.s. and $X \in \mathcal{F}(S(1))$, then $X^* = A^*f_1X^*$.*

Proof. Clearly, $X \in \mathcal{K}_2$, so $X^*/A^*f_1X^* \cong (X\varepsilon_2)^* = 0$ by Proposition 5.3. □

Theorem 5.7. *If A is s.K.s.s., then right standard A-modules are mapped to left proper standard A^*-modules, and left proper standard A-modules are mapped to right standard A^*-modules by the functor* Ext^*_A, *that is,* $\operatorname{Ext}^*_A(\Delta^\circ(i)) \cong \overline{\Delta}_{A^*}(i)$ *and* $\operatorname{Ext}^*_A(\overline{\Delta}^\circ(i)) \cong \Delta_{A^*}(i)$.

Proof. We provide here the proof only for right standard modules. The statement about the left proper standard modules can be proved similarly. Applying Theorem 2.1, we use induction on the number of simple modules.

For a local algebra, the module $\Delta(1)$ is projective and $\operatorname{Ext}^*_A(\Delta(1)) = S^\circ_{A^*}(1) = \overline{\Delta}^\circ_{A^*}(1)$. So we may assume that A is not local and the statement holds for C_2. We recall that $\operatorname{Ext}^h_A(\Delta(i), S(j)) = 0$ for all $h \geqslant 0$ and $i \geqslant j$. Besides, it is easy to see that $\Delta(i) \in \mathcal{K}$.

Suppose that $i \geqslant 2$. Then $\operatorname{Ext}^*_{C_2}(\Delta(i)\varepsilon_2) \cong \operatorname{Ext}^*_{C_2}(\Delta_{C_2}(i))$ and they are isomorphic to $\overline{\Delta}^\circ_{C_2^*}(i)$ by the inductive hypothesis. On the other hand, $A^* f_1 \operatorname{Ext}^*_A(\Delta(i)) = 0$ because $\operatorname{Ext}^h_A(\Delta(i), S(1)) = 0$ for all $h \geqslant 0$, so we get $\overline{\Delta}^\circ_{C_2^*}(i) \cong \overline{\Delta}^\circ_{A^*}(i)$ as A^*-modules, since $C_2^* \cong A^*/A^* f_1 A^*$ by Lemma 5.5. Finally, Proposition 5.3 yields $\operatorname{Ext}^*_A(\Delta(i)) \cong \overline{\Delta}^\circ_{A^*}(i)$.

It is left to be shown that $\operatorname{Ext}^*_A(\Delta(1)) \cong \overline{\Delta}^\circ_{A^*}(1)$. Since $\Delta(1) \in \mathcal{K}$, the module $\operatorname{Ext}^*_A(\Delta(1))$ is a graded module generated in degree 0. It is also clear that it has a one-dimensional degree 0 part, and since $\operatorname{Ext}^h_A(\Delta(1), S(1)) = 0$ if $h \geqslant 1$, we see that $\operatorname{Ext}^*_A(\Delta(1))$ is a homomorphic image of $\overline{\Delta}^\circ_{A^*}(1)$. Consider the Ext^*_A-image of the short exact sequence $0 \to \operatorname{rad}\Delta(1) \to \Delta(1) \to S(1) \to 0$, which is the exact sequence

$$0 \to \operatorname{Ext}^*_A(\operatorname{rad}\Delta(1))[1] \to \operatorname{Ext}^*_A(S(1)) \to \operatorname{Ext}^*_A(\Delta(1)) \to 0$$

in A^*-grmod by Lemma 5.2. This sequence shows that there is an epimorphism $P^\circ_{A^*}(1) \to \operatorname{Ext}^*_A(\Delta(1))$, whose kernel is isomorphic to $\operatorname{Ext}^*_A(\operatorname{rad}\Delta(1))$. By Lemma 5.6, $\operatorname{Ext}^*_A(\operatorname{rad}\Delta(1)) = A^* f_1 \operatorname{Ext}^*_A(\operatorname{rad}\Delta(1))$ because $\operatorname{rad}\Delta(1)$ is in $\mathcal{F}(S(1))$. Thus $\operatorname{Ext}^*_A(\Delta(1)) \cong \overline{\Delta}^\circ_{A^*}(1)$. □

Next, we want to show that $r\mathcal{K}$ is mapped into $\mathcal{F}(\overline{\Delta}^\circ_{A^*})$. In particular, this will imply that A^* is a standardly stratified algebra with respect to the opposite order of idempotents. In the proof, we use induction on the number of simple modules, so for the induction step we need to show that for $X \in \mathcal{K}$ the trace of the first projective A^*-module in X^* is filtered by $\overline{\Delta}^\circ_{A^*}(1)$.

Lemma 5.8. *If A is s.K.s.s. and $X \in \mathcal{F}(S(1))$, then $A^* f_1 X^*$ is filtered by $\overline{\Delta}^\circ_{A^*}(1)$.*

Proof. First, we observe that if $X \in \mathcal{F}(S(1))$, then X has a Δ-cover. That is, there exists an epimorphism $\Delta(X) \to X$ such that its kernel is contained in $\operatorname{rad}\Delta(X)$

and $\Delta(X)$ is isomorphic to a direct sum of copies of $\Delta(1)$. Indeed, if we take the projective cover $P(X) \to X$, then it factors through $P(X) \to P(X)/P(X)\varepsilon_2 A \cong \bigoplus \Delta(1)$.

Let us apply the functor Ext^*_A to the short exact sequence

$$0 \to X' \to \Delta(X) \to X \to 0.$$

This yields the exact sequence

$$0 \to (X')^*[1] \to X^* \to (\Delta(X))^* \to 0$$

by Lemma 5.2. Since X' also belongs to $\mathcal{F}(S(1))$, we can continue the procedure to get

$$X^* \supseteq (X')^* \supseteq (X'')^* \supseteq \ldots \supseteq (X^{(i)})^* \supseteq \ldots,$$

where $(X^{(i)})^*$ is identified with its image in $(X^*)_{\geqslant i}$. Thus, the intersection of the chain is 0 and the factors are isomorphic to $\mathrm{Ext}^*_A(\Delta(X^{(i)})) \cong \bigoplus \overline{\Delta}^{\circ}_{A^*}(1)$. $\square$

Proposition 5.9. *Suppose that A is s.K.s.s. and $X \in \mathcal{K}_2$. Then the short exact sequence $0 \to \widetilde{X} \to X \to \overline{X} \to 0$ yields an exact sequence in A^*-* grfmod

$$0 \to N[1] \to A^* f_1 \overline{X}^* \to A^* f_1 X^* \to A^* f_1 \widetilde{X}^* \to N \to 0 \tag{10}$$

with morphisms of degree 0 *and* $N = A^* f_1 N$.

