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SOLVABILITY CLASSES FOR CORE PROBLEMS IN MATRIX TOTAL LEAST SQUARES MINIMIZATION

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Abstract. Linear matrix approximation problems $AX \approx B$ are often solved by the total least squares minimization (TLS). Unfortunately, the TLS solution may not exist in general. The so-called core problem theory brought an insight into this effect. Moreover, it simplified the solvability analysis if B is of column rank one by extracting a core problem having always a unique TLS solution. However, if the rank of B is larger, the core problem may stay unsolvable in the TLS sense, as shown for the first time by Hnětynková, Plešinger, and Sima (2016). Full classification of core problems with respect to their solvability is still missing. Here we fill this gap. Then we concentrate on the so-called composed (or reducible) core problems that can be represented by a composition of several smaller core problems. We analyze how the solvability class of the components influences the solvability class of the composed problem. We also show on an example that the TLS solvability class of a core problem may be in some sense improved by its composition with a suitably chosen component. The existence of irreducible problems in various solvability classes is discussed.

Keywords: linear approximation problem; core problem theory; total least squares; classification; (ir)reducible problem

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1. Introduction

1.1. The core problem theory. Let us consider a *linear approximation problem*

$$AX \approx B, \quad \text{where } A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times d},\ X \in \mathbb{R}^{n\times d} \tag{1.1}$$

are matrices representing the system matrix of a discretized model, observation ma-

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trix of measurements (together forming the data matrix $[B, A]$), and the matrix of unknowns, respectively. For simplicity we usually assume $\mathcal{R}(B) \not\subseteq \mathcal{R}(A)$ and $\mathcal{R}(B) \not\subseteq \mathcal{N}(A^{\mathrm{T}})$, otherwise the problem has either a solution in a classical sense $AX = B$ with $X \equiv A^{\dagger}B$, or the column spaces of both matrices are orthogonal $A^{\mathrm{T}}B = 0$ and it makes no sense to approximate columns of B by columns of A, (where $\mathcal{R}(K)$, $\mathcal{N}(K)$, and $K^{\dagger}$ denote respectively the range, null-space, and Moore–Penrose pseudoinverse of K).

The *core problem theory* developed in [8], [4], [5] gives the following. For every (1.1), there exist orthogonal matrices $P \in \mathbb{R}^{m\times m}$, $P^{\mathrm{T}} = P^{-1}$, $Q \in \mathbb{R}^{n\times n}$, $Q^{\mathrm{T}} = Q^{-1}$, $R \in \mathbb{R}^{d\times d}$, $R^{\mathrm{T}} = R^{-1}$ so that

$$(P^{\mathrm{T}}AQ)(Q^{\mathrm{T}}XR) \equiv \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \approx \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} \equiv (P^{\mathrm{T}}BR), \tag{1.2}$$

with conforming partitioning of matrices (i.e., in particular, A_{11} and B_1 have the same number of rows) satisfying the following three conditions:

(CP1) The matrix A_{11} is of *full column rank.*

(CP2) The matrix B_1 is of *full column rank.*

(CP3) Let A_{11} have ξ distinct nonzero singular values with multiplicities μ_j and $\mu_{\xi+1} \equiv \dim(\mathcal{N}(A_{11}^{\mathrm{T}}))$, and let U_j' be matrices having orthonormal bases of left singular vector subspaces of A_{11} as their columns.
The matrix $(U_j')^{\mathrm{T}}B_1$ is of *full row rank* μ_j for $j = 1, \ldots, \xi, \xi + 1$.

In [8] and [4], it was shown, that (CP1)–(CP3) are equivalent to the minimality of $[B_1, A_{11}]$ (and maximality of A_{22}) over all orthogonal transformations giving the same zero-nonzero block structure of the system and observation matrices. Note that [8] focuses on the case $d = 1$, i.e., when B and therefore also B_1 are vectors, while [4] focuses on the matrix right-hand side case $d > 1$. The minimally dimensioned subproblem

$$A_{11}X_{11} \approx B_1 \tag{1.3}$$

is called the *core problem* (within (1.1)) and (1.2) is the *core problem revealing transformation.*

1.2. The total least squares minimization. Problems of the form (1.1) are solved in many applications by using plenty of different approaches, usually based on least squares techniques. Total least squares (TLS) minimization represents one of them. It typically seeks for

$$\min_{G\in\mathbb{R}^{m\times d},\, E\in\mathbb{R}^{m\times n}} \|[G, E]\|_F \quad \text{subject to } \mathcal{R}(B+G) \subseteq \mathcal{R}(A+E) \tag{1.4}$$

(where $\|K\|_F$ denotes the Frobenius norm of K). Then any matrix X^{TLS} satisfying

$$(A+E)X^{\mathrm{TLS}} = B + G$$

is called the TLS solution of (1.1).

The TLS problem differs from the basic (ordinary) LS in including a correction E of the model matrix A into the minimization formulation. Problems, for which a TLS solution represents better approximation than a LS solution have been widely discussed in the literature in the past decades. A nice overview can be found, e.g., in [10], Chapter 1.2 or [7]. For example, the TLS approach is advantageous in classical errors-in-variables (EIV) models, where the aim is to reveal the existing unknown model (representing relations between variables) from its approximation A rather than obtaining a precise approximation of X, or in cases where model errors are significantly larger than observation errors. The TLS method is applied (under various names) in areas such as experimental modal analysis, system identification, signal processing, image processing or chemometrics, see [7] for references, where LS often fails to give reliable approximations.

However, allowing corrections of A in (1.4) has significant impact on the solvability of the minimization problem. While LS solution always exists (and one can uniquely select a solution with minimum norm), this is no longer true for TLS. The existence and uniqueness of X^{TLS} has been analyzed in many papers starting from [1], [10], [12], [13], and in particular [14]. Moreover, the so-called nongeneric solution was defined in [10] for cases where the standard TLS solution does not exist or is complicated to construct (as revealed and explained later in [3]). The question of TLS solvability of a general problem (1.4) was finally resolved in [14] and [3]. In particular, [3] introduced a novel full classification of problems (1.1) with respect to their TLS solvability. The problems (1.1) are there divided into four *solvability classes* and for each of them the (non)existence and (non)uniqueness of the TLS solution is proved. Thus, the solvability class of a given problem reveals how its approximate solution can be computed, and what is the meaning of this solution in terms of the original data.

The TLS minimization (1.4) employs the Frobenius, i.e., orthogonally invariant norm, and the core problem revealing transformation (1.2) is an orthogonal transformation. Thus the TLS minimizations applied to the original and transformed problems result in the same minima (up to the transformation). Taking into account the zero blocks in the transformed right-hand side (1.2), it is reasonable to put $X_{12} = 0$, $X_{21} = 0$, $X_{22} = 0$. Consequently, using the core reduction as a sort of preprocessing of the data A, B, it is obvious that we in fact need to solve the single nontrivial and typically smaller subproblem—the core problem (1.3). The link between the TLS solution of the core problem and the TLS or non-generic solution of the original problem if $d = 1$ was explained in [8]. There it was also proved that the core problem with $d = 1$ is always uniquely TLS solvable. For problems with $d > 1$, the first attempts of clarification were published in [2]. In particular it was shown that if $d > 1$, the *core problem may stay unsolvable* in the TLS sense. However, complete classification of core problems with respect to their solvability is still missing. Such knowledge would indicate in which cases the core reduction simplifies the solvability of the TLS problem, and clarify the meaning of the TLS solution of the core problem with respect to the original data. Thus we study this open question here.

1.3. Contribution of this work. In this paper we present some further pieces of the missing mosaic. We show which solvability classes are possible for core problems with $d = 2$ and $d > 2$, resulting in *full solvability classification of core problems* with respect to the number of their right-hand sides. Then we concentrate on the so-called *composed (or reducible) core problems* introduced in [2]. Such problems can be equivalently represented by a composition of several (in some sense block independent) core problems of smaller dimensions. Assuming the solvability classes of the components are known, we analyze feasible solvability classes of the resulting composed problem. We also show on an example that the TLS solvability of a core problem may be in some sense improved by its composition with a suitably chosen component. For completeness, examples of irreducible problems in various solvability classes are presented.

The text is organized as follows. Section 2 recapitulates the TLS classification, the previous TLS solvability results for core problems, and the core problem composition. Section 3 gives the full solvability classification of core problems with respect to the number of their right-hand sides. Section 4 analyzes solvability classes in the course of core problems composing. Section 5 comments on the irreducible core problems, and Section 6 concludes the paper.

2. Recapitulation of known results

2.1. Classification of TLS problems. First of all we briefly recall the above-mentioned full classification of problems with respect to their TLS solvability developed in [3]. It employs the singular value decompositions (SVD) of the data matrix $[B, A] \in \mathbb{R}^{m\times(n+d)}$ (we assume $m \geqslant n + d$ for simplicity; in the other case one can add zero rows to the data matrix, which is equivalent to adding $(n + d) - m$ zero singular values). Let

$$[B, A] = U\Sigma V^{\mathrm{T}}, \quad \text{where } \Sigma = \begin{bmatrix} \operatorname{diag}(\sigma_1, \ldots, \sigma_{n+d}) \\ 0 \end{bmatrix} \in \mathbb{R}^{m\times(n+d)}, \tag{2.1}$$

let q $(0 \leqslant q \leqslant n)$ and e $(1 \leqslant e \leqslant d)$ be the *left-* and *right-multiplicity* of σ_{n+1}, e.g.,

$$\sigma_{n-q} > \underbrace{\sigma_{n-q+1} = \ldots = \sigma_{n+1} = \ldots = \sigma_{n+e}}_{(q+e)\text{-tuple singular value}} > \sigma_{n+e+1} \tag{2.2}$$

in the typical case (if $q = n$ or $e = d$, then σ_{n-q} or σ_{n+e+1} do not exist, respectively). The classification is then based on ranks of individual blocks of V,

$$V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{matrix} \}\, d \\ \}\, n \end{matrix} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \end{bmatrix} \begin{matrix} \}\, d \\ \}\, n \end{matrix} \tag{2.3}$$
$$\qquad\qquad\qquad\qquad \underbrace{\phantom{V_{11}}}_{n-q}\ \underbrace{\phantom{V_{12}}}_{q+e}\ \underbrace{\phantom{V_{13}}}_{d-e}$$

(if $q = n$ or $e = d$, then $[V_{11}^{\mathrm{T}}, V_{21}^{\mathrm{T}}]^{\mathrm{T}}$ or $[V_{13}^{\mathrm{T}}, V_{23}^{\mathrm{T}}]^{\mathrm{T}}$ have no columns, respectively). Then (1.1) with the minimization (1.4) belongs to the class:

- $\mathcal{F}$ if $\operatorname{rank}([V_{12}, V_{13}]) = d$ (so-called *generic* problem), in particular to its sub-class:
  - $\mathcal{F}_1$ if $\operatorname{rank}(V_{12}) = e$,
  - $\mathcal{F}_2$ if $\operatorname{rank}(V_{12}) > e$ and $\operatorname{rank}(V_{13}) = d - e$, or
  - $\mathcal{F}_3$ if $\operatorname{rank}(V_{13}) < d - e$ (i.e., $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$); or
- $\mathcal{S}$ if $\operatorname{rank}([V_{12}, V_{13}]) < d$ (so-called *non-generic* problem).

