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EXPLICIT FINITE ELEMENT ERROR ESTIMATES FOR NONHOMOGENEOUS NEUMANN PROBLEMS

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Abstract. The paper develops an explicit a priori error estimate for finite element solution to nonhomogeneous Neumann problems. For this purpose, the hypercircle over finite element spaces is constructed and the explicit upper bound of the constant in the trace theorem is given. Numerical examples are shown in the final section, which implies the proposed error estimate has the convergence rate as 0.5.

Keywords: finite element methods; nonhomogeneous Neumann problems; explicit error estimates

MSC 2010: 65N15, 65N30

1. Introduction

The Steklov type differential equation problem involves the Neumann boundary conditions. It models various physical phenomena, for example, the vibration modes of a structure in contact with an incompressible fluid [4] and the antiplane shearing on a system of collinear faults under slip-dependent friction law [8]. There is wide literature on numerical schemes to solve this type of problems by using, for example, the finite element method (FEM), see [6], [13]. Also, the Steklov type eigenvalue problem is a fundamental problem in mathematics. For example, the optimal constant appearing in the trace theorem for Sobolev spaces is given by the smallest eigenvalue of a Steklov type eigenvalue problem raised to the power $-\frac{1}{2}$, see

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e.g. [17]. Efforts have been made on bounding eigenvalues by using conforming or non-conforming FEMs, see [13], [19].

Most of the existing literature focuses on the convergence analysis of the discrete solution, while there has been very rare work on the explicit bound of the solution error. Recently, in the newly developed field of verified computing, the quantitative error estimate (e.g. explicit values of error) is desired. For example, the explicit values or bounds of the error constants are required in solution verification of non-linear partial differential equations, see e.g. [18].

In this paper, we apply the finite element method to solving the Steklov type differential equation and provide an a priori error estimate for the FEM solution. The main idea in developing a priori error estimation can be regarded as a direct extension of the one proposed by Liu in [16], where the a priori error estimation is constructed by using the hypercircle method for homogeneous boundary conditions. Such ideas can be further tracked back to the one of Kikuchi in [10], where a posteriori error estimation is considered. This a priori estimate can be used for bounding an eigenvalue in the framework proposed by [14] and it will be the topic of a forthcoming paper.

The rest of this paper is organized as follows. In Section 2, we describe the problem to be considered. In Section 3, we construct the hypercircle over FEM spaces, based on which we deduce computable error estimates. In Section 4, we discuss the constant appearing in the trace theorem and propose the explicit a priori error estimate for nonhomogeneous Neumann problems. In Section 5, the computation results are presented.

2. Preliminaries

Throughout this paper, we use the standard notation (see e.g. [3]) for the Sobolev spaces $H^m(\Omega)$ $(m > 0)$. The Sobolev space $H^0(\Omega)$ coincides with $L^2(\Omega)$. Denote by $\|v\|_{L^2}$ or $\|v\|_0$ the L^2 norm of $v \in L^2(\Omega)$; by $|v|_{H^m(\Omega)}$ and $\|v\|_{H^m(\Omega)}$ the seminorm and norm in $H^m(\Omega)$, respectively. Symbol $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $(L^2(\Omega))^2$. The space $H(\mathrm{div},\Omega)$ is defined by

$$H(\mathrm{div},\Omega) := \{q \in (L^2(\Omega))^2 \mid \mathrm{div}\, q \in L^2(\Omega)\}.$$

We are concerned with the model problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = f & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, $\partial/\partial n$ is the outward unit normal derivative on the boundary $\partial\Omega$.

A weak formulation of the above problem is to find $u \in V = H^1(\Omega)$ such that

$$a(u, v) = b(f, v) \quad \forall\, v \in V, \tag{2.2}$$

where

$$a(u, v) = \int_\Omega (\nabla u \nabla v + uv)\,\mathrm{d}x, \quad b(f, v) = \int_{\partial\Omega} fv\,\mathrm{d}s.$$

We define $\|u\|_b = b(u, u)^{1/2}$.

We also have the following regularity result for the solution of problem (2.1), see for example [9].