Proof. We apply $\mathrm{Hom}_A(-, \widehat{S})$ to $0 \to \widetilde{X} \to X \to \overline{X} \to 0$ and get the long exact sequence

$$\ldots \xrightarrow{\delta_h} \mathrm{Ext}^h_A(\overline{X}, \widehat{S}) \to \mathrm{Ext}^h_A(X, \widehat{S}) \to \mathrm{Ext}^h_A(\widetilde{X}, \widehat{S}) \xrightarrow{\delta_{h+1}} \mathrm{Ext}^{h+1}_A(\overline{X}, \widehat{S}) \to \ldots.$$

The sequence $\overline{X}^* \to X^* \to \widetilde{X}^*$ is exact and we may add to it the respective kernel and cokernel to get

$$0 \to N[1] \to \overline{X}^* \to X^* \to \widetilde{X}^* \to N \to 0,$$

where N is the graded left A^*-module whose degree h part is

$$\begin{aligned} N_h &= \mathrm{coker}(\mathrm{Ext}^h_A(X, \widehat{S}) \to \mathrm{Ext}^h_A(\widetilde{X}, \widehat{S})) \\ &= \ker(\mathrm{Ext}^{h+1}_A(\overline{X}, \widehat{S}) \to \mathrm{Ext}^{h+1}_A(X, \widehat{S})). \end{aligned}$$

We still need to show that $A^*f_1N = N$. Since both X and $\widetilde{X}$ are in $\mathcal{K}_2$, we can apply Proposition 3.12 to get $X^*/A^*f_1X^* \cong (X\varepsilon_2)^* \cong (\widetilde{X}\varepsilon_2)^* \cong \widetilde{X}^*/A^*f_1\widetilde{X}^*$. Hence, we have the following commutative exact diagram:

$$\begin{tikzcd}
 & A^*f_1X^* \arrow[r] \arrow[d] & A^*f_1\widetilde{X}^* \arrow[r] \arrow[d] & N \arrow[d] & \\
 & X^* \arrow[r] \arrow[d, "q_X"] & \widetilde{X}^* \arrow[r] \arrow[d, "q_{\widetilde{X}}"] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (X\varepsilon_2)^* \arrow[r] \arrow[d] & (\widetilde{X}\varepsilon_2)^* \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

The snake lemma gives us that $A^*f_1\widetilde{X}^* \to N$ is an epimorphism and so $N = A^*f_1N$. Finally, we can extend the upper row to get

$$0 \to N[1] \to \overline{X}^* \to A^*f_1X^* \to A^*f_1\widetilde{X}^* \to N \to 0,$$

where $\overline{X}^* \in \mathcal{F}(\overline{\Delta}^\circ(1))$ by Lemma 5.8, so Lemma 4.1 gives $\overline{X}^* = A^*f_1\overline{X}^*$. □

Theorem 5.10. *If A is s.K.s.s. and $X \in \mathcal{K}_2$, then $A^*f_1X^* \in \mathcal{F}(\overline{\Delta}^\circ_{A^*}(1))$.*

P r o o f. Consider the following chain of submodules:

$$A^*f_1X^* \supseteq A^*f_1(X^*)_{\geqslant 1} \supseteq \ldots \supseteq A^*f_1(X^*)_{\geqslant h} \supseteq \ldots.$$

We claim that the factor modules

$$A^*f_1(X^*)_{\geqslant h}/A^*f_1(X^*)_{\geqslant h+1} \cong A^*f_1\Omega_h(X)^*/A^*f_1(\Omega_h(X)^*)_{\geqslant 1}$$

are isomorphic to finite direct powers of $\overline{\Delta}^\circ_{A^*}(1)$. As Proposition 2.6 of [10] implies that $\Omega_h(X) \in \mathcal{K}_2$ for all $h \geqslant 0$, it suffices to deal with the case $h = 0$. For this, we show the isomorphism

$$A^*f_1X^*/A^*f_1(X^*)_{\geqslant 1} \cong A^*f_1\overline{X}^*/A^*f_1(\overline{X}^*)_{\geqslant 1}. \tag{11}$$

Consider sequence (10) for the module X. Then $(N[1])_0 = 0$, and by Proposition 5.9, $N[1] = A^*f_1N[1]$, so we have $N[1] \subseteq A^*f_1(\overline{X}^*)_{\geqslant 1} \cong A^*f_1\Omega(\overline{X})^*$. The space $(A^*f_1\widetilde{X}^*)_0 = \operatorname{Hom}_A(\widetilde{X}, S(1))$ is zero, thus the map $A^*f_1\overline{X}^* \to A^*f_1X^*$ induces an isomorphism

$$A^*f_1\overline{X}^*/(A^*f_1\overline{X}^*)_{\geqslant 1} \cong A^*f_1X^*/(A^*f_1X^*)_{\geqslant 1}$$

and these modules are isomorphic to a direct power $(S^{\circ}_{A^*}(1))^t$. Thus, the projective cover $(P^{\circ}_{A^*})^t \to A^*f_1X^*/A^*f_1(X^*)_{\geqslant 1}$ can be factored through $(\overset{\circ}{\overline{\Delta}}_{A^*}(1))^t$, which is isomorphic to $A^*f_1\overline{X}^*/A^*f_1(\overline{X}^*)_{\geqslant 1}$ by Lemmas 5.8 and 4.4. So

$$A^*f_1\overline{X}^*/A^*f_1(\overline{X}^*)_{\geqslant 1} \longrightarrow A^*f_1X^*/A^*f_1(X^*)_{\geqslant 1} \tag{12}$$

is a graded epimorphism of degree 0.

Since $\overline{X} \in \mathcal{F}(S(1)) \subset \mathcal{K}_2$, its syzygy $\Omega(\overline{X}) \in \mathcal{K}_2$ according to Proposition 2.6 of [10], so $\Omega(\overline{X})^*/A^*f_1\Omega(\overline{X})^* \cong \big(\Omega(\overline{X})\varepsilon_2\big)^*$ by Proposition 5.3.