The problem has a TLS solution if and only if it belongs to $\mathcal{F}_1 \cup \mathcal{F}_2$, as shown in [3]. Thus problems in $\mathcal{F}_3 \cup \mathcal{S}$ (i.e., even the generic problems in $\mathcal{F}_3$) have no TLS solution. This classification has been recently extended to TLS formulations with an arbitrary unitarily invariant norm in (1.4), see [11].

Note that the so-called *classical TLS algorithm* (see [10], [3]) returns the TLS solution only for problems from $\mathcal{F}_1$, moreover it always returns the solution minimal in both the Frobenius and spectral norms. For problems from $\mathcal{F}_2$, the algorithm requires a small modification (see [6]), but it is not able to return the minimal norm solution in general.

2.2. Solvability of core problems. The key result proved in [8] for $d = 1$ is the following: The core problem with single right-hand side has always the unique TLS solution X_{11}^{TLS}. Moreover, its back-transformation $X = Q\left[(X_{11}^{\mathrm{TLS}})^{\mathrm{T}}, 0\right]^{\mathrm{T}} R^{\mathrm{T}}$ (since $d = 1$, R becomes equal to 1 or -1) is the (unique or minimum norm) TLS solution of the original problem (if it is TLS solvable), or the so-called (unique or minimum norm) nongeneric solution (otherwise).

In the context of solvability classification, it was shown in [3] that a problem $AX \approx B$ with a single right-hand side belongs either to $\mathcal{F}_1$ or $\mathcal{S}$, and the core problem $A_{11}X_{11} \approx B_1$ with a single right-hand side belongs always to $\mathcal{F}_1$. (Recall that all problems in $\mathcal{F}_1$ are TLS solvable, whereas in $\mathcal{S}$ they are not.) Note that in [2] it was also shown that *any core problem* (i.e., with $d = 1$ as well as $d > 1$) in $\mathcal{F}_1$ has a *unique TLS solution.*

Since the solution of the original problem and the core problem within are closely linked, authors of [8] say that for $d = 1$ the core problem contains *only the necessary and all the sufficient information* for solving the original problem in the TLS sense. Therefore, the transition from the original general problem (GP) to the core problem (CP) is called the *core problem reduction.* To simplify the exposition, we schematically describe this by the diagram:

$$(\mathsf{GP}, 1, \mathcal{F}_1 \text{ or } \mathcal{S}) \xrightarrow[\text{reduction}]{\text{core problem}} (\mathsf{CP}, 1, \mathcal{F}_1), \tag{2.4}$$

where the first component of each triplet identifies whether we deal with general or core problem, the second component specifies the number of its right-hand sides d, and the last component denotes its solvability class. In the general case $d \geqslant 1$, such scheme takes the form:

$$(\mathsf{GP}, d, \text{any class}) \xrightarrow[\text{reduction}]{\text{core problem}} (\mathsf{CP}, \overline{d}, \text{unknown class}), \quad d \geqslant \overline{d} \geqslant 1, \tag{2.5}$$

since nothing is known about the resulting class of the core problem.

2.3. Composing of core problems. In [2], it was shown that we can *compose the core problems* as follows. If $A_{11}^{(l)} X_{11}^{(l)} \approx B_1^{(l)}$, $l = \alpha, \beta$, represent two core problems (i.e., each satisfies (CP1)–(CP3)), then the problem

$$A_{11} X_{11} \equiv \left(P^{\mathrm{T}} \begin{bmatrix} A_{11}^{(\alpha)} & 0 \\ 0 & A_{11}^{(\beta)} \end{bmatrix} Q \right) X_{11} \approx \left(P^{\mathrm{T}} \begin{bmatrix} B_1^{(\alpha)} & 0 \\ 0 & B_1^{(\beta)} \end{bmatrix} R \right) \equiv B_{11}, \tag{2.6}$$

where P, Q, R are orthogonal matrices, also satisfies (CP1)–(CP3) and therefore represents a core problem. We call such a core problem *composed or reducible.* Schematically, we describe the composition by the sign "$\boxplus$" with the particular summands indexed by small Greek letters from the beginning of the alphabet.

The relationship between $X_{11}^{(\alpha)}$, $X_{11}^{(\beta)}$, and X_{11} is not clear, except for some special cases. In particular, it was shown by examples in [2] that there *exist two components* such that

$$(\mathsf{CP}, 1, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\beta = (\mathsf{CP}, 2, \mathcal{F}_1) \text{ or } (\mathsf{CP}, 2, \mathcal{F}_2) \text{ or } (\mathsf{CP}, 2, \mathcal{S}). \tag{2.7}$$

Further, there *exist three components* such that

$$(\mathsf{CP}, 1, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\beta \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma = (\mathsf{CP}, 3, \mathcal{F}_3). \tag{2.8}$$

Thus the core problem with $d > 1$ can belong to any of the four solvability classes. Note that not every core problem with $d > 1$ can be written as a composition of single right hand-side core problems. In [2], an example of irreducible $\mathcal{F}_2$ core problem was presented.

Even though we have excluded compatible problems (i.e., with $\mathcal{R}(B) \subseteq \mathcal{R}(A)$) and "fully incompatible" problems (i.e., with $\mathcal{R}(B) \subseteq \mathcal{N}(A^{\mathrm{T}})$, or equivalently $\mathcal{R}(B) \perp \mathcal{R}(A)$ or $A^{\mathrm{T}}B = 0$), a component of a core problem can still have such properties. If we try to find the core problem within a fully incompatible problem, we see that B_1 is square invertible, and formally A_{11} has no columns, i.e., the data matrix takes the form $[B_1, A_{11}] = B_1$. Such *degenerated core problem* can play a role of a component (which cannot be approximated and only increases the residual) in a composed problem. The degenerated component is always of $\mathcal{F}_1$. For illustration, we give examples of the *proper incompatible*, *compatible*, and *degenerated core problems* (or their components) $A_{11}X_{11} \approx B_1$, $A_{11} \in \mathbb{R}^{m\times n}$, $B_1 \in \mathbb{R}^{m\times d}$, with $d = 1$. Their so-called *SVD forms* always look like

$$[B_1, A_{11}] = \left[\begin{array}{c|cccc} b_1 & \varsigma_1 & & & \\ b_2 & & \varsigma_2 & & \\ \vdots & & & \ddots & \\ b_n & & & & \varsigma_n \\ b_{n+1} & 0 & 0 & \dots & 0 \end{array}\right], \quad \left[\begin{array}{c|cccc} b_1 & \varsigma_1 & & & \\ b_2 & & \varsigma_2 & & \\ \vdots & & & \ddots & \\ b_n & & & & \varsigma_n \end{array}\right], \quad \text{and} \quad [\, b_1 \,],$$

respectively, where $b_j \neq 0$ and $\varsigma_j > \varsigma_{j+1} > 0$. Clearly $m = n + 1$, n, and 1 in these three respective cases, and $n = 0$ in the last one.

3. Solvability classes of core problems with respect to the number of their right-hand sides

The single right-hand side core problem always belongs to the class $\mathcal{F}_1$, see [8]. Examples of $\mathcal{F}_2$, and $\mathcal{S}$ core problems are in (2.7) built up from two single right-hand components, whereas $\mathcal{F}_3$ core problem in (2.8) is built up from three, see [2]. This motivates a question whether the number of right-hand sides d restricts the available classes of core problems not only for $d = 1$ but also for $d > 1$. We analyze this below.

3.1. Core problems with two right-hand sides. The following theorem gives all possible classes for $d = 2$.

Theorem 3.1. *Let $A_{11}X_{11} \approx B_1$, $B_1 \in \mathbb{R}^{m\times d}$, be a core problem with $d = 2$ right-hand sides. Then the core problem belongs to the class $\mathcal{F}_1$, $\mathcal{F}_2$, or $\mathcal{S}$. Equivalently, the core problem with $d = 2$ cannot belong to the class $\mathcal{F}_3$.*

P r o o f. Recalling that there exist composed core problems with $d = 2$ in $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{S}$ (see (2.7)), we only need to exclude $\mathcal{F}_3$.