Lemma 2.1. *If $f \in L^2(\partial\Omega)$, then $u \in H^{1+r/2}(\Omega)$; if $f \in H^{1/2}(\partial\Omega)$, then $u \in H^{1+r}(\Omega)$; here, $r \in (\frac{1}{2}, 1]$, especially $r = 1$ when Ω is convex and $r < \pi/\omega$ (with ω being the largest inner angle of Ω) otherwise.*

Finite element approximation. Let $\mathcal{T}_h$ be a shape regular triangulation of the domain Ω. For each element $K \in \mathcal{T}_h$, denote by h_K the longest edge length of K and define the mesh size h by

$$h := \max_{K \in \mathcal{T}_h} h_K.$$

Define by E_h the set of edges of the triangulation and by $E_{h,\Gamma}$ the set of edges on the boundary of Ω. The finite element space $V^h (\subset V)$ consists of piecewise linear and continuous functions. Assume that $\dim(V^h) = n$. The conforming finite element approximation of (2.2) is defined as follows: Find $u_h \in V^h$ such that

$$a(u_h, v_h) = b(f, v_h) \quad \forall\, v_h \in V^h. \tag{2.3}$$

In this paper, the following classical finite element spaces will also be used in constructing the a priori estimate.

(i) Piecewise constant function spaces X^h and X^h_Γ are defined as

$$X^h := \{v \in L^2(\Omega) \mid v \text{ is constant on each element } K \text{ of } \mathcal{T}_h\},$$
$$X^h_\Gamma := \{v \in L^2(\Gamma) \mid v \text{ is constant on each edge e} \in E_{h,\Gamma}\}.$$

(ii) Raviart-Thomas FEM space W^h is defined as

$$W^h := \{p_h \in H(\mathrm{div}, \Omega) \mid p_h = (a_K + c_K x, b_K + c_K y) \text{ in } K \in \mathcal{T}_h\},$$

where a_K, b_K, c_K are constants on an element K.

The space $W^h_{f_h}$ is a shift of W^h corresponding to $f_h \in X^h_\Gamma$

$$W^h_{f_h} := \{p_h \in W^h \mid p_h \cdot n = f_h \in X^h_\Gamma \text{ on } \Gamma\}.$$

3. The hypercircle

In this section, we first present two hypercircles which can be used to facilitate the error estimate.

Consider the boundary value problem[1]

$$\begin{cases} -\Delta u + \alpha u = g & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = f & \text{on } \Gamma \end{cases} \tag{3.1}$$

with α being a positive constant and $g \in L^2(\Omega)$. A weak formulation of the above problem is to find $u \in V = H^1(\Omega)$ such that

$$\int_\Omega (\nabla u \nabla v + \alpha u v)\,\mathrm{d}x = \int_\Omega g v\,\mathrm{d}x + b(f, v) \quad \forall\, v \in V. \tag{3.2}$$

Corresponding to problem (3.1), the following hypercircle holds, see e.g. page 185 of [5].

Theorem 3.1. *Let u be a solution to problem* (3.2). *For $v \in H^1(\Omega)$ and $v = 0$ on Γ suppose that $\sigma \in H(\mathrm{div}, \Omega)$ satisfies*

$$\sigma \cdot n = f \quad \textit{on } \Gamma \quad \textit{and} \quad \mathrm{div}\, \sigma + g = \alpha v.$$

Then we have

$$\|\nabla(u - v)\|_0^2 + \|\nabla u - \sigma\|_0^2 + 2\alpha\|u - v\|_0^2 = \|\nabla v - \sigma\|_0^2. \tag{3.3}$$

Proof. The expansion of $\|\nabla v - \sigma\|_0^2 = \|(\nabla v - \nabla u) + (\nabla u - \sigma)\|_0^2$ tells that

$$\|\nabla v - \sigma\|_0^2 = \|\nabla v - \nabla u\|_0^2 + \|\nabla u - \sigma\|_0^2 + 2(\nabla u - \sigma, \nabla(v - u)).$$

Let $w := v - u$. From the definition of u in (3.2), we have

$$(\nabla u, \nabla w) = b(f, w) + \int_\Omega (g - \alpha u) w\,\mathrm{d}x. \tag{3.4}$$

[1] The boundary condition can be extended to a mixed one. For example, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\partial u/\partial n = f_1$ on Γ_1, $u = f_2$ on Γ_2, and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Also, by applying Green's theorem to the term with σ, we have

$$(\sigma, \nabla w) = \int_{\partial\Omega} (\sigma \cdot n) w \, \mathrm{d}s - \int_{\Omega} \operatorname{div} \sigma w \, \mathrm{d}x = b(f, w) - \int_{\Omega} (\alpha v - g) w \, \mathrm{d}x. \tag{3.5}$$

By taking (3.4)–(3.5), we have $(\nabla u - \sigma, \nabla(v - u)) = \alpha \|v - u\|_0^2$, which leads to the conclusion of the theorem. □

However, it is usually difficult to construct σ such that $\operatorname{div} \sigma + g = \alpha v$ holds for general v and g. Below we establish a revised hypercircle over finite element spaces. As a preparation, let us introduce two projection operators: π_h and $\pi_{h,\Gamma}$.