For the sequence $0 \to \widetilde{X} \to X \to \overline{X} \to 0$ (with $\widetilde{X} \overset{t}{\leqslant} X$), the horseshoe lemma gives the exact sequence $0 \to \omega(X) \to \Omega(X) \to \Omega(\overline{X}) \to 0$ of the syzygies. Apply $\mathrm{Hom}(\varepsilon_2 A, -)$ to get $0 \to \omega(X)\varepsilon_2 \to \Omega(X)\varepsilon_2 \to \Omega(\overline{X})\varepsilon_2 \to 0$, where $\omega(X)\varepsilon_2 = \widetilde{\omega}(X)\varepsilon_2 \cong \Omega(X\varepsilon_2) \in \mathcal{C}_{C_2}$. Since $\widetilde{\omega}(X) \to \Omega(X)$ is a top embedding by Lemma 3.2, $\widetilde{\omega}(X)\varepsilon_2$ is a top submodule of $\Omega(X)\varepsilon_2$ according to Lemma 1.4 of [10]. By Lemma 5.1, the last sequence is mapped by $\mathrm{Ext}^*_{C_2}$ to the exact sequence

$$0 \to \big(\Omega(\overline{X})\varepsilon_2\big)^* \to (\Omega(X)\varepsilon_2)^* \to (\Omega(X\varepsilon_2))^* \to 0.$$

Thus, we found an injective graded morphism of degree 0 from

$$\big(\Omega(\overline{X})\varepsilon_2\big)^* \cong \Omega(\overline{X})^*/A^*f_1\Omega(\overline{X})^* \cong (A^*f_1\overline{X}^*/A^*f_1(\overline{X}^*)_{\geqslant 1})_{\geqslant 1}$$

to

$$(\Omega(X)\varepsilon_2)^* \cong \Omega(X)^*/A^*f_1\Omega(X)^* \cong (A^*f_1X^*/A^*f_1(X^*)_{\geqslant 1})_{\geqslant 1}.$$

But the epimorphism in (12) induces an epimorphism from the former to the latter, so taking into account that all levels of the modules have finite dimension, these factor modules must be isomorphic as stated in (11). Then Lemmas 5.8 and 4.4 finish the proof. □

Theorem 5.11. *If A is a standard Koszul standardly stratified algebra and $X \in r\mathcal{K}$, then $X^* \in \mathcal{F}(\overset{\circ}{\overline{\Delta}}_{A^*})$. In particular, if X is a top extension of simple and standard modules, then X^* is $\overset{\circ}{\overline{\Delta}}_{A^*}$-filtered.*

P r o o f. The first statement follows by induction using Theorems 5.4 and 5.10, while the second is a consequence of Proposition 3.4 because simple and standard modules obviously belong to $r\mathcal{K}$. □

Theorem 5.12. *If A is a standard Koszul standardly stratified algebra, then its homological dual A^* is a standardly stratified algebra.*

P r o o f. Semisimple A-modules belong to $r\mathcal{K}$, thus ${}_{A^*}A^* = \widehat{S}^* \in \mathcal{F}(\overline{\Delta}^{\circ}_{A^*})$. $\square$

6. $\overline{\Delta}$-filtered algebras

In this section, we focus on the left module category of a standard Koszul standardly stratified algebra. To keep our notation simple, we investigate the right modules over an algebra A, whose opposite algebra A° is a standard Koszul standardly stratified algebra, so $A_A \in \mathcal{F}(\overline{\Delta})$.

We would like to prove theorems analogous to those of the previous section. However, to handle the asymmetry of the left and the right module category of A, we have to consider a narrower subclass $\mathcal{K}^+ \subseteq \mathcal{K}$ of modules. It is defined with additional restrictions as

$$\mathcal{K}^+ = \{X \in \mathcal{K}\colon\ \widetilde{\omega}_h(X) \in \mathcal{C}_A \text{ and } \overline{\omega}_h(X) \cong \bigoplus S(1) \text{ for all } h \geqslant 0\}.$$

We also introduce the recursive version of $\mathcal{K}^+$ as

$$r\mathcal{K}^+ = \{X \in \mathcal{K}^+\colon\ X\varepsilon_i \in \mathcal{K}^+_{C_i} \text{ for all } i\}.$$

We shall prove that the functor Ext^*_A maps the subclass $r\mathcal{K}^+$ into $\mathcal{F}(\Delta^\circ_{A^*})$. Furthermore, we show that $r\mathcal{K}^+$ is closed under top extensions and also that simple and proper standard modules belong to this class.

Lemma 6.1. *If A° is s.K.s.s. and $X \in \mathcal{K}^+$, then $\omega(X)$ and $\widetilde{\omega}(X)$ also belong to $\mathcal{K}^+$.*

P r o o f. According to Corollary 3.3, both modules $\omega(X)$ and $\widetilde{\omega}(X)$ are in $\mathcal{K}_2$. By definition, $\widetilde{\omega}(X)$ is also Koszul, and it is a top submodule of $\omega(X)$. So we have the exact sequence

$$0 \to \widetilde{\omega}(X) \to \omega(X) \to \overline{\omega}(X) \to 0$$

with a top embedding, where $\widetilde{\omega}(X)$ and $\overline{\omega}(X) \cong \bigoplus S(1)$ are Koszul, so their top extension $\omega(X)$ is also Koszul by Lemma 2.4 of [2]. The remaining conditions hold by the recursive definition of ω_h. $\square$

Proposition 6.2. *If A° is s.K.s.s., the classes $\mathcal{K}^+$ and $r\mathcal{K}^+$ are closed under top extensions.*

Proof. Suppose that $X, Z \in \mathcal{K}^+$ and we have the short exact sequence

$$0 \to X \xrightarrow{t} Y \to Z \to 0$$

with a top embedding. First we show that in this case, $\widetilde{Y}$ is a top extension of $\widetilde{Z}$ by $\widetilde{X}$. As $\widetilde{X} \overset{t}{\leqslant} Y$, the sequence $0 \to X/\widetilde{X} \to Y/\widetilde{X} \to Z \to 0$ is a top extension (cf. Lemma 1.3 of [10]). The first term is a direct sum of copies of $S(1)$, so the sequence splits and we get $Y/\widetilde{X} \cong \overline{X} \oplus Z$. This yields $\overline{Y} \cong \overline{X} \oplus \overline{Z} \cong \bigoplus S(1)$ and it also implies $\widetilde{Y}/\widetilde{X} \cong \widetilde{Z}$. That is, the sequence

$$0 \to \widetilde{X} \to \widetilde{Y} \to \widetilde{Z} \to 0 \tag{13}$$

is exact, where $\widetilde{X} \overset{t}{\leqslant} \widetilde{Y}$, so $\widetilde{Y} \in \mathcal{C}_A$ according to Lemma 2.4 of [2]. The application of the horseshoe lemma to the sequence (13) gives the short exact sequence $0 \to \omega(X) \to \omega(Y) \to \omega(Z) \to 0$ of the syzygies. By the Koszul property of $\widetilde{X}$, it is a top extension. Using Lemma 6.1, we can show by induction that $\widetilde{\omega}_h(Y)$ and $\omega_h(Y)$ satisfy the prescribed conditions of $\mathcal{K}^+$ for every h. Finally, (13) gives a top extension $0 \to X\varepsilon_2 \to Y\varepsilon_2 \to Z\varepsilon_2 \to 0$ by Lemma 1.4 of [10], so a recursive argument shows that $Y \in r\mathcal{K}^+$. $\square$