Assume by contradiction that there exists a core problem with $d = 2$ in $\mathcal{F}_3$. The classification is based on the ranks of blocks of V (see (2.3)), and the class $\mathcal{F}_3$ is characterized by $\operatorname{rank}([V_{12}, V_{13}]) = d$ and $\operatorname{rank}(V_{13}) < d - e$, where e ($1 \leqslant e \leqslant d$) is the right-multiplicity of the singular value σ_{n+1}. Since $d = 2$, we have $e \in \{1, 2\}$. The inequality $\operatorname{rank}(V_{13}) < d - e = 2 - e$ then implies that

$$e = 1, \quad \operatorname{rank}(V_{13}) = 0, \quad \text{and} \quad V_{13} \in \mathbb{R}^{2\times 1}. \tag{3.1}$$

Because the number of columns of V_{13} is equal to the sum of multiplicities of singular values strictly smaller than σ_{n+1}, we see that there is only one simple (possibly zero) singular value with this property, i.e., $\sigma_{n+1} > \sigma_{n+2} \geqslant 0$. Here we need to use another property of core problems that has not been mentioned yet:

(CP5) Let $[B_1, A_{11}]$ have χ distinct nonzero singular values with multiplicities ϱ_j and $\varrho_{\chi+1} \equiv \dim(\mathcal{N}([B_1, A_{11}]))$, and let V_j' be matrices having orthonormal bases of left singular vector subspaces of $[B_1, A_{11}]$ as their columns.
The leading $d \times \varrho_j$ submatrix of V_j' is of *full column rank* ϱ_j for $j = 1, \ldots, \chi, \chi + 1$; see [5] and [2].

We see that $[V_{13}^{\mathrm{T}}, V_{23}^{\mathrm{T}}]^{\mathrm{T}}$ is one of the matrices V_j', and V_{13} is one of the $d \times \varrho_j$ blocks. Therefore, V_{13} has linearly independent columns, i.e., is of rank one which is in contradiction with (3.1). $\square$

Note that in the case of composed core problem (i.e., having two single right-hand side components), this theorem directly implies that, schematically:
$\forall(\mathsf{CP}, 1, \mathcal{F}_1)_\alpha$, $\forall(\mathsf{CP}, 1, \mathcal{F}_1)_\beta$,

$$(\mathsf{CP}, 1, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\beta = (\mathsf{CP}, 2, \mathcal{F}_1),\ (\mathsf{CP}, 2, \mathcal{F}_2),\ \text{or}\ (\mathsf{CP}, 2, \mathcal{S}),$$

or equivalently

$$(\mathsf{CP}, 1, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\beta \neq (\mathsf{CP}, 2, \mathcal{F}_3).$$

3.2. Core problems with three and more right-hand sides. First we prove a theorem stating that it is always possible to compose a general core problem with a single right-hand side component without changing the solvability class.

Theorem 3.2. *Let $A_{11}^{(\alpha)} X_{11}^{(\alpha)} \approx B_1^{(\alpha)}$, $A_{11}^{(\alpha)} \in \mathbb{R}^{m_\alpha \times n_\alpha}$, $B_1^{(\alpha)} \in \mathbb{R}^{m_\alpha \times d_\alpha}$ be a core problem (that will serve as a component) and let it be in the class $\mathcal{C} \in \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{S}\}$.*

Then there exists a single right-hand side component $A_{11}^{(\beta)} X_{11}^{(\beta)} \approx B_1^{(\beta)}$, $A_{11}^{(\beta)} \in \mathbb{R}^{m_\beta \times n_\beta}$, $B_1^{(\beta)} \in \mathbb{R}^{m_\beta \times 1}$ *such that the composed core problem*

$$A_{11} X_{11} \equiv \left(P^{\mathrm{T}} \begin{bmatrix} A_{11}^{(\alpha)} & 0 \\ 0 & A_{11}^{(\beta)} \end{bmatrix} Q \right) X_{11} \approx \left(P^{\mathrm{T}} \begin{bmatrix} B_1^{(\alpha)} & 0 \\ 0 & B_1^{(\beta)} \end{bmatrix} R \right) \equiv B_{11},$$

is also in the class $\mathcal{C}$.

Schematically: $\forall (\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha$, $\exists (\mathsf{CP}, 1, \mathcal{F}_1)_\beta$ such that

$$(\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\beta = (\mathsf{CP}, d_\alpha + 1, \mathcal{C}),$$

where $\mathcal{C} \in \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{S}\}$.

Proof. Let $\sigma_i^{(\alpha)}$, $i = 1, \dots n_\alpha + d_\alpha$, be the singular values of the α-component $[B_1^{(\alpha)}, A_{11}^{(\alpha)}]$. Denote q_l, e_l the left- and right-multiplicity of the singular value of interest, i.e., $\sigma_{n_l+1}^{(\alpha)}$. Construct a core problem representing the β-component $[B_1^{(\beta)}, A_{11}^{(\beta)}]$ arbitrarily with the only restriction that

$$\sigma_{n_\beta+1}^{(\beta)} = \sigma_{n_\alpha+1}^{(\alpha)}.$$

Since $d_\beta = 1$, the singular values of the β-component are simple and thus the left- and right-multiplicity of $\sigma_{n_\beta+1}^{(\beta)}$ is $q_\beta = 0$, $e_\beta = 1$. Then in the partitioning of the $V^{(l)}$ matrix from the SVDs of the extended matrices, we get

$$V_1^{(\alpha)} = [\underbrace{V_{11}^{(\alpha)}}_{n_\alpha - q_\alpha}, \underbrace{V_{12}^{(\alpha)}}_{q_\alpha + e_\alpha}, \underbrace{V_{13}^{(\alpha)}}_{d_\alpha - e_\alpha}] \} \, d_\alpha, \quad V_1^{(\beta)} = [\underbrace{V_{11}^{(\beta)}}_{n_\beta}, \underbrace{V_{12}^{(\beta)}}_{1}] \} \, 1,$$

here $V_{13}^{(\beta)}$ does not exist (it has zero columns). Moreover, $V_{12}^{(\beta)} = v_{1,n_\beta+1}^{(\beta)} \neq 0$. Then, similarly to (3.3),

$$\begin{aligned} [V_{11}, V_{12}, V_{13}] &= R^{\mathrm{T}} \begin{bmatrix} V_1^{(\alpha)} & 0 \\ 0 & V_1^{(\beta)} \end{bmatrix} \Psi, \\ &= R^{\mathrm{T}} \left[\begin{array}{cc|cc|c} V_{11}^{(\alpha)} & 0 & V_{12}^{(\alpha)} & 0 & V_{13}^{(\alpha)} \\ 0 & V_{11}^{(\beta)} & 0 & V_{12}^{(\beta)} & 0 \end{array} \right] \left[\begin{array}{c|c|c} \Psi_{11} & & \\ \hline & I & \\ \hline & & I \end{array} \right]. \end{aligned}$$

Clearly,

$$\begin{aligned} \operatorname{rank}(V_{12}) &= \operatorname{rank}\left(R^{\mathrm{T}} \begin{bmatrix} V_{12}^{(\alpha)} & 0 \\ 0 & v_{1,n_\beta+1}^{(\beta)} \end{bmatrix} \right) = \operatorname{rank}(V_{12}^{(\alpha)}) + 1, \\ \operatorname{rank}(V_{13}) &= \operatorname{rank}\left(R^{\mathrm{T}} \begin{bmatrix} V_{13}^{(\alpha)} \\ 0 \end{bmatrix} \right) = \operatorname{rank}(V_{13}^{(\alpha)}), \\ \text{and} \quad \operatorname{rank}([V_{12}, V_{13}]) &= \operatorname{rank}([V_{12}^{(\alpha)}, V_{13}^{(\alpha)}]) + 1, \end{aligned}$$

where $V_{12} \in \mathbb{R}^{d\times(q+e)}$, $V_{13} \in \mathbb{R}^{d\times(d-e)}$, $d \equiv d_\alpha + 1$, $d - e = d_\alpha - e_\alpha$ so $e \equiv e_\alpha + 1$, and $q + e = q_\alpha + e_\alpha + 1$ so $q \equiv q_\alpha$. Thus the α-component $[B_1^{(\alpha)}, A_{11}^{(\alpha)}]$ and the composed core problem $[B_1, A_{11}]$ are of the same class. □

Consequently, applying the theorem to examples of core problems with $d = 2$ from [2], see (2.7), we find there exist core problems with $d = 3$ in $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{S}$. Recalling the example (2.8), we see that for $d = 3$ there exist core problems in all four solvability classes. For $d > 3$, we can proceed analogously giving full solvability classification summarized in Table 1. Note that for any given $d > 1$ and any feasible class, we can find a composed core problem having only single right-hand side components. This result is interesting in view of the fact that any core problem with $d = 1$ belongs to $\mathcal{F}_1$ (the set of problems having always the TLS solution).

d	Classes			
1	$\mathcal{F}_1$	—	—	—
2	$\mathcal{F}_1$	$\mathcal{F}_2$	—	$\mathcal{S}$
3 and more	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{S}$

Table 1. Core problem with d right-hand sides belongs to one of the following classes.

3.3. Note on composing identical components. In general, it is not known what is the relation between the class of a composed problem and the classes of its components. Now we show that when a core problem is composed with itself, the solvability class cannot change. The theorem gives another way how to construct composed core problems in selected classes.

Theorem 3.3. *Let $A_{11}X_{11} \approx B_1$ be a core problem. If it is composed of two (or more) identical components $A_{11}^{(\alpha)} X_{11}^{(\alpha)} \approx B_1^{(\alpha)}$, then the core problem and its component belong to the same class.*

Schematically:

$$\forall(\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha,\ (\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha \boxplus (\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha = (\mathsf{CP}, 2d_\alpha, \mathcal{C}),$$
$$\text{and thus also} \quad \mathop{\boxplus}_{i=1}^{k} (\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha = (\mathsf{CP}, kd_\alpha, \mathcal{C}),$$

where $\mathcal{C} \in \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{S}\}$.