▷ For $g \in L^2(\Omega)$ define the projection $\pi_h \colon\ L^2(\Omega) \to X^h$ such that

$$(g - \pi_h g, v_h) = 0 \quad \forall\, v_h \in X^h.$$

The error estimate for π_h is given by

$$\|g - \pi_h g\|_0 \leqslant C_0 h |g|_{H^1(\Omega)} \quad \forall\, g \in H^1(\Omega). \tag{3.6}$$

Here $C_0 := \max\limits_{K \in \mathcal{T}_h} C_0(K)/h$ depends on the triangulation and has an explicit upper bound. For example, in [12], [15] it is shown that the optimal constant is given by $C_0(K) = h_K / j_{1,1}$, where $j_{1,1} \approx 3.83171$ denotes the first positive root of the Bessel function J_1. Upper bounds of C_0 for concrete triangles can be found e.g. in [11], [15].

▷ For $f \in L^2(\Gamma)$, define the projection $\pi_{h,\Gamma} \colon\ L^2(\Gamma) \to X_\Gamma^h$,

$$b(f - \pi_{h,\Gamma} f, v_h) = 0 \quad \forall\, v_h \in X_\Gamma^h.$$

Theorem 3.2. *Given $f_h \in X_\Gamma^h$, let $\tilde{u} \in V$ and $\tilde{u}_h \in V^h$ be solutions to the following variational problems, respectively,*

$$a(\tilde{u}, v) = b(f_h, v) \quad \forall\, v \in V, \tag{3.7}$$

$$a(\tilde{u}_h, v_h) = b(f_h, v_h) \quad \forall\, v_h \in V^h. \tag{3.8}$$

Then for $p_h \in W_{f_h}^h$ satisfying $\operatorname{div} p_h = \pi_h \tilde{u}_h$ we have the revised hypercircle

$$\begin{aligned} \|\nabla \tilde{u}_h - p_h\|_{L^2}^2 = \|\tilde{u} - \tilde{u}_h\|_{H^1(\Omega)}^2 \\ + \|\nabla \tilde{u} - p_h\|_{L^2}^2 + \|\tilde{u} - \tilde{u}_h\|_{L^2}^2 + 2((\pi_h - I)(\tilde{u} - \tilde{u}_h), (\pi_h - I)\tilde{u}_h), \end{aligned}$$

where I is the identity operator.

P r o o f. Rewriting $\nabla\tilde{u}_h - p_h$ by $(\nabla\tilde{u}_h - \nabla\tilde{u}) + (\nabla\tilde{u} - p_h)$, we have

$$\|\nabla\tilde{u}_h - p_h\|^2_{L^2} = \|\nabla\tilde{u}_h - \nabla\tilde{u}\|^2_{L^2} + \|\nabla\tilde{u} - p_h\|^2_{L^2} + 2(\nabla\tilde{u}_h - \nabla\tilde{u}, \nabla\tilde{u} - p_h).$$

Notice that

$$\begin{aligned}(\nabla\tilde{u}_h - \nabla\tilde{u}, \nabla\tilde{u} - p_h) &= (\tilde{u}_h - \tilde{u}, -\tilde{u} + \pi_h(\tilde{u}_h))\\ &= (\tilde{u}_h - \tilde{u}, -\tilde{u} + \tilde{u}_h - \tilde{u}_h + \pi_h(\tilde{u}_h)) = \|\tilde{u}_h - \tilde{u}\|^2_{L^2} + (\tilde{u}_h - \tilde{u}, -\tilde{u}_h + \pi_h(\tilde{u}_h)).\end{aligned}$$

Thus, from the definition of π_h we get the conclusion. $\square$

The following theorem gives a computable error estimate for $f_h \in X^h_\Gamma$.