Proposition 6.3. *If A° is s.K.s.s. and $X \in \mathcal{K}^+$, then $X^*/A^*f_1X^* \cong (X\varepsilon_2)^*$.*

Proof. In view of Propositions 3.11 and 3.12, it is enough to show that the modules $Y_h(X)$ defined in Proposition 3.5 by the short exact sequences

$$0 \to \widetilde{\omega}_h(X) \xrightarrow{\alpha_h} \Omega_h(X) \xrightarrow{\beta_h} Y_h(X) \to 0 \tag{14}$$

are Koszul for all h. We prove this by induction on h. The module $Y_0(X) = \overline{\omega}_0(X) = \overline{X}$ is semisimple, hence Koszul. Now we assume that $Y_h(X) \in \mathcal{C}_A$. By assumption, $X \in \mathcal{K}^+$, so $\widetilde{\omega}_h(X)$ is Koszul for all h. If we apply Lemma 5.1 to the sequence (14), we get that $\Omega_h(X)^* \to \widetilde{\omega}_h(X)^*$ is an epimorphism, in particular, $\operatorname{Hom}_A(\Omega_{h+1}(X), \widehat{S}) \to \operatorname{Hom}_A(\omega_{h+1}(X), \widehat{S})$ is surjective. It means that in the induced sequence of the syzygies

$$0 \to \omega_{h+1}(X) \to \Omega_{h+1}(X) \to \Omega(Y_h(X)) \to 0$$

we also get a top embedding. If we factor out the submodule $\widetilde{\omega}_{h+1}(X)$ (which is a top submodule both in the first and the middle terms), then by Lemma 1.3 of [10], we get that the sequence

$$0 \to \overline{\omega}_{h+1}(X) \to Y_{h+1}(X) \to \Omega(Y_h(X)) \to 0$$

also has a top embedding. The first term is semisimple, hence Koszul, and $\Omega(Y_h(X)) \in \mathcal{C}_A$ follows from the inductive hypothesis. By Lemma 2.4 of [2], their top extension $Y_{h+1}(X)$ is also in $\mathcal{C}_A$. $\square$

Applying the proposition recursively, we immediately get the trace filtration of X^* for modules X of $r\mathcal{K}^+$.

Theorem 6.4. *If A° is s.K.s.s. and $X \in r\mathcal{K}^+$, then $X^*/A^*(f_1+\ldots+f_{i-1})X^* \cong (X\varepsilon_i)^*$ for all $i \geqslant 1$.*

Lemma 6.5. *Suppose that A° is s.K.s.s., $X, Y \in \operatorname{mod} A$ and $Y \in \mathcal{F}(\nabla)$, i.e. Y is filtered by costandard modules. Then the map $\operatorname{Ext}_A^h(X,Y) \to \operatorname{Ext}_A^h(\widetilde{X},Y)$ induced by the natural embedding $\widetilde{X} \to X$ is an isomorphism for $h \geqslant 1$.*

Proof. We take the short exact sequence $0 \to \widetilde{X} \to X \to \overline{X} \to 0$ and apply the functor $\operatorname{Hom}_A(-,Y)$. In the long exact sequence

$$\ldots \to \operatorname{Ext}_A^h(\overline{X},Y) \to \operatorname{Ext}_A^h(X,Y) \to \operatorname{Ext}_A^h(\widetilde{X},Y) \to \operatorname{Ext}_A^{h+1}(\overline{X},Y) \to \ldots,$$

$\operatorname{Ext}_A^h(\overline{X},Y) = 0$ for $h \geqslant 0$ because $\operatorname{Ext}_A^h(S(1),\nabla(1)) = \operatorname{Ext}_A^h(\overline{\Delta}(1),\nabla(1)) = 0$ if A° is standardly stratified (cf. Theorem 3.1 of [3]). $\square$

Lemma 6.6. *Let $h \geqslant n$, where n is the number of simple A-modules. If A° is s.K.s.s. and $X \in \mathcal{K}_2$, then $\operatorname{Hom}_A(\omega_h(X), S(1)) = 0$. Consequently, $A^* f_1 \omega_n(X)^* = 0$.*

Proof. As $\mathcal{K}_2$ is closed under ω, we only have to deal with the case when $h = n$. Let $0 \to \omega_n(X) \to P(\widetilde{\omega}_{n-1}(X)) \to \widetilde{\omega}_{n-1}(X) \to 0$ be the first step of a projective resolution of $\widetilde{\omega}_{n-1}(X)$. Then $\operatorname{Hom}_A(P(\widetilde{\omega}_{n-1}(X)),\nabla(1)) = 0$ and so $\operatorname{Ext}_A^1(\widetilde{\omega}_{n-1}(X),\nabla(1)) \cong \operatorname{Hom}_A(\omega_n(X),\nabla(1))$. This and Lemma 6.5 yield

$$\begin{aligned}\operatorname{Hom}_A(\omega_n(X),\nabla(1)) &\cong \operatorname{Ext}_A^1(\widetilde{\omega}_{n-1}(X),\nabla(1)) \cong \operatorname{Ext}_A^1(\omega_{n-1}(X),\nabla(1)) \cong \ldots \\ \ldots &\cong \operatorname{Ext}_A^{n-1}(\omega(X),\nabla(1)) \cong \operatorname{Ext}_A^n(\widetilde{X},\nabla(1)).\end{aligned}$$

Since A° is standardly stratified, the injective dimension of $\nabla(1)$ is less than n (cf. Lemma 3.2 of [6]), giving $\operatorname{Hom}_A(\omega_n(X),\nabla(1)) \cong \operatorname{Ext}_A^n(\widetilde{X},\nabla(1)) = 0$. Thus $\operatorname{Hom}_A(\omega_n(X),S(1)) = 0$.