Proof. The statement holds trivially for compatible and degenerated components. Therefore, we focus on the proper incompatible components. Recall that

$$[B_1, A_{11}] = P^{\mathrm{T}} \left[\begin{array}{ccc|ccc} B_1^{(\alpha)} & 0 & 0 & A_{11}^{(\alpha)} & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & B_1^{(\alpha)} & 0 & 0 & A_{11}^{(\alpha)} \end{array}\right] \left[\begin{array}{c|c} R & 0 \\ \hline 0 & Q \end{array}\right]$$

$$= P^{\mathrm{T}} \left[I_k \otimes B_1^{(\alpha)} \;\middle|\; I_k \otimes A_{11}^{(\alpha)} \right] \left[\begin{array}{c|c} R & 0 \\ \hline 0 & Q \end{array} \right],$$

where "$\otimes$" denotes the Kronecker product; $A_{11}^{(\alpha)} \in \mathbb{R}^{m_\alpha \times n_\alpha}$, $B_{11}^{(\alpha)} \in \mathbb{R}^{m_\alpha \times d_\alpha}$, $n \equiv kn_\alpha$, $m \equiv km_\alpha$, and $d \equiv kd_\alpha$. Consider the full SVD $[B_1^{(\alpha)}, A_{11}^{(\alpha)}] = U^{(\alpha)} \Sigma^{(\alpha)} (V^{(\alpha)})^{\mathrm{T}}$ with square $U^{(\alpha)}$ and $V^{(\alpha)}$, with partitionings

$$(3.2) \qquad V^{(\alpha)} = \begin{bmatrix} V_1^{(\alpha)} \\ V_2^{(\alpha)} \end{bmatrix} \begin{matrix} \} \, d_\alpha \\ \} \, n_\alpha \end{matrix}, \quad \text{and} \quad V_1^{(\alpha)} = [\, V_{11}^{(\alpha)},\ V_{12}^{(\alpha)},\ V_{13}^{(\alpha)} \,]$$

as in (2.3). This immediately gives the SVD of the composed problem in the form

$$[B_1, A_{11}] = \underbrace{(P^{\mathrm{T}}(I_k \otimes U^{(\alpha)})\Pi)}_{U} \underbrace{(\Pi^{\mathrm{T}}(I_k \otimes \Sigma^{(\alpha)})\Psi)}_{\Sigma} \underbrace{\left(\left[\begin{array}{c|c} R & 0 \\ \hline 0 & Q \end{array} \right]^{\mathrm{T}} \left[\begin{array}{c} I_k \otimes V_1^{(\alpha)} \\ \hline I_k \otimes V_2^{(\alpha)} \end{array} \right] \Psi \right)^{\mathrm{T}}}_{V},$$

where Π and Ψ are permutation matrices sorting the singular values in the nonincreasing order on the diagonal of Σ. Since the permutations realize the commutation of the Kronecker product

$$\Pi^{\mathrm{T}}(I_k \otimes \Sigma^{(\alpha)})\Psi = \Sigma^{(\alpha)} \otimes I_k,$$

where Σ is square, we have simply $\Pi = \Psi$, see [9]. Note that multiplicities of all singular values are in the composed problem k-times larger than in its component.

Let us focus on V and denote $v_{:,j}^{(\alpha)}$ the jth column of $V_1^{(\alpha)}$. Then we get

$$(3.3) \qquad \begin{aligned} V_1 = [\, V_{11}, V_{12}, V_{13} \,] &= R^{\mathrm{T}}(I_k \otimes V_1^{(\alpha)})\Psi \\ &= R^{\mathrm{T}} \left[\, I_k \otimes v_{:,1}^{(\alpha)}, I_k \otimes v_{:,2}^{(\alpha)}, \ldots, I_k \otimes v_{:,n_\alpha+d_\alpha}^{(\alpha)} \,\right]. \end{aligned}$$

Clearly, the dimensions of V_{ij} in (3.3) are k-times larger than the dimensions of $V_{ij}^{(\alpha)}$ in (3.2). From the structure of the last matrix, and since R is orthogonal, we see that

$$\operatorname{rank}(V_{ij}) = \operatorname{rank}(RV_{ij}) = k \cdot \operatorname{rank}(V_{ij}^{(\alpha)}),$$

i.e., also the ranks of V_{ij} are k-times larger than the ranks of $V_{ij}^{(\alpha)}$.

Since the solvability classification is based on multiplicities of singular values, ranks and sizes of the blocks (in particular on the relations between these quantities), and all these quantities are in the composed problem just k-times larger, the component and the composed problem must belong to the same class. $\square$

Theorems 3.2 and 3.3 formulate basic relations between solvability classes in the course of core problems composing in two special cases. Further results are given in the next section.

In all cases discussed previously (see Theorems 3.2 and 3.3, and examples (2.7)), a composition of core problems leads to a composed problem with the same or worse TLS solvability on the scale

$$\mathcal{F}_1 \text{ (the best)}—\mathcal{F}_2—\mathcal{F}_3—\mathcal{S} \text{ (the worst)}.$$

Recall that $\mathcal{F}_1$ problems always have a TLS solution (that can be computed by the classical TLS algorithm), and core problems have a unique TLS solution; $\mathcal{F}_2$ problems also have a TLS solution (that cannot be simply computed by the classical TLS algorithm); $\mathcal{F}_3$ problems are still generic, but they have no TLS solution; and $\mathcal{S}$ problems are nongeneric and have no TLS solution. Such scale naturally corresponds to "removing the linear independence" from the upper right corner of V (see (2.3) and the classification below) and motivates the question whether the composition always worsens the TLS solvability. First we build up an illustrative example, then some general statements follow.

4.1. Does the composition always worsen the TLS solvability? The following example illustrates that composition of core problems can counter-intuitively improve the TLS solvability class. First, we give a particular example of an $\mathcal{F}_1$ single right-hand side core problem. Then we start to compose it to obtain more complicated problems.

E x a m p l e 4.1. Consider the approximation problem

$$\begin{bmatrix} a_l s \\ b_l c \end{bmatrix} x \approx \begin{bmatrix} a_l c \\ -b_l s \end{bmatrix}, \quad \text{where} \quad a_l > b_l > 0, \qquad (4.1)$$

$$s = \sin(\varphi),\ c = \cos(\varphi),\ \varphi \neq \tfrac{1}{2}\pi k,\ k \in \mathbb{Z}.$$

Then

$$[B_1^{(l)}, A_{11}^{(l)}] \equiv \begin{bmatrix} a_l c & a_l s \\ -b_l s & b_l c \end{bmatrix} = I_2 \begin{bmatrix} a_l & 0 \\ 0 & b_l \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix}^{\mathrm{T}}$$

is in principle the SVD of the extended matrix. Since $m_l = 2$, $n_l = 1$, $d_l = 1$, so $\sigma_{n_l+1}^{(l)} = b_l$ is simple, so $q_l = 0$, $e_l = 1$, and $V_1^{(l)} = [c, s]$, $V_{12} = [s]$, and V_{13} has no columns. Consequently (4.1) is of class $\mathcal{F}_1$ and has a unique TLS solution.

To show that (4.1) is a core problem, we need to verify that it satisfies (CP1)–(CP3). Clearly $A_{11}^{(l)}$ as well as $B_1^{(l)}$ are of full column rank, i.e., (CP1) and (CP2) hold. Employing the SVD

$$A_{11}^{(l)} = \left(\frac{1}{\sqrt{(a_l s)^2 + (b_l c)^2}} \begin{bmatrix} a_l s & -b_l c \\ b_l c & a_l s \end{bmatrix} \right) \begin{bmatrix} \sqrt{(a_l s)^2 + (b_l c)^2} \\ 0 \end{bmatrix} [\,1\,]^{\mathrm{T}},$$

it is easy to see that both

$$(U_1')^{\mathrm{T}} B_1^{(l)} = \left(\frac{1}{\sqrt{(a_l s)^2 + (b_l c)^2}} \begin{bmatrix} a_l s \\ b_l c \end{bmatrix} \right) B_1^{(l)} = \frac{(a_l^2 - b_l^2) cs}{\sqrt{(a_l s)^2 + (b_l c)^2}},$$
$$(U_2')^{\mathrm{T}} B_1^{(l)} = \left(\frac{1}{\sqrt{(a_l s)^2 + (b_l c)^2}} \begin{bmatrix} -b_l c \\ a_l s \end{bmatrix} \right) B_1^{(l)} = \frac{-2 a_l b_l}{\sqrt{(a_l s)^2 + (b_l c)^2}}$$

are (one-by-one) full row rank matrices, i.e., (CP3) is satisfied. Consequently (4.1) is a core problem of the class $\mathcal{F}_1$.

Now we take two particular choices of the parameters a_l, b_l in the example above, such that the composition of (4.1) with a single right-hand side degenerated component results in a core problem in $\mathcal{S}$ and $\mathcal{F}_1$, respectively.

Example 4.2. Consider the core problem (4.1) with $l = \alpha$, $a_\alpha = 3$ and $b_\alpha = 2$. Consider the core problem (4.1) with $l = \beta$, $a_\beta = 5$, $b_\beta = 1$. Compositions of these problems with the same degenerated component $[B_1^{(\gamma)}, A_{11}^{(\gamma)}] = [B_1^{(\gamma)}] = [4]$ (belonging also to $\mathcal{F}_1$), gives composed core problems with the following SVDs

$$\left[\begin{array}{cc|c} 3c & 0 & 3s \\ -2s & 0 & 2c \\ 0 & 4 & 0 \end{array}\right] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \left[\begin{array}{c|c|c} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{array}\right] \left[\begin{array}{c|c|c} 0 & c & -s \\ 1 & 0 & 0 \\ \hline 0 & s & c \end{array}\right]^{\mathrm{T}}, \tag{4.2}$$

$$\left[\begin{array}{cc|c} 5c & 0 & 5s \\ -1s & 0 & 1c \\ 0 & 4 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{c|c|c} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c|c|c} c & 0 & -s \\ 0 & 1 & 0 \\ \hline s & 0 & c \end{array}\right]^{\mathrm{T}}, \tag{4.3}$$

respectively. The partitioning (2.3) of the matrices V is suggested by the lines. Then (4.2) is of class $\mathcal{S}$, while (4.3) remains in the class $\mathcal{F}_1$.

Thus we have two proper incompatible core problems (both with $d = 2$) which we now compose together.