Theorem 3.3. *Given $f_h \in X^h_\Gamma$, let $\tilde{u} \in V$ and $\tilde{u}_h \in V^h$ be solutions to* (3.7) *and* (3.8), *then the following computable error estimate holds*

$$\|\tilde{u} - \tilde{u}_h\|_{H^1(\Omega)} \leqslant \kappa_h \|f_h\|_b.$$

Here κ_h is defined by

$$\kappa_h := \max_{f_h \in X^h_\Gamma \setminus \{0\}} \frac{Y(f_h, p_h, \beta)}{\|f_h\|_b},$$

where

$$Y^2(f_h, p_h, \beta) := (2 + \beta + 1/\beta)(C_0 h)^4 \|\nabla\tilde{u}_h\|^2_0 + (1 + 1/\beta)\|\nabla\tilde{u}_h - p_h\|^2_0 \quad \forall\, \beta > 0$$

and $p_h \in W^h_{f_h}$ satisfies $\operatorname{div} p_h = \pi_h \tilde{u}_h$.

P r o o f. From the hypercircle and (3.6), we get

$$\begin{aligned}\|\tilde{u} - \tilde{u}_h\|^2_{H^1(\Omega)} &\leqslant \|\nabla\tilde{u}_h - p_h\|^2_{L^2} - 2((\pi_h - I)(\tilde{u} - \tilde{u}_h), (\pi_h - I)\tilde{u}_h)\\ &\leqslant \|\nabla\tilde{u}_h - p_h\|^2_0 + 2C_0 h \|\nabla(\tilde{u} - \tilde{u}_h)\|_0 \|(I - \pi_h)\tilde{u}_h\|_0\\ &\leqslant \|\nabla\tilde{u}_h - p_h\|^2_0 + 2(C_0 h)^2 \|\nabla(\tilde{u} - \tilde{u}_h)\|_0 \|\nabla\tilde{u}_h\|_0.\end{aligned} \tag{3.9}$$

Define $x := \|\tilde{u} - \tilde{u}_h\|_{H^1(\Omega)}$, $A := 2(C_0 h)^2 \|\nabla\tilde{u}_h\|_0$, $B := \|\nabla\tilde{u}_h - p_h\|_0$. By solving the inequality $x^2 \leqslant B^2 + Ax$, one can easily deduce that

$$\|\tilde{u} - \tilde{u}_h\|_{H^1(\Omega)} \leqslant Y(f_h, p_h, \beta) \tag{3.10}$$

for any $\beta > 0$. By further varying f_h in X^h_Γ, we draw the conclusion about κ_h. $\square$

R e m a r k 3.1. The selection of p_h in Theorem 3.3 is not unique. A proper p_h will be determined in Section 4.3. In practical computation, since the first term in $Y(f_h, p_h, \beta)$ has higher order convergence, we can take $\beta > 1$ to have a smaller value of κ_h.

4.1. Trace theorem. This section is devoted to providing the explicit bound for the constant in the trace theorem.

Let us follow the method in [2] to show the explicit value of constants related to the trace theorem.

Theorem 4.1. *Let e be an edge of triangle element K. Given $u \in V_e(K)$, we have the trace theorem*

$$\|u\|_{L^2(e)} \leqslant 0.574\sqrt{\frac{|e|}{|K|}}h_K|u|_{H^1(K)},$$

where $V_e(K) := \{v \in H^1(K) \mid \int_e v \,\mathrm{d}s = 0\}$.

Proof. Suppose P_1, P_2, P_3 are the vertices of K and $e := P_1P_2$. For any $u \in H^1(K)$ the Green theorem leads to

$$\int_K ((x,y) - P_3)\cdot\nabla(u^2)\,\mathrm{d}K = \int_{\partial K}((x,y) - P_3)\cdot nu^2\,\mathrm{d}s - \int_K 2u^2\,\mathrm{d}K.$$

For the term $((x,y) - P_3)\cdot n$, we have

$$((x,y) - P_3)\cdot n = \begin{cases} 0 & \text{on } P_1P_3,\ P_2P_3, \\ 2|K|/|e| & \text{on } e. \end{cases} \tag{4.1}$$

Thus,

$$\begin{aligned} 2\frac{|K|}{|e|}\int_e u^2\,\mathrm{d}s &= \int_K 2u^2\,\mathrm{d}K + \int_K ((x,y) - P_3)\cdot\nabla(u^2)\,\mathrm{d}K \\ &\leqslant \int_K 2u^2\,\mathrm{d}K + 2h_K\int_K |u||\nabla u|\,\mathrm{d}K \\ &\leqslant 2\|u\|_{0,K}^2 + 2h_K\|u\|_{0,K}\|\nabla u\|_{0,K}. \end{aligned}$$