We obtained that $\omega_h(X) = \widetilde{\omega}_h(X)$ for all $h \geqslant n$, hence $\operatorname{Ext}_A^t(\omega_n(X),S(1)) \cong \operatorname{Hom}_A(\Omega_t(\omega_n(X)),S(1)) = \operatorname{Hom}_A(\omega_{n+t}(X),S(1)) = 0$ for $t \geqslant 0$, proving the second statement. $\square$

Theorem 6.7. *If A° is s.K.s.s. and $X \in r\mathcal{K}^+$, then $X^* \in \mathcal{F}(\Delta^\circ_{A^*})$.*

P r o o f. In view of Theorem 6.4, we only have to show that $A^* f_1 X^*$ is projective when $X \in \mathcal{K}^+$. Applying the functor Ext^*_A to the short exact sequence $0 \to \widetilde{X} \to X \to \overline{X} \to 0$ gives the exact sequence

$$0 \to \overline{X}^* \to X^* \to \widetilde{X}^* \to 0.$$

Since $\overline{X} = \bigoplus S(1)$, we have the exact sequence

$$0 \to A^* f_1 \overline{X}^* \to A^* f_1 X^* \to A^* f_1 \widetilde{X}^* \to 0,$$

where $A^* f_1 \overline{X}^*$ is projective. Furthermore, $\operatorname{Hom}_A(\widetilde{X}, S(1)) = 0$, so $A^* f_1 \widetilde{X}^* \cong A^* f_1 \Omega(\widetilde{X})^* = A^* f_1 \omega(X)^*$. We get that $A^* f_1 X^*$ is projective if $A^* f_1 \omega(X)^*$ is projective. We have seen in Lemma 6.1 that $\mathcal{K}^+$ is closed under ω, while $A^* f_1 \omega_n(X)^*$ is zero by Lemma 6.6. By induction, $A^* f_1 \omega_h(X)^*$ is also projective for all $0 \leqslant h \leqslant n$. $\square$

In the remaining part of this section, we want to show that $\overline{\Delta}(i) \in r\mathcal{K}^+$ and $S(i) \in r\mathcal{K}^+$ for all $i \geqslant 1$.

Theorem 6.8. *If A° is s.K.s.s., then the proper standard modules are in $r\mathcal{K}^+$.*

P r o o f. The centralizer algebras of A° are standard Koszul standardly stratified algebras and $\overline{\Delta}(i)\varepsilon_2 \cong \overline{\Delta}_{C_2}(i)$ for all i (see Theorem 2.1). This means that it is enough to see that $\overline{\Delta}(i) \in \mathcal{K}^+$ for all indices i.

If $i = 1$, then $\overline{\Delta}(1) = S(1) \in \mathcal{C}_A$, and $\omega_h(S(1)) = 0$ for $h \geqslant 1$. If $i \geqslant 2$, then $\operatorname{Ext}^h_A(\overline{\Delta}(i), S(1)) = 0$ for $h \geqslant 0$, so $\widetilde{\omega}_h(\overline{\Delta}(i)) = \omega_h(\overline{\Delta}(i)) = \Omega_h(\overline{\Delta}(i))$, which is Koszul by assumption, and we also have $\overline{\omega}_h(\overline{\Delta}(i)) = 0$. $\square$

Now, we focus on simple modules. Since $\overline{\Delta}(1) \cong S(1)$, it suffices to deal with simple modules S which are not isomorphic to $S(1)$. All simple A-modules belong to $\mathcal{K}_2$, so by Corollary 3.3, $\omega_h(S) \in \mathcal{K}_2$ for all h.

We consider the canonical embeddings $e^h \colon \widetilde{\omega}_h(S) \to \omega_h(S)$ and $i \colon S(1) \to \nabla(1)$. These morphisms give rise for every h to a commutative diagram:

(15)
$$\begin{tikzcd}
(\omega_{h+1}(S), S(1))^0 \arrow[r, "\cong"] \arrow[d, "i'", "\cong"'] & (\widetilde{\omega}_h(S), S(1))^1 \arrow[d, "\widetilde{i}"] & (\omega_h(S), S(1))^1 \arrow[l, two heads, "e'"'] \arrow[r, "\cong"] \arrow[d, "i'"] & {} \\
(\omega_{h+1}(S), \nabla(1))^0 \arrow[r, "\cong"] & (\widetilde{\omega}_h(S), \nabla(1))^1 & (\omega_h(S), \nabla(1))^1 \arrow[l, two heads, "\widetilde{e}"', "\cong"] \arrow[r, "\cong"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
{} & (\omega_1(S), S(1))^h \arrow[l, "e'"'] \arrow[r, "\cong"] \arrow[d, "i'"] & (S, S(1))^{h+1} \arrow[d, "\widetilde{i}"] \\
\cdots & (\omega_1(S), \nabla(1))^h \arrow[l, "\widetilde{e}"', "\cong"] \arrow[r, "\cong"] & (S, \nabla(1))^{h+1}
\end{tikzcd}$$

where $(X,Y)^k$ stands for $\operatorname{Ext}_A^k(X,Y)$ if $k > 0$, while $(X,Y)^0$ denotes the space $\operatorname{Hom}_A(X,Y)$. For simplicity, we also omit the indices of the maps in the diagram. Proposition 6.9 shows that in diagram (15), the marked morphisms are indeed epimorphisms and isomorphisms, respectively.

Proposition 6.9. *If A° is s.K.s.s., then the induced maps of the diagram* (15) *have the following properties:*

(1) $\widetilde{e}\colon \operatorname{Ext}_A^k(\omega_j(S),\nabla(1)) \to \operatorname{Ext}_A^k(\widetilde{\omega}_j(S),\nabla(1))$ *is an isomorphism for all* $k \geqslant 1$ *and* $j \geqslant 0$.

(2) *The maps* $\operatorname{Ext}_A^k(\omega_{j+1}(S),X) \to \operatorname{Ext}_A^{k+1}(\widetilde{\omega}_j(S),X)$ *are isomorphisms for all* $k,j \geqslant 0$ *if* $X \in \mathcal{F}(S(1))$, *in particular, when* $X = S(1)$ *or* $\nabla(1)$. *Consequently, the map* $\widetilde{i}\colon \operatorname{Ext}_A^1(\widetilde{\omega}_h(S),S(1)) \to \operatorname{Ext}_A^1(\widetilde{\omega}_h(S),\nabla(1))$ *is injective for all* $h \geqslant 0$.