Example 4.3. Consider the core problems (4.2) and (4.3). Their composition results in a composed core problem with the following extended matrix and its SVD:

$$[B_1, A_{11}] = \left[\begin{array}{cccc|cc} B_1^{(\alpha)} & 0 & & & A_{11}^{(\alpha)} & \\ 0 & B_1^{(\gamma)} & & & 0 & \\ & & B_1^{(\beta)} & 0 & & A_{11}^{(\beta)} \\ & & 0 & B_1^{(\gamma)} & & 0 \end{array}\right] = \left[\begin{array}{cccc|cc} 3c & 0 & & & 3s & \\ -2s & 0 & & & 2c & \\ 0 & 4 & & & 0 & \\ & & 5c & 0 & & 5s \\ & & -1s & 0 & & 1c \\ & & 0 & 4 & & 0 \end{array}\right]$$

$$
= \left[\begin{array}{cccccc}
 & & 0 & 1 & 0 & \\
 & & 0 & 0 & 1 & \\
 & & 1 & 0 & 0 & \\
1 & 0 & & & & 0 \\
0 & 0 & & & & 1 \\
0 & 1 & & & & 0
\end{array}\right]
\left[\begin{array}{c|cc|ccc}
5 & 0 & & & & 0 \\
0 & 4 & & & & 0 \\
 & & 4 & 0 & 0 & \\
 & & 0 & 3 & 0 & \\
 & & 0 & 0 & 2 & \\
0 & 0 & & & & 1
\end{array}\right]
\left[\begin{array}{c|cc|ccc}
 & & 0 & c & -s & \\
 & & 1 & 0 & 0 & \\
c & 0 & & & & -s \\
0 & 1 & & & & 0 \\
\hline
 & & 0 & s & c & \\
s & 0 & & & & c
\end{array}\right]^{\mathrm{T}}.
$$

The partitioning (2.3) of V is again suggested by the lines. Clearly, we got a core problem with $d = 4$ that is of the class $\mathcal{F}_3$.

If we denote problems (4.2) and (4.3) as δ- and ε-component, respectively, the composition above can be schematically expressed as follows:

$$
\underbrace{((\mathsf{CP}, 1, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma)}_{(\mathsf{CP}, 2, \mathcal{S})_\delta} \boxplus \underbrace{((\mathsf{CP}, 1, \mathcal{F}_1)_\beta \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma)}_{(\mathsf{CP}, 2, \mathcal{F}_1)_\varepsilon} = (\mathsf{CP}, 4, \mathcal{F}_3).
$$

Now we look at the whole process the other way. Having in hand a problem of the class $\mathcal{S}$ (i.e., nongeneric one), its composition with a suitable $\mathcal{F}_1$ problem may result in a problem in $\mathcal{F}_3$ (i.e., it becomes generic). This can be seen as a form of *correction*, or *improvement* of the δ-component in terms of TLS solvability classes. Such improvement can be done in general, which will be investigated in the next section.

R e m a r k 4.4. Since the core problems composition is associative and commutative (up to a permutation of components), the problem from Example 4.3 can also be expressed as follows (classes of the intermediate problems or components can be seen directly by crossing out suitable rows and columns of the SVD in Example 4.3):

$$
\begin{aligned}
(\mathsf{CP}, 4, \mathcal{F}_3) &= \underbrace{((\mathsf{CP}, 1, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\beta)}_{(\mathsf{CP}, 2, \mathcal{F}_1)_{\alpha\boxplus\beta}} \boxplus \underbrace{((\mathsf{CP}, 1, \mathcal{F}_1)_\gamma \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma)}_{(\mathsf{CP}, 2, \mathcal{F}_1)_{\gamma\boxplus\gamma}} \\
&= \underbrace{((\mathsf{CP}, 1, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\beta \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma)}_{(\mathsf{CP}, 3, \mathcal{S})_{\alpha\boxplus\beta\boxplus\gamma}} \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma \\
&= \underbrace{((\mathsf{CP}, 1, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma)}_{(\mathsf{CP}, 3, \mathcal{F}_3)_{\alpha\boxplus\gamma\boxplus\gamma}} \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\beta \\
&= \underbrace{((\mathsf{CP}, 1, \mathcal{F}_1)_\beta \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\gamma)}_{(\mathsf{CP}, 3, \mathcal{F}_1)_{\beta\boxplus\gamma\boxplus\gamma}} \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\alpha.
\end{aligned}
$$

The first and the last row show that a composition of two $\mathcal{F}_1$ (in the first row one proper incompatible and one degenerated; in the last row two proper incompatible) components may result in an $\mathcal{F}_3$ problem. Recall that for two single right-hand side (i.e., $\mathcal{F}_1$) components, such composition is not possible (see Theorem 3.1 and the comment below), and therefore it was not observed in [2].

4.2. Improvement of nongeneric problems. The following theorem shows that it is always possible to move a nongeneric (i.e., class $\mathcal{S}$) core problem to the class of generic problems by composing it with another problem representing a sort of correction of the measured data, see Example 4.3.

Theorem 4.5. *Let $A_{11}^{(\alpha)} X_{11}^{(\alpha)} \approx B_1^{(\alpha)}$, $A_{11}^{(\alpha)} \in \mathbb{R}^{m_\alpha \times n_\alpha}$, $B_1^{(\alpha)} \in \mathbb{R}^{m_\alpha \times d_\alpha}$ be a core problem (that will serve as a component) and let it be in the class $\mathcal{S}$. Then there exists a component $A_{11}^{(\beta)} X_{11}^{(\beta)} \approx B_1^{(\beta)}$, $A_{11}^{(\beta)} \in \mathbb{R}^{m_\beta \times n_\beta}$, $B_1^{(\beta)} \in \mathbb{R}^{m_\beta \times d_\beta}$ such that the composed core problem*

$$A_{11} X_{11} \equiv \left(P^{\mathrm{T}} \begin{bmatrix} A_{11}^{(\alpha)} & 0 \\ 0 & A_{11}^{(\beta)} \end{bmatrix} Q \right) X_{11} \approx \left(P^{\mathrm{T}} \begin{bmatrix} B_1^{(\alpha)} & 0 \\ 0 & B_1^{(\beta)} \end{bmatrix} R \right) \equiv B_{11},$$

is in the class $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Schematically: $\forall (\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha$, $\exists (\mathsf{CP}, d_\beta, \mathcal{C})_\beta$ so that

$$(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{C})_\beta = (\mathsf{CP}, d_\alpha + d_\beta, \mathcal{F}),$$

where $\mathcal{C} \in \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{S}\}$ and $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.

Proof. Let $[\, B_1^{(\alpha)}, A_{11}^{(\alpha)} \,] = U^{(\alpha)} \Sigma^{(\alpha)} (V^{(\alpha)})^{\mathrm{T}}$ be the SVD with the partitioning (2.3) of $V^{(\alpha)}$. Further, let

$$\sigma_1^{(\alpha)} \geqslant \sigma_2^{(\alpha)} \geqslant \ldots \geqslant \sigma_{n_\alpha - q_\alpha}^{(\alpha)},$$

be the singular values of $V_{11}^{(\alpha)}$. Let k be the number of distinct singular values of $V_{11}^{(\alpha)}$ with the multiplicities ϱ_j, $j = 1, \ldots, k$; i.e., $\sum_{j=1}^{k} \varrho_j = n_\alpha - q_\alpha$. Consider also a partitioning of $V_{11}^{(\alpha)}$ with respect to these multiplicities,

$$V_{11}^{(\alpha)} = [\, V_{11,1}^{(\alpha)}, V_{11,2}^{(\alpha)}, \ldots, V_{11,k}^{(\alpha)} \,] \in \mathbb{R}^{d_\alpha \times (n_\alpha - q_\alpha)}, \quad \text{with } V_{11,j}^{(\alpha)} \in \mathbb{R}^{d_\alpha \times \varrho_j}$$

being of full column ranks. Since the α-component is nongeneric, i.e., of class $\mathcal{S}$, $[V_{12}^{(\alpha)}, V_{13}^{(\alpha)}]$ has linearly dependent rows. Let t be defined so that

$$\begin{aligned} &\operatorname{rank}([\, V_{11,t}^{(\alpha)}, V_{11,t+1}^{(\alpha)}, \ldots, V_{11,k}^{(\alpha)}, V_{12}^{(\alpha)}, V_{13}^{(\alpha)} \,]) = d_\alpha, \quad \text{and} \\ &\operatorname{rank}([\, V_{11,t+1}^{(\alpha)}, \ldots, V_{11,k}^{(\alpha)}, V_{12}^{(\alpha)}, V_{13}^{(\alpha)} \,]) < d_\alpha. \end{aligned} \tag{4.4}$$

Now we construct a suitable β-component. Consider an arbitrary β-component such that it belongs to $\mathcal{F}_1$ (thus $[\,V_{12}^{(\beta)}, V_{13}^{(\beta)}\,] \in \mathbb{R}^{d_\beta \times d_\beta}$ is square invertible and $q_\beta = 0$) and

$$\sigma^{(\beta)}_{n_\beta+1} \equiv \sigma^{(\alpha)}_{\varrho_1+\ldots+\varrho_{t-1}+1} = \ldots = \sigma^{(\alpha)}_{\varrho_1+\ldots+\varrho_{t-1}+\varrho_t},$$

i.e., the e_β-tuple singular value of the β-component corresponding to $V_{12}^{(\beta)}$ is equal to the ϱ_t-tuple singular value of the α-component corresponding to $V_{11,t}^{(\alpha)}$. Then the block $V_1 \in \mathbb{R}^{d\times(n+d)}$ with $d = d_\alpha + d_\beta$, $n = n_\alpha + n_\beta$, from the SVD of the composed problem takes the form

$$V_1 = [V_{11}, V_{12}, V_{13}] = \left[\begin{matrix} [V^{(\alpha)}_{11,1}, \ldots, V^{(\alpha)}_{11,t-1}] & 0 \\ 0 & V^{(\beta)}_{11} \end{matrix}\right| \underbrace{\left|\begin{matrix} V^{(\alpha)}_{11,t} & 0 \\ 0 & V^{(\beta)}_{12} \end{matrix}\right|}_{\varrho_t + e_\beta} \underbrace{\left.\begin{matrix} [V^{(\alpha)}_{11,t+1}, \ldots, V^{(\alpha)}_{11,k}], V^{(\alpha)}_{12}, V^{(\alpha)}_{13} & 0 \\ 0 & V^{(\beta)}_{13} \end{matrix}\right]}_{(\varrho_{t+1}+\ldots+\varrho_k)+(d_\alpha+q_\alpha)+(d_\beta-e_\beta)} \left[\begin{array}{c|c|c} \Psi_{11} & & \\ \hline & I & \\ \hline & & \Psi_{13} \end{array}\right].$$