Since $u \in V_e(K)$, we have

$$\int_e u^2\,\mathrm{d}s \leqslant \int_e (u - \pi_h u)^2\,\mathrm{d}s \leqslant \frac{|e|}{|K|}(\|u - \pi_h u\|_{0,K}^2 + h_K\|u - \pi_h u\|_{0,K}\|\nabla u\|_{0,K}).$$

By further applying the estimation of π_h in (3.6), we have

$$\|u\|_{L^2(e)} \leqslant \sqrt{\frac{1}{3.8317^2} + \frac{1}{3.8317}}\sqrt{\frac{|e|}{|K|}}h_K\|\nabla u\|_{0,K} \leqslant 0.574\sqrt{\frac{|e|}{|K|}}h_K\|\nabla u\|_{0,K}.$$

□

R e m a r k 4.1. Almost the same result is shown in [2], where higher dimensional elements are considered. Since a sharper bound for π_h is utilized here, the constant 0.574 obtained in Theorem 4.1 is smaller than the one in [2] (about 0.648).

R e m a r k 4.2. Numerial computations indicate that when the lengths of two edges P_1P_3, P_2P_3 are fixed as h, the constant C in the estimate $\|u\|_e \leqslant Ch\|\nabla u\|_{0,K}$ for all $u \in V_e(K)$ will tend to 0 when the length of the third edge $e := P_1P_2$ tends to 0. However, this behavior of the constant C cannot be deduced from Theorem 4.1.

4.2. Explicit a priori error estimates.

Theorem 4.2. *Let u and $\tilde{u}$ be solutions to* (2.2) *and* (3.7), *respectively, with f_h taken as $f_h := \pi_{h,\Gamma} f$. Then the following error estimate holds:*

$$\|u - \tilde{u}\|_{H^1(\Omega)} \leqslant C_1(h)\|(I - \pi_{h,\Gamma})f\|_b,$$

where

$$C_1(h) = \max_{e \in E_{h,\Gamma}} \left\{0.574\sqrt{\frac{|e|}{|K|}}h_K\right\}.$$

P r o o f. Setting $v = u - \tilde{u}$ in (2.2) and (3.7), we have

$$a(u - \tilde{u}, u - \tilde{u}) = b(f - f_h, u - \tilde{u}) = b((I - \pi_{h,\Gamma})f, (I - \pi_{h,\Gamma})(u - \tilde{u})).$$

From the Schwartz inequality and Theorem 4.1, we get

$$\begin{aligned} \|u - \tilde{u}\|^2_{H^1(\Omega)} &\leqslant \|(I - \pi_{h,\Gamma})f\|_b \|(I - \pi_{h,\Gamma})(u - \tilde{u})\|_b \\ &\leqslant C_1(h)\|(I - \pi_{h,\Gamma})f\|_b |u - \tilde{u}|_{H^1(\Omega)}, \end{aligned}$$

which implies the conclusion. □

Now, we are ready to formulate and prove the explicit a priori error estimate.

Theorem 4.3. *Let u and u_h be solutions to* (2.2) *and* (2.3), *respectively. Then the following error estimates hold:*

$$\|u - u_h\|_{H^1(\Omega)} \leqslant M_h\|f\|_b, \tag{4.2}$$

$$\|u - u_h\|_b \leqslant M_h^2\|f\|_b \tag{4.3}$$

with $M_h := \sqrt{(C_1(h))^2 + \kappa_h^2}$.

P r o o f. The estimation in (4.2) can be obtained by applying Theorems 3.3 and 4.2:

$$\begin{aligned}\|u-u_h\|_{H^1(\Omega)} &\leqslant \|u-\tilde{u}\|_{H^1(\Omega)} + \|\tilde{u}-\tilde{u}_h\|_{H^1(\Omega)} \\ &\leqslant C_1(h)\|(I-\pi_{h,\Gamma})f\|_b + \kappa_h\|f_h\|_b \\ &\leqslant \sqrt{(C_1(h))^2+\kappa_h^2}\|f\|_b.\end{aligned}$$

The error estimate (4.3) can be obtained by the Aubin-Nitsche duality technique. □