(3) $\widetilde{i}\colon \operatorname{Ext}_A^k(\widetilde{\omega}_j(S),S(1)) \to \operatorname{Ext}_A^k(\widetilde{\omega}_j(S),\nabla(1))$ *and* $i'\colon \operatorname{Ext}_A^k(\omega_j(S),S(1)) \to \operatorname{Ext}_A^k(\omega_j(S),\nabla(1))$ *are epimorphisms for all* $j \geqslant 0$ *and* $k \geqslant 0$.

(4) $e'\colon \operatorname{Ext}_A^1(\omega_h(S),S(1)) \to \operatorname{Ext}_A^1(\widetilde{\omega}_h(S),S(1))$ *is surjective for all* $h \geqslant 0$.

P r o o f. (1) The first statement follows immediately from Lemma 6.5.

(2) Apply $\operatorname{Hom}_A(-,X)$ to $0 \to \omega_{j+1}(S) \to P(\widetilde{\omega}_j(S)) \to \widetilde{\omega}_j(S) \to 0$, which is the first step of the minimal projective resolution of $\widetilde{\omega}_j(S)$, to get

$$\begin{aligned}\ldots \to \operatorname{Ext}_A^k(P(\widetilde{\omega}_j(S)),X) &\to \operatorname{Ext}_A^k(\omega_{j+1}(S),X)\\ &\to \operatorname{Ext}_A^{k+1}(\widetilde{\omega}_j(S),X) \to \operatorname{Ext}_A^{k+1}(P(\widetilde{\omega}_j(S)),X) \to \ldots .\end{aligned}$$

Here $\operatorname{Ext}_A^k(P(\widetilde{\omega}_j(S),X) = 0$ if $k \geqslant 1$ because $P(\widetilde{\omega}_j(S))$ is projective, and $\operatorname{Hom}_A(P(\widetilde{\omega}_j(S)),X) = 0$ since $P(\widetilde{\omega}_j(S)) = P(\widetilde{\omega}_j(S))\varepsilon_2 A$. These give the required isomorphisms, while the left exactness of $\operatorname{Hom}_A(\omega_{j+1}(S),-)$ implies the second part.

(3) First, we note that as $\widetilde{e}$ is an isomorphism, the surjectivity of i' implies the surjectivity of $\widetilde{i}$ for every pair (k,j). Thus, we may prove the surjectivity of the two maps simultaneously. We use induction on j.

The algebra A° is standard Koszul, so the left module $\Delta^\circ(1)$ lies in $\mathcal{C}_{A^\circ}$. In view of Proposition 2.7 of [2] (or rather its "K-dual version"), $\Delta^\circ(1) \in \mathcal{C}_{A^\circ}$ implies that the natural maps $\operatorname{Ext}_A^k(S,S(1)) \to \operatorname{Ext}_A^k(S,\nabla(1))$ are epimorphisms for all k. This provides the base case $(k,0)$ of the induction.

Suppose that the statement is proved for the pair $(k+1,j-1)$. The inductive hypothesis gives the surjectivity of $\widetilde{i}$, and hence the surjectivity of i' in the diagram below.

$$\begin{tikzcd}
\operatorname{Ext}_A^k(\omega_j(S),S(1)) \arrow[r, "\cong"] \arrow[d, "i'"] & \operatorname{Ext}_A^{k+1}(\widetilde{\omega}_{j-1}(S),S(1)) \arrow[d, "\widetilde{i}"] \\
\operatorname{Ext}_A^k(\omega_j(S),\nabla(1)) \arrow[r, "\cong"] & \operatorname{Ext}_A^{k+1}(\widetilde{\omega}_{j-1}(S),\nabla(1))
\end{tikzcd}$$

(4) The fourth statement is a consequence of the first three. □

Proposition 6.10. *Let A° be s.K.s.s. and S a simple A-module not isomorphic to $S(1)$. The homomorphism $\alpha_{k-1,0}\colon \omega_{k+h}(S) \to \Omega_k(\omega_h(S))$ induced by α_{k-1} of formula* (14) *applied to $X = \omega_h(S)$ is a top embedding for all k.*

P r o o f. Let $k \geqslant 1$ be arbitrary. The map $\alpha_{k-1}\colon \widetilde{\omega}_{k+h-1}(S) \to \Omega_{k-1}(\omega_h(S))$ is a top embedding by Proposition 3.5, and this implies that $\Omega(\widetilde{\omega}_{k+h-1}(S)) = \omega_{k+h}(S)$ is mapped into $\Omega_k(\omega_h(S))$ injectively.

To see that $\alpha_{k-1,0}$ is a top embedding, we will show that the induced map $\alpha^*_{k-1,0}\colon \mathrm{Hom}_A(\Omega_k(\omega_h(S)), \widehat{S}) \to \mathrm{Hom}_A(\omega_{k+h}(S), \widehat{S})$ is surjective. By Proposition 3.5, the restriction of $\alpha_{k-1,0}$ to $\widetilde{\omega}_{k+h}(S) \subseteq \omega_{k+h}(S)$ is a top embedding, or what is equivalent, $\mathrm{Hom}_A(\Omega_k(\omega_h(S)), \widehat{S}\varepsilon_2 A) \overset{\alpha^*_{k-1,0}}{\longrightarrow} \mathrm{Hom}_A(\omega_{k+h}(S), \widehat{S}\varepsilon_2 A)$ is an epimorphism. Thus, we only need to show that $\mathrm{Hom}_A(\Omega_k(\omega_h(S)), S(1)) \overset{\alpha^*_{k-1,0}}{\longrightarrow} \mathrm{Hom}_A(\omega_{k+h}(S), S(1))$ is an epimorphism. Consider the following commutative diagram.

$$\begin{tikzcd}
\cdots & (\widetilde{\omega}_j(S), S(1))^l \arrow[l, "\cong"'] & (\omega_j(S), S(1))^l \arrow[l, "e'"'] & (\widetilde{\omega}_{j-1}(S), S(1))^{l+1} \arrow[l, "\cong"'] & \cdots \arrow[l, "e'"'] \\
\cdots \arrow[r, equal] & (\Omega_l(\widetilde{\omega}_j(S)), S(1))^0 \arrow[u, "E^l_{\widetilde{\omega}_j(S)}"'] & (\Omega_l(\omega_j(S)), S(1))^0 \arrow[l, "(e^j_{l-1})^*"'] \arrow[u, "E^l_{\omega_j(S)}"'] \arrow[r, equal] & (\Omega_{l+1}(\widetilde{\omega}_{j-1}(S)), S(1))^0 \arrow[u, "E^{l+1}_{\widetilde{\omega}_{j-1}(S)}"'] & \cdots \arrow[l]
\end{tikzcd}$$