To align the blocks suggested by the vertical lines with the partitioning $[V_{11}, V_{12}, V_{13}]$, the $(n+1)$st (i.e., the dth last) column of V_1 has to be in $\begin{bmatrix} V^{(\alpha)}_{11,t} & 0 \\ 0 & V^{(\beta)}_{12} \end{bmatrix}$. Equivalently

$$\begin{aligned} d = d_\alpha + d_\beta &> (\varrho_{t+1}+\ldots+\varrho_k) + (d_\alpha + q_\alpha) + (d_\beta - e_\beta), \quad \text{i.e.,} \\ e_\beta &> (\varrho_{t+1}+\ldots+\varrho_k) + q_\alpha. \end{aligned}$$

Recall that also $e_\beta \leqslant d_\beta$, see (2.2)–(2.3). Thus, put

$$\begin{aligned} e_\beta &\equiv (\varrho_{t+1}+\ldots+\varrho_k) + q_\alpha + 1, \quad \text{and} \\ d_\beta &\equiv (\varrho_{t+1}+\ldots+\varrho_k) + q_\alpha + 1 + \Delta, \quad \Delta \geqslant 0. \end{aligned}$$

Then $V_{13}^{(\beta)} \in \mathbb{R}^{d_\beta\times\Delta}$ and $V_{13} \in \mathbb{R}^{(d_\alpha+d_\beta)\times((\varrho_{t+1}+\ldots+\varrho_k)+(d_\alpha+q_\alpha)+(d_\beta-e_\beta))} \equiv \mathbb{R}^{d\times(d-1)}$. We see that blocks are aligned and the $(n+1)$st (dth last) column of V_1 is exactly the last column of V_{12}. Since (4.4) is of full row rank d_α and $[\,V_{12}^{(\beta)}, V_{13}^{(\beta)}\,]$ is square invertible of rank d_β,

$$[V_{12}, V_{13}] = \left[\begin{matrix} V^{(\alpha)}_{11,t} & 0 \\ 0 & V^{(\beta)}_{12} \end{matrix}\right|\left.\begin{matrix} [V^{(\alpha)}_{11,t+1}, \ldots, V^{(\alpha)}_{11,k}], V^{(\alpha)}_{12}, V^{(\alpha)}_{13} & 0 \\ 0 & V^{(\beta)}_{13} \end{matrix}\right] \left[\begin{array}{c|c} I & \\ \hline & \Psi_{13} \end{array}\right]$$

is also of full row rank $d = d_\alpha + d_\beta$, and thus the composed problem is of class $\mathcal{F}$.

It remains to show that there always exists a β-component satisfying all the requested properties. We take the simplest one,

$$[B_1^{(\beta)}, A_{11}^{(\beta)}] = [B_1^{(\beta)}] \equiv \sigma^{(\alpha)}_{\varrho_1+\ldots+\varrho_{t-1}+1} I_{\varrho_{t+1}+\ldots+\varrho_k+q_\alpha+1}, \tag{4.5}$$

i.e., $n_\beta = 0$ (it is a degenerated component), $m_\beta = d_\beta = e_\beta = (\varrho_{t+1}+\ldots+\varrho_k)+q_\alpha+1$, $\Delta = 0$, and $\sigma^{(\beta)}_{n_\beta+1} \equiv \sigma^{(\beta)}_1 = \sigma^{(\alpha)}_{\varrho_1+\ldots+\varrho_{t-1}+1}$ with the multiplicity e_β. The matrix $V^{(\beta)}$ from the SVD of $[B^{(\beta)}_1, A^{(\beta)}_{11}]$ contains only the block $V^{(\beta)}_{12}$ (the other blocks have no rows or columns, see (2.3) and the classification below). Moreover, $V^{(\beta)} = V^{(\beta)}_{12} = I_{\varrho_{t+1}+\ldots+\varrho_k+q_\alpha+1}$ is obviously square invertible. □

Note that we proved slightly stronger variant of Theorem 4.5. Instead of looking for a general β-component, we restricted ourselves first only to the class $\mathcal{F}_1$, and then only to the degenerated (class $\mathcal{F}_1$) components. However, such restriction was used only for simplicity and it is not necessary (see in particular Example 4.3).

Recall further the definition of t in (4.4). Instead of t, we may use any ϱ_τ and $V^{(\alpha)}_{11,\tau}$, $1 \leqslant \tau \leqslant t$, in the roles of ϱ_t and $V^{(\alpha)}_{11,t}$ for the construction of a β-component in the proof. In particular, we may simply use a degenerated β-component in the form[1] $[B^{(\beta)}_1, A^{(\beta)}_{11}] = [B^{(\beta)}_1] \equiv \sigma^{(\alpha)}_1 I_{n_\alpha+1}$ instead of (4.5). Our choice in (4.5) is in some sense the minimal one (since t is maximal among all τ's, $\Delta = 0$ is minimal among all Δ's, and both minimize the dimensions of the β-component).

Moreover, the resulting composed problem has in its SVD the block V_{13} that contains $\begin{bmatrix} V^{(\alpha)}_{12} & V^{(\alpha)}_{13} \\ 0 & 0 \end{bmatrix}$ as a submatrix. Since $[V^{(\alpha)}_{12}, V^{(\alpha)}_{13}] \in \mathbb{R}^{d_\alpha \times (d_\alpha+q_\alpha)}$, $q_\alpha \geqslant 0$, has linearly dependent rows and the number of its columns is larger than or equal to the number of columns, it has also linearly dependent columns. Thus also $\begin{bmatrix} V^{(\alpha)}_{12} & V^{(\alpha)}_{13} \\ 0 & 0 \end{bmatrix}$ and in particular V_{13} have linearly dependent columns. Consequently, the problem composed in the proof above does not belong to the classes $\mathcal{F}_1$ and $\mathcal{F}_2$. We actually proved that, schematically:

$\forall (\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha$, $\exists$ $(\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta$ so that

$$(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta = (\mathsf{CP}, d_\alpha + d_\beta, \mathcal{F}_3),$$

where the β-component is degenerated. This motivates a general result as follows.

Let us return back to the original, less restricted case: If we compose the α-component of the class $\mathcal{S}$ with an arbitrary β-component so that the resulting composed problem is in $\mathcal{F}$, then (see in particular (4.4)) $\begin{bmatrix} V^{(\alpha)}_{11,t} \\ 0 \end{bmatrix}$ and $\begin{bmatrix} V^{(\alpha)}_{12} & V^{(\alpha)}_{13} \\ 0 & 0 \end{bmatrix}$ have to be submatrices of $[V_{12}, V_{13}]$. Since the singular value corresponding to $V^{(\alpha)}_{11,t}$ is strictly larger than the singular value corresponding to $V^{(\alpha)}_{12}$, $\begin{bmatrix} V^{(\alpha)}_{12} & V^{(\alpha)}_{13} \\ 0 & 0 \end{bmatrix}$ is a submatrix of V_{13}. Consequently (as discussed above), if the composition results in an

[1] Note that the so-called TLS algorithm when applied to the composed problem with this choice of a β-component returns a zero output.

$\mathcal{F}$ problem, it always belongs to $\mathcal{F}_3$. The classes $\mathcal{F}_1$ and $\mathcal{F}_2$ are not available. We formulate this observation as a corollary.

Corollary 4.6. *Let $A_{11}^{(\alpha)}X_{11}^{(\alpha)} \approx B_1^{(\alpha)}$ be a core problem in the class $\mathcal{S}$, and let $A_{11}^{(\beta)}X_{11}^{(\beta)} \approx B_1^{(\beta)}$ be an arbitrary core problem. Their composition cannot result in a problem in the class $\mathcal{F}_1$ or $\mathcal{F}_2$.*

Schematically: $\forall(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha$, $\forall(\mathsf{CP}, d_\beta, \mathcal{C})_\beta$,

$$(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{C})_\beta \neq (\mathsf{CP}, d_\alpha + d_\beta, \mathcal{F}_1), (\mathsf{CP}, d_\alpha + d_\beta, \mathcal{F}_2),$$

where $\mathcal{C} \in \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{S}\}$.

In other words, we are able to *move a class $\mathcal{S}$ (nongeneric) problem to the class $\mathcal{F}_3$* (generic, but without a TLS solution), but *no better result* is achievable by employing the approach above. The TLS solvability of a nongeneric core problem cannot be improved by its composition with another core problem.

4.3. Available and unavailable classes. Table 2 summarizes all the known available compositions of two core problems in terms of classes, see (2.7), Theorems 3.3, 3.2, Example 4.3, and Remark 4.4.

$\boxplus$	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{S}$
$\mathcal{F}_1$	$\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, or $\mathcal{S}$ $\star$	sym.	sym.	sym.
$\mathcal{F}_2$	$\mathcal{F}_2$	$\mathcal{F}_2$	sym.	sym.
$\mathcal{F}_3$	$\mathcal{F}_3$		$\mathcal{F}_3$	sym.
$\mathcal{S}$	$\mathcal{F}_3$ or $\mathcal{S}$ $\star$			$\mathcal{S}$ $\star$

Table 2. List of *known available* compositions of two core problems (components) in terms of classes. Stars ($\star$) denote cases where all four possible results have been analyzed (cf. Table 3). The table is symmetric.

On the contrary, at the end of the previous section we have found for the first time a combination (of classes of components and a class of the resulting composed problem) that is not achievable. Consequently, it is clear that all 40 combinations are not available for core problem compositions. The following theorems discuss two more such cases. First we prove the assertion of Corollary 4.6 also for $\mathcal{F}_3$ problems. Then we show that a combination of two $\mathcal{S}$ class core problems results in a composed problem belonging again to $\mathcal{S}$.