R e m a r k 4.3. The result (4.2) of Theorem 4.3 provides an explicit *a priori* error estimation for the FEM solutions, which is based on the *a posteriori* error estimation in (3.10). Notice that in (3.10), by taking any explicit p_h and β, we have the following explicit *a posteriori* bound for the FEM solution:

$$\|u-u_h\|_{H^1(\Omega)} \leqslant C_1(h)\|(I-\pi_{h,\Gamma})f\|_b + Y(f_h,p_h,\beta). \tag{4.4}$$

Similar results about *a posteriori* error estimation can be found in [1], [2], [10]. In [1], [10], the homogeneous Dirichlet boundary condition is considered. In [2], the nonhomogeneous Neumann boundary condition is considered and (4.4) can be regarded as a special case of [2].

4.3. Computation of κ_h. The quantity κ_h is evaluated in two steps.

First, for fixed f_h, we deduce explicit forms of $\tilde{u}_h \in V^h$ and $p_h \in W^h$ which appear in the definition of $Y(f_h,p_h,\beta)$. According to the standard theories of the conforming FEM and the Raviart-Thomas FEM, see e.g. [7], we solve the following two problems:

(a) Find $\tilde{u}_h \in V^h$ such that

$$a(\tilde{u}_h,v_h) = b(f_h,v_h) \quad \forall\, v_h \in V^h.$$

(b) Let $\tilde{u}_h$ be the solution of (a). Find $p_h \in W^h_{f_h}$ and $\varrho_h \in X^h$, $c \in \mathbb{R}$ such that

$$\begin{cases}(p_h,\tilde{p}_h)+(\varrho_h,\operatorname{div}\tilde{p}_h)+(\varrho_h,d)=0 & \forall\,\tilde{p}_h\in W^h_0,\ \forall\, d\in\mathbb{R},\\ (\operatorname{div}p_h,\tilde{q}_h)+(c,\tilde{q}_h)=(\pi_h(\tilde{u}_h),\tilde{q}_h) & \forall\,\tilde{q}_h\in X^h,\end{cases}$$

where $W^h_0 := \{p_h \in W^h \mid p_h\cdot n = 0 \in X^h_\Gamma \text{ on } \Gamma\}$.

Notice that the solution p_h of (b) depends on f_h. Let us rewrite $Y(f_h,p_h,\beta)$ as $Y(f_h,\beta)$. Second, we find f_h that maximizes the value of $Y(f_h,\beta)/\|f_h\|_b$ by solving an eigenvalue problem. By using the solutions of (a) and (b), $Y(f_h, \beta)$ and $\|f_h\|_b$ can be formulated by

$$Y^2(f_h, \beta) = x^{\mathrm{T}} A x \quad \text{and} \quad \|f_h\|_b^2 = x^{\mathrm{T}} B x,$$

where x is the coefficient vector of f_h with respect to the basis of X_Γ^h, and A, B are symmetric matrices to be determined upon the selection of the basis of the FEM spaces. Thus, the value of κ_h^2 is given by the maximum eigenvalue of the problem

$$Ax = \lambda Bx.$$

For detailed solution of this eigenvalue problem we refer to [16], where an analogous problem is described.

5. Numerical examples

In this section, several numerical tests are presented. The constant κ_h is computed for problem (2.1) and four different domains. For each domain a sequence of uniformly refined finite element meshes is considered. If κ_{2h} and κ_h are computed on two consecutive meshes, then the convergence rate is estimated numerically as

$$\kappa_h\text{-rate} := \frac{\log(\kappa_{2h}/\kappa_h)}{\log 2}.$$

5.1. The unit square. We consider the problem (2.1) on the unit square domain $\Omega = (0,1) \times (0,1)$. In the numerical experiment, we set $\beta = 0.1, 1, 10, 100$, and 1000. The dependency of κ_h on β is displayed in Figure 1, which illustrates that larger β gives smaller κ_h. However, the definition of $Y(f_h, \beta)$ clearly shows that β cannot be too large.

Computed quantities κ_h, $C_1(h)$, and M_h for the case $\beta = 100$ are shown in Table 1. The estimated convergence rate of κ_h, denoted by κ_h-rate, is close to 0.5.

h	κ_h	$C_1(h)$	M_h	κ_h-rate
$\sqrt{2}/4$	0.4143	0.574	0.7079	–
$\sqrt{2}/8$	0.2973	0.4059	0.5031	0.4788
$\sqrt{2}/16$	0.2110	0.2870	0.3562	0.4947
$\sqrt{2}/32$	0.1493	0.2029	0.2519	0.4990

Table 1. Computed quantities for the square and $\beta = 100$.