By Corollary 3.6, $\mathrm{Hom}_A(\Omega_k(\omega_h(S)), S(1)) \overset{\alpha^*_{k-1,0}}{\longrightarrow} \mathrm{Hom}_A(\omega_{k+h}(S), S(1))$ is surjective if the bottom row of the diagram is surjective. This is equivalent to the surjectivity of the top row, which comes from the top row of diagram (15) by reversing the isomorphisms. Hence it can be factored as

$$\begin{aligned}
\mathrm{Ext}^k_A(\omega_h(S), S(1)) &\overset{i'}{\to} \mathrm{Ext}^k_A(\omega_h(S), \nabla(1)) \overset{\widetilde{e}}{\to} \mathrm{Ext}^k_A(\widetilde{\omega}_h(S), \nabla(1)) \\
&\overset{\cong}{\to} \mathrm{Ext}^{k-1}_A(\omega_{h+1}(S), \nabla(1)) \overset{\widetilde{e}}{\to} \ldots \overset{\widetilde{e}}{\to} \mathrm{Ext}^1_A(\widetilde{\omega}_{k+h-1}(S), \nabla(1)) \\
&\overset{\widetilde{i}^{-1}}{\to} \mathrm{Ext}^1_A(\widetilde{\omega}_{k+h-1}(S), S(1)),
\end{aligned}$$

where i' is an epimorphism, while the other maps are isomorphisms, so the composition is surjective. $\square$

Theorem 6.11. *If A° is s.K.s.s., then the simple A-modules are in $r\mathcal{K}^+$.*

P r o o f. In view of Theorem 2.1, it suffices to show that simple A-modules belong to $\mathcal{K}^+$. We also know that $S(1) \in r\mathcal{K}^+$ by Theorem 6.8. So we only have to prove the statement for a simple module S, which is not isomorphic to $S(1)$.

We show first that $\widetilde{\omega}_h(S) \in \mathcal{C}^1_A$ for all h. Applying Proposition 6.10 to $\alpha_{h,0}\colon \Omega(\widetilde{\omega}_h(S)) = \omega_{h+1}(S) \to \Omega_{h+1}(S)$ and using $S \in \mathcal{C}_A$, we get $\alpha_{h,0}(\Omega(\widetilde{\omega}_h(S))) \overset{t}{\leqslant}$

$\Omega_{h+1}(S) \overset{t}{\leqslant} \operatorname{rad} P_h(S)$. As $\alpha_{h,0}(\Omega(\widetilde{\omega}_h(S))) \subseteq \alpha_{h,0}(\operatorname{rad} P(\widetilde{\omega}_h(S))) \subseteq \operatorname{rad} P_h(S)$, it follows that $\Omega(\widetilde{\omega}_h(S))$ is a top submodule of $\operatorname{rad} P(\widetilde{\omega}_h(S))$.

To prove that $\overline{\omega}_h(S) = \omega_h(S)/\widetilde{\omega}_h(S)$ is semisimple, in fact, isomorphic to $\bigoplus S(1)$, we only need that $\operatorname{Hom}_A(\omega_h(S), S(1)) \to \operatorname{Hom}_A(\omega_h(S), \nabla(1))$ is surjective, and this was proved in the third part of Proposition 6.9.

Finally, we show that $\omega_h(S) \in \mathcal{C}_A$ by backwards induction. For $h \geqslant n$, Lemma 6.6 gives that $\Omega(\widetilde{\omega}_h(S)) = \omega_{h+1}(S) = \widetilde{\omega}_{h+1}(S)$, so every syzygy of $\omega_h(S)$ is in $\mathcal{C}_A^1$. Thus, $\widetilde{\omega}_h(S) = \omega_h(S) \in \mathcal{C}_A$ if $h \geqslant n$. On the other hand, if $\omega_h(S) \in \mathcal{C}_A$, then in the exact sequence $0 \to \widetilde{\omega}_h(S) \to \omega_h(S) \to \overline{\omega}_h(S) \to 0$ (with top embedding) both the first and the third terms are Koszul. Hence, by Lemma 2.4 of [2], $\omega_h \in \mathcal{C}_A$. Together with the first part of the proof, this gives $\widetilde{\omega}_{h-1} \in \mathcal{C}_A$. $\square$

We point out that Theorems 6.4 and 6.11 imply that ${}_{A^*}A^*$ is filtered by standard modules. Actually, this gives an alternative proof for Theorem 5.12.

Finally, the combination of the results of Proposition 6.2 and Theorems 6.7, 6.8 and 6.11 provides the following theorem.

Theorem 6.12. *If A° is a standard Koszul standardly stratified algebra and X is a top extension of standard and simple modules, then X^* is filtered by standard A^*-modules.*

7. Examples

We conclude our work with a few examples. Some of them point out differences between the behaviour of quasi-hereditary algebras and standardly stratified algebras, while others show that some of our results can not be strengthened.

Example 7.1. In [4], it was shown that the classes $\mathcal{K}_2$ and $\mathcal{K}$ coincide when A is standard Koszul and quasi-hereditary. It was also shown that, in this context, the class $\mathcal{K}$ is closed under the operation ω. In our case, both properties fail. In this example, A is standard Koszul and standardly stratified, X belongs to $\mathcal{K}_2$ but it is not Koszul. It is also easy to check that $Y \in \mathcal{K}$ but $\omega(Y) = X \notin \mathcal{K}$.

$$A_A = \begin{array}{c} 1 \\ \shortmid \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ \shortmid \\ 1 \\ \shortmid \\ 1 \end{array} \oplus \begin{array}{ccc} & 3 & \\ \diagup & & \diagdown \\ 1 & & 2 \\ \diagdown & & \diagup \\ & 1 & \end{array}, \quad X = P(2)/\operatorname{soc} P(2) = \begin{array}{c} 2 \\ \shortmid \\ 1 \end{array}, \quad Y = \begin{array}{ccc} 1 & & 3 \\ & \diagdown \; \diagup & \\ & 1 & \end{array}$$

Example 7.2. This example shows that on the Δ-filtered side, the simple modules do not have to be in $\mathcal{K}^+$, even $\widetilde{\omega}(S)$ does not have to be Koszul for each simple module S.