Theorem 4.7. *Let $A_{11}^{(\alpha)}X_{11}^{(\alpha)} \approx B_1^{(\alpha)}$ be a core problem in the class $\mathcal{F}_3$, and let $A_{11}^{(\beta)}X_{11}^{(\beta)} \approx B_1^{(\beta)}$ be an arbitrary core problem. Their composition cannot result in a problem in the class $\mathcal{F}_1$ or $\mathcal{F}_2$.*

Schematically: $\forall(\mathsf{CP}, d_\alpha, \mathcal{F}_3)_\alpha$, $\forall(\mathsf{CP}, d_\beta, \mathcal{C})_\beta$,

$$(\mathsf{CP}, d_\alpha, \mathcal{F}_3)_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{C})_\beta \neq (\mathsf{CP}, d_\alpha + d_\beta, \mathcal{F}_1),\ (\mathsf{CP}, d_\alpha + d_\beta, \mathcal{F}_2),$$

where $\mathcal{C} \in \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{S}\}$.

P r o o f. First of all note that the assertion is trivially true for $\mathcal{F}_3$ problems which are composed, and contain an $\mathcal{S}$ component (use Corollary 4.6 and the associativity of core problem composing). Now consider a general $\mathcal{F}_3$ problem as the α-component with partitioning of the matrix of right singular vectors as usual. Then the blocks of $V_1^{(\alpha)} = \left[V_{11}^{(\alpha)}, V_{12}^{(\alpha)}, V_{13}^{(\alpha)}\right] \in \mathbb{R}^{d_\alpha \times (n_\alpha + d_\alpha)}$ satisfy:

$$\begin{aligned} [V_{12}^{(\alpha)}, V_{13}^{(\alpha)}] \in \mathbb{R}^{d_\alpha \times (d_\alpha + q_\alpha)} & \quad \text{is of full row rank } d_\alpha, \text{ and} \\ V_{13}^{(\alpha)} \in \mathbb{R}^{d_\alpha \times (d_\alpha - e_\alpha)} & \quad \text{has linearly dependent columns (and rows, } e_\alpha \geqslant 1). \end{aligned}$$

Recall that $V_{12}^{(\alpha)}$ corresponds to the singular value $\sigma_{n_\alpha+1}^{(\alpha)}$ with multiplicity $q_\alpha + e_\alpha$. Consider also the SVDs of the β-component and of the composed core problem, in particular the matrices $V_1^{(\beta)} \in \mathbb{R}^{d_\beta \times (n_\beta + d_\beta)}$ and $V_1 = [V_{11}, V_{12}, V_{13}] \in \mathbb{R}^{d \times (n+d)}$. Clearly,

$$V_1 = \begin{bmatrix} V_1^{(\alpha)} & 0 \\ 0 & V_1^{(\beta)} \end{bmatrix} \Psi = \begin{bmatrix} V_{11}^{(\alpha)} & V_{12}^{(\alpha)} & V_{13}^{(\alpha)} & 0 \\ 0 & 0 & 0 & V_1^{(\beta)} \end{bmatrix} \Psi,$$

where the permutation matrix Ψ sorts the singular values originated in both components into nonincreasing order. Thus Ψ does not change the ordering of columns of V_1 originated in one particular component, it only interlaces them with the columns originated in the other component.

Assume that the composed problem is in the class $\mathcal{F}$. Then $[V_{12}, V_{13}]$ is of full row rank. Since the α-component is of $\mathcal{F}_3$ and $V_{13}^{(\alpha)}$ has linearly dependent rows, $\begin{bmatrix} V_{12}^{(\alpha)} & V_{13}^{(\alpha)} \\ 0 & 0 \end{bmatrix}$ is a submatrix of $[V_{12}, V_{13}]$. Thus σ_{n+1} (the singular value corresponding to the V_{12} block of the composed problem) satisfies $\sigma_{n+1} \geqslant \sigma_{n_\alpha+1}$. Since $V_{13}^{(\alpha)}$ corresponds to singular values strictly smaller than σ_{n+1}, $\begin{bmatrix} V_{13}^{(\alpha)} \\ 0 \end{bmatrix}$ is a submatrix of V_{13}. Since $V_{13}^{(\alpha)}$ has linearly dependent columns, V_{13} has linearly dependent columns as well. Consequently, the composed problem cannot belong to $\mathcal{F}_1$ or $\mathcal{F}_2$. □

Theorem 4.5, Corollary 4.6, and Theorem 4.7 together are of particular importance. They show that while class $\mathcal{S}$ problems can be moved to $\mathcal{F}_3$ (but no better improvement is possible), $\mathcal{F}_3$ problems cannot be improved further. Consequently, *the set of $\mathcal{F}_3$ and $\mathcal{S}$ core problems is in some sense closed with respect to compositions with core problems from other classes.* This indicates that the distinction between $\mathcal{F}_3$ and $\mathcal{S}$ problems is rather artificial, as it originated in the generic—nongeneric classification introduced in [10]. Recall that in both $\mathcal{F}_3$ and $\mathcal{S}$, the TLS solution does not exist. Now we show that the class $\mathcal{S}$ is closed in a slightly weaker sense.

Theorem 4.8. *Composition of two (or more) class $\mathcal{S}$ core problems always results in a class $\mathcal{S}$ problem.*

Schematically: $\forall(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha$, $\forall(\mathsf{CP}, d_\beta, \mathcal{S})_\beta$,

$$(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{S})_\beta = (\mathsf{CP}, d_\alpha + d_\beta, \mathcal{S}),$$

or equivalently

$$(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{S})_\beta \neq (\mathsf{CP}, d_\alpha + d_\beta, \mathcal{F}), \quad \mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3.$$

P r o o f. Let $A_{11}^{(l)} X_{11}^{(l)} \approx B_1^{(l)}$, $A_{11}^{(l)} \in \mathbb{R}^{m_l \times n_l}$, $B_1^{(l)} \in \mathbb{R}^{m_l \times d_l}$ for $l = \alpha, \beta$ be two core problems in the class $\mathcal{S}$. Consider their SVDs $[B_1^{(l)}, A_{11}^{(l)}] = U^{(l)} \Sigma^{(l)} (V^{(l)})^{\mathrm{T}}$, with the partitionings

$$V^{(l)} = \begin{bmatrix} V_1^{(l)} \\ V_2^{(l)} \end{bmatrix} = \begin{bmatrix} V_{11}^{(l)} & V_{12}^{(l)} & V_{13}^{(l)} \\ V_{21}^{(l)} & V_{22}^{(l)} & V_{23}^{(l)} \end{bmatrix} \begin{matrix} \}\, d_l \\ \}\, n_l \end{matrix} \qquad \operatorname{rank}([\,V_{12}^{(l)}, V_{13}^{(l)}\,]) < d_l.$$

We are interested in the singular values $\sigma_{n_l+1}^{(l)}$, $l = \alpha, \beta$. There are two cases: Either $\sigma_{n_\alpha+1}^{(\alpha)} = \sigma_{n_\beta+1}^{(\beta)}$, or $\sigma_{n_\alpha+1}^{(\alpha)} > \sigma_{n_\beta+1}^{(\beta)}$ (the third case $\sigma_{n_\alpha+1}^{(\alpha)} < \sigma_{n_\beta+1}^{(\beta)}$ is essentially the same as the second, only with the exchanged roles of α- and β-components).

Case 1. Let $\sigma_{n_\alpha+1}^{(\alpha)} = \sigma_{n_\beta+1}^{(\beta)}$. Then the SVD of

$$[B_1, A_{11}] = P^{\mathrm{T}} \left[\begin{array}{cc|cc} B_1^{(\alpha)} & 0 & A_{11}^{(\alpha)} & 0 \\ 0 & B_1^{(\beta)} & 0 & A_{11}^{(\beta)} \end{array}\right] \left[\begin{array}{c|c} R & 0 \\ \hline 0 & Q \end{array}\right]$$

gives V with the structure

$$V_1 = R^{\mathrm{T}} \left[\begin{array}{cc|cc|cc} V_{11}^{(\alpha)} & 0 & V_{12}^{(\alpha)} & 0 & V_{13}^{(\alpha)} & 0 \\ 0 & V_{11}^{(\beta)} & 0 & V_{12}^{(\beta)} & 0 & V_{13}^{(\beta)} \end{array}\right] \left[\begin{array}{c|c|c} \Psi_{11} & & \\ \hline & I & \\ \hline & & \Psi_{13} \end{array}\right] \in \mathbb{R}^{d \times (n+d)},$$

where $n \equiv n_\alpha + n_\beta$, $d \equiv d_\alpha + d_\beta$. It remains to verify whether the vertical lines correspond to the partitioning of $V_1 = [V_{11}, V_{12}, V_{13}]$ with respect to σ_{n+1}, i.e., whether σ_{n+1} is the singular value $\sigma_{n_\alpha+1}^{(\alpha)} = \sigma_{n_\beta+1}^{(\beta)}$.

Since $V_{11}^{(l)} \in \mathbb{R}^{d_l \times (n_l - q_l)}$, we have $\begin{bmatrix} V_{11}^{(\alpha)} & 0 \\ 0 & V_{11}^{(\beta)} \end{bmatrix} \in \mathbb{R}^{d \times (n - q_\alpha - q_\beta)}$. Because $q_l \geqslant 0$, we have $n - q_\alpha - q_\beta < n + 1$, i.e., the $(n+1)$th column of V_1 does not belong to the first block. Similarly, from $V_{13}^{(l)} \in \mathbb{R}^{d_l \times (d_l - e_l)}$ we get $\begin{bmatrix} V_{13}^{(\alpha)} & 0 \\ 0 & V_{13}^{(\beta)} \end{bmatrix} \in \mathbb{R}^{d \times (d - e_\alpha - e_\beta)}$. Because $e_l \geqslant 1$, then $d > d - e_\alpha - e_\beta$, i.e., the $(n+1)$th column (which is actually also the dth last column of V_1) does not belong to this last block.