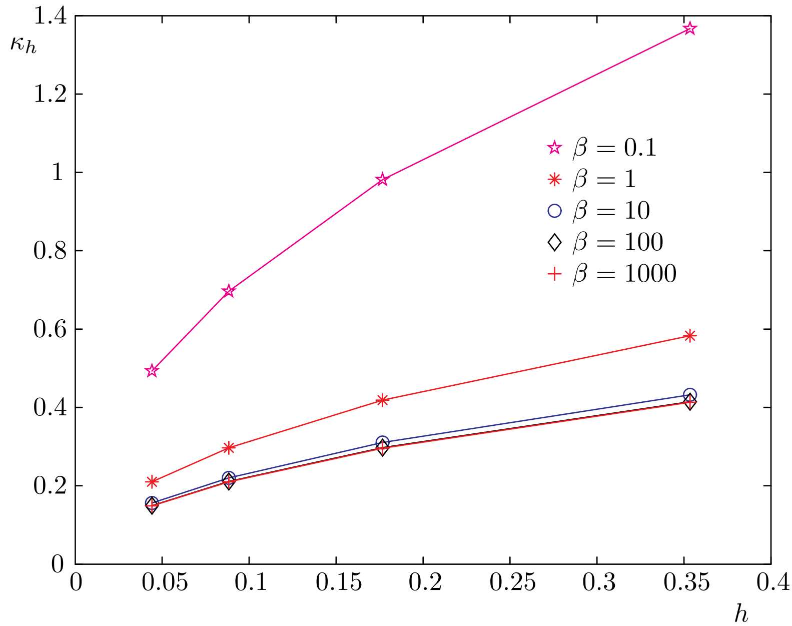

Figure 1. The dependence of κ_h on β (unit square).

5.2. Right triangle, equilateral triangle, and the L-shaped domain. In this example, three domains are considered, namely, the isosceles right triangle with unit legs, the unit equilateral triangle, and the L-shaped domain $\Omega = (0,1) \times (0,1) \setminus [\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$. The results for $\beta = 100$ are displayed in Tables 2–4, respectively. For all domains the convergence rate of κ_h is close to 0.5.

h	κ_h	$C_1(h)$	M_h	κ_h-rate
$\sqrt{2}/4$	0.4448	0.6826	0.8147	–
$\sqrt{2}/8$	0.3107	0.4827	0.5741	0.5176
$\sqrt{2}/16$	0.2197	0.3413	0.4059	0.5000
$\sqrt{2}/32$	0.1554	0.2413	0.2870	0.4995

Table 2. Computed quantities for the isosceles right triangle and $\beta = 100$.

h	κ_h	$C_1(h)$	M_h	κ_h-rate
$1/4$	0.3783	0.4361	0.5773	–
$1/8$	0.2696	0.3084	0.4096	0.4887
$1/16$	0.1909	0.2181	0.2898	0.4980
$1/32$	0.1350	0.1542	0.2049	0.4999

Table 3. Computed quantities for the equilateral triangle and $\beta = 100$.

h	κ_h	$C_1(h)$	M_h	κ_h-rate
$\sqrt{2}/4$	0.4872	0.574	0.7529	–
$\sqrt{2}/8$	0.3432	0.4059	0.5315	0.5055
$\sqrt{2}/16$	0.2439	0.2870	0.3766	0.4928
$\sqrt{2}/32$	0.1734	0.2029	0.2669	0.4922

Table 4. Computed quantities for the L-shaped domain and $\beta = 100$.

6. Conclusion

In this paper, by applying the technique of the hypercircle method, we successfully construct the explicit a priori error estimate for the FEM solution of nonhomogeneous Neumann problems. By following the framework proposed by the second author in [14], the a priori error estimate obtained here can be used in bounding eigenvalues of the Steklov type eigenvalue problems. The expected rate of convergence of M_h is 1 in case the solution is smooth enough. In this paper, only the H^1 regularity is required in the analysis, and both the theoretical results, see Theorem 4.2, and numerical tests confirm the suboptimal convergence rate 0.5 for M_h as well as κ_h. It is an interesting problem whether the rate of convergence can be improved or not for general $f \in L_2(\partial\Omega)$.

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