$$A_A = \begin{array}{ccccc} & & 1 & & \\ & \diagup & & \diagdown & \\ 1 & & & & \\ | & & & & 2 \\ 1 & & & & \\ & \diagdown & & \diagup & \\ & & 1 & & \end{array} \oplus \begin{array}{c} 2 \\ 1 \end{array} \oplus \begin{array}{ccccc} & & 3 & & \\ & \diagup & & \diagdown & \\ 1 & & & & \\ | & & & & 2 \\ 1 & & & & \\ & \diagdown & & \diagup & \\ & & 1 & & \end{array} \oplus \begin{array}{ccccc} & & 4 & & \\ & \diagup & & \diagdown & \\ 1 & & & & \\ | & & & & 3 \\ 1 & & & & \\ & \diagdown & & \diagup & \\ & & 1 & & \end{array}, \quad S(4) \notin \mathcal{K}^+,\ \widetilde{\omega}(S(4)) \notin \mathcal{C}_A.$$

Example 7.3. None of the defining conditions of the class $\mathcal{K}^+$ can be omitted in Proposition 6.3. Consider the algebra A, whose regular representation is the following:

$$A_A = \begin{array}{ccccc} & & 1 & & \\ & \diagup & & \diagdown & \\ 1 & & & & 2 \\ & \diagdown & & \diagup & \\ & & 1 & & \end{array} \oplus \begin{array}{c} 2 \\ | \\ 1 \end{array} \oplus \begin{array}{c} 3 \\ | \\ 2 \\ | \\ 1 \end{array}, \quad X = \begin{array}{ccccc} 1 & & & & 3 \\ & \diagdown & & \diagup & \\ & & 2 & & \end{array}, \quad Y = \begin{array}{c} 1 \\ | \\ 1 \end{array}$$

Here, A° is standard Koszul and standardly stratified, $X \in \mathcal{K}$, and $\overline{\omega}_k(X)$ is semisimple for all k but $\widetilde{X} \notin \mathcal{C}_A$. The A^*-module $A^* f_1 X^*$ is not projective:

$${}_{A^*}A^* = \begin{array}{ccccc} & & 1 & & \\ & \diagup & & \diagdown & \\ & 1 & & 2 & \\ \diagup & & \diagdown \ \diagup & & \\ 2 & & 1 & & \\ \diagdown & & \diagup \ \diagdown & & \\ & 1 & & 2 & \\ \diagup & & \diagdown \ \diagup & & \\ \vdots & & \vdots & & \end{array} \oplus \begin{array}{ccccc} & & 2 & & \\ & & | & & \\ & & 1 & & \\ & \diagup & & \diagdown & \\ & 1 & & 2 & \\ \diagup & & \diagdown \ \diagup & & \\ 2 & & 1 & & \\ \diagdown & & \diagup \ \diagdown & & \\ & 1 & & 2 & \\ \diagup & & \diagdown \ \diagup & & \\ \vdots & & \vdots & & \end{array} \oplus \begin{array}{c} 3 \\ | \\ 2 \end{array} \quad \text{and} \quad X^* = \begin{array}{ccccccc} & & & 1 & & & 3 \\ & & \diagup & & \diagdown & \diagup & \\ & 1 & & & & 2 & \\ \diagup & & & & & & \\ 2 & & & & & & \end{array}$$

On the other hand, Y is not semisimple but satisfies all the other conditions prescribed by the definition of $\mathcal{K}^+$, and $Y^* \cong \overline{\Delta}^{\circ}_{A^*}(1) \neq P^{\circ}_{A^*}(1)$.

Example 7.4. The map q defined in Section 3 does not have to be an epimorphism if $X \notin \mathcal{K}_2$. In our next example, the A-module X fails to be in $\mathcal{K}_2$ because $X\varepsilon_2 \notin \mathcal{C}_{C_2}$. Here $\mathrm{Ext}^h_A(X, S(4)) = 0$ for all h but $\mathrm{Ext}^1_{C_2}(X\varepsilon_2, S(4)\varepsilon_2) \neq 0$.

$$A_A = \begin{array}{ccccc} & & 1 & & \\ & \diagup & | & \diagdown & \\ 3 & & 1 & & 2 \\ & & \diagdown & \diagup & \\ & & & 3 & \end{array} \oplus \begin{array}{c} 2 \\ | \\ 3 \\ | \\ 4 \end{array} \oplus \begin{array}{c} 3 \\ | \\ 4 \end{array} \oplus 4, \quad X = \begin{array}{ccccc} 1 & & & & 2 \\ & \diagdown & & \diagup & \\ & & 3 & & \end{array}$$

To see that the other defining condition of $\mathcal{K}_2$ is also necessary consider the (hereditary) algebra A, whose regular representation is

$$A_A = \begin{array}{c} 1 \\ | \\ 2 \end{array} \oplus 2.$$

Here $P(1)\varepsilon_2 \in \mathcal{C}_{C_2}$ but $P(1)\varepsilon_2 A \overset{t}{\nleq} P(1)$, so $P(1) \notin \mathcal{K}_2$. It is easy to check that $\mathrm{Ext}^*_A(P(1)) = S^{\circ}_{A^*}(1)$ and $\mathrm{Ext}^*_{C_2}(P(1)\varepsilon_2) \neq 0$.

Example 7.5. Our last example shows that in general $\ker q_X \neq A^* f_1 X^*$, even if A satisfies $\varepsilon_i J^2 \varepsilon_i = \varepsilon_i J \varepsilon_i J \varepsilon_i$ for all i and $X \in \mathcal{K}$ (see Proposition 3.12). We take the algebra A and the A-module X for which

$$A_A = \begin{array}{ccc} & 1 & \\ \diagup & & \diagdown \\ 1 & & 2 \\ | & & | \\ 2 & & 2 \\ | & & \\ 2 & & \end{array} \oplus \begin{array}{c} 2 \\ | \\ 2 \end{array}, \qquad X = \begin{array}{ccc} & & 1 \\ & & | \\ 2 & & 1 \\ & \diagdown \; \diagup & \\ & 2 & \end{array}.$$

Here A° is standard Koszul and standardly stratified. The A-module X is in $\mathcal{K}$ but $A^* f_1 X^* \neq \ker q_X$ as

$$_{A^*}A^* = \begin{array}{ccc} & 1 & \\ \diagup & & \diagdown \\ 1 & & 2 \\ | & & \\ 1 & & \\ | & & \\ \vdots & & \end{array} \oplus \begin{array}{c} 2 \\ | \\ 2 \\ | \\ \vdots \end{array}, \qquad X^* = \begin{array}{c} 1 \\ | \\ 2 \end{array} \oplus \begin{array}{c} 2 \\ | \\ 2 \end{array} \quad \text{and} \quad q_X(X^*) = S(2).$$

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