Consequently, $\sigma_{n+1} = \sigma_{n_\alpha+1}^{(\alpha)} = \sigma_{n_\beta+1}^{(\beta)}$ and it has multiplicity $q + e$, where $q \equiv q_\alpha + q_\beta$ is its left-, and $e \equiv e_\alpha + e_\beta$ is its right-multiplicity. Since both $[V_{12}^{(l)}, V_{13}^{(l)}]$ for $l = \alpha, \beta$, have linearly dependent rows, $[V_{11}, V_{12}]$ has linearly dependent rows as well, i.e., $\operatorname{rank}([V_{11}, V_{12}]) < d$. Finally, the composed problem is of the class $\mathcal{S}$.

Case 2. Let $\sigma_{n_\alpha+1}^{(\alpha)} > \sigma_{n_\beta+1}^{(\beta)}$. Then the SVD of the extended matrix gives V with much more complicated structure of V_1. Here the relations between $\sigma_1^{(\beta)}, \ldots, \sigma_{n_\beta}^{(\beta)}$ and $\sigma_{n_\alpha+1}^{(\alpha)}$ have to be taken into account. In particular there may be singular values strictly larger than, equal to, and smaller than $\sigma_{n_\alpha+1}^{(\alpha)}$. To reflect this, we introduce the formal partitioning

$$V_{11}^{(\beta)} = [V_{11\mathsf{A}}^{(\beta)}, V_{11\mathsf{B}}^{(\beta)}, V_{11\mathsf{C}}^{(\beta)}] \in \mathbb{R}^{d_\beta \times (n_\beta - q_\beta)}$$

without specifying the dimensions of the individual blocks. Then

$$V_1 = R^{\mathrm{T}} \left[\begin{array}{cc|cc|cccc} V_{11}^{(\alpha)} & 0 & V_{12}^{(\alpha)} & 0 & V_{13}^{(\alpha)} & 0 & 0 & 0 \\ 0 & V_{11\mathsf{A}}^{(\beta)} & 0 & V_{11\mathsf{B}}^{(\beta)} & 0 & V_{11\mathsf{C}}^{(\beta)} & V_{12}^{(\beta)} & V_{13}^{(\beta)} \end{array}\right] \left[\begin{array}{c|c|c} \Psi_{11} & & \\ \hline & I & \\ \hline & & \Psi_{13} \end{array}\right],$$

but the partitioning suggested by the vertical lines *may not* correspond to the partitioning of $V_1 = [V_{11}, V_{12}, V_{13}]$ with respect to σ_{n+1}. However, the number of columns of the first suggested block is less than, or equal to $n - q_\alpha - q_\beta$. Since $q_l \geqslant 0$, we have $n - q_\alpha - q_\beta < n + 1$ and thus the $(n+1)$st column of V_1 is either in the second, or in the third of the suggested blocks. The matrix $[V_{12}, V_{13}]$ is then in general a *submatrix* of the matrix formed by the last two suggested blocks.

Since $[V_{12}^{(\alpha)}, V_{13}^{(\alpha)}]$ has linearly dependent rows, the matrix formed by the last two suggested blocks has linearly dependent rows, i.e., it is of the rank strictly smaller than d. Therefore, any of its submatrices is of rank strictly smaller than d, and in particular $\operatorname{rank}([V_{11}, V_{12}]) < d$. Thus the composed problem is of class $\mathcal{S}$. □

Table 2 of known available compositions of core problems (in terms of classes) can now be complemented by a list of known unavailable compositions in Table 3, see Corrolary 4.6 and Theorem 4.8. Both tables together indicate combinations that require further investigation.

$\boxplus$	$\mathcal{F}_1$	$\mathcal{F}_2$	$\mathcal{F}_3$	$\mathcal{S}$
$\mathcal{F}_1$	— *	sym.	sym.	sym.
$\mathcal{F}_2$			sym.	sym.
$\mathcal{F}_3$	$\mathcal{F}_1$ and $\mathcal{F}_2$	$\mathcal{F}_1$ and $\mathcal{F}_2$	$\mathcal{F}_1$ and $\mathcal{F}_2$	sym.
$\mathcal{S}$	$\mathcal{F}_1$ and $\mathcal{F}_2$ *	$\mathcal{F}_1$ and $\mathcal{F}_2$	$\mathcal{F}_1$ and $\mathcal{F}_2$	$\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ *

Table 3. List of *known unavailable* compositions of two core problems (components) in terms of classes. Stars ($\star$) denote cases where all four possible results have been analyzed (cf. Table 2). The table is symmetric.

5. Existence of irreducible core problems in various classes

All particular examples of core problems discussed in the previous sections (e.g., when filling up Table 2) have been composed from single right-hand side components. However, in [2] it was shown that there exists an irreducible (nondecomposable) core problem with $d = 2$ in $\mathcal{F}_2$. For completeness, we show by examples that there exist irreducible core problems with $d = 2$ also in $\mathcal{F}_1$ and $\mathcal{S}$. Recall that an $\mathcal{F}_3$ problem with $d = 2$ does not exist, see Table 1.

Example 5.1. Consider three problems $A_{11}X_1 \approx B_1$, $A_{11} \in \mathbb{R}^{4\times 2}$, $B_1 \in \mathbb{R}^{4\times 2}$ given in forms of SVDs of their extended matrices:

$$[B_1, A_{11}] = I_4 \left[\begin{array}{cc|c|c} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \left(\frac{1}{3} \left[\begin{array}{cc|c|c} -1 & -3 & \sqrt{3} & \sqrt{3} \\ 3 & -1 & \sqrt{3} & -\sqrt{3} \\ \hline \sqrt{3} & \sqrt{3} & 1 & 3 \\ \sqrt{3} & -\sqrt{3} & -3 & 1 \end{array}\right] \right)^{\mathrm{T}}, \tag{5.1}$$

$$[B_1, A_{11}] = I_4 \left[\begin{array}{c|cc|c} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \left(\frac{1}{3} \left[\begin{array}{c|cc|c} -1 & -3 & \sqrt{3} & \sqrt{3} \\ 3 & -1 & \sqrt{3} & -\sqrt{3} \\ \hline \sqrt{3} & \sqrt{3} & 1 & 3 \\ \sqrt{3} & -\sqrt{3} & -3 & 1 \end{array}\right] \right)^{\mathrm{T}}, \tag{5.2}$$

$$[B_1, A_{11}] = I_4 \left[\begin{array}{cc|c|c} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \left(\frac{1}{2} \left[\begin{array}{cc|c|c} \sqrt{2} & \sqrt{2} & 0 & 0 \\ -1 & 1 & -1 & 1 \\ \hline 0 & 0 & \sqrt{2} & \sqrt{2} \\ 1 & -1 & -1 & 1 \end{array}\right] \right)^{\mathrm{T}}. \tag{5.3}$$

The second problem has already been presented in [3] and [2], it is included for completeness. Note that the matrix of the left singular vectors may be chosen arbitrarily, we use I_4 for simplicity. The partitioning of the right-most matrices of the right singular vectors corresponds to (2.3). Clearly, the problems above belong to the class $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{S}$, respectively.

Now we show that they represent core problems. Since all three matrices $[B_1, A_{11}]$ are of full column rank, A_{11} and B_1 are also of full column rank. Thus the problems satisfy (CP1) and (CP2). Matrices A_{11} have simple singular values

$$\varsigma_{1,2} = \frac{1}{2}\sqrt{25 \pm 3\sqrt{2}}, \quad \varsigma_{1,2} = \sqrt{4 \pm \frac{3\sqrt{5}}{8}}, \quad \varsigma_{1,2} = \sqrt{5 \pm \frac{\sqrt{2}\sqrt{59}}{4}},$$

respectively. It is easy to find their left and right singular vectors (e.g., by using MATLAB with Symbolic Math Toolbox)[2], and to verify that (CP3) is satisfied as well. Consequently, all problems represent core problems with the SVD forms

$$\begin{bmatrix} b_{11} & b_{12} & \varsigma_1 & 0 \\ b_{21} & b_{22} & 0 & \varsigma_2 \\ b_{31} & b_{32} & 0 & 0 \\ b_{41} & b_{42} & 0 & 0 \end{bmatrix}, \quad \varsigma_1 > \varsigma_2 > 0, \tag{5.4}$$

where the only two free parameters (up to sign changes) are hidden in:

▷ the transformation of the right-hand side $B_1 = \widetilde{B}_1 G_R^{\mathrm{T}}$ by some orthogonal matrix $G_R^{\mathrm{T}} = G_R^{-1} \in \mathbb{R}^{2\times 2}$; and

▷ the choice of the orthonormal basis (let it be stored in the columns of the matrix U_3') of the two-dimensional $\mathcal{N}(A_{11}^{\mathrm{T}})$, i.e., $U_3' = \widetilde{U}_3' G_L^{\mathrm{T}}$, $G_L^{\mathrm{T}} = G_L^{-1} \in \mathbb{R}^{2\times 2}$.

Both of them involve the left bottom block of (5.4), in particular

$$\begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} = (U_3')^{\mathrm{T}} B_1 = G_L((\widetilde{U}_3')^{\mathrm{T}} \widetilde{B}_1) G_R^{\mathrm{T}}. \tag{5.5}$$

It remains to show that the problems are irreducible. In general, if a core problem is composed, its SVD form must be composable from SVD forms of its individual components. Recalling that any single right-hand side component in the SVD form has the right-hand side with all entries being nonzero (see [8]), the right-hand side of a composed core problem in the SVD form (5.4) must be orthogonally transformable to a chess-board-like pattern of zero and (strictly) nonzero blocks. Consequently, if $[B_1, A_{11}]$ is composed then there exist orthogonal matrices (elementary Givens rotations) G_L and G_R transforming (5.5) to a chess-board structured $(\widetilde{U}_3')^{\mathrm{T}} \widetilde{B}_1$. Since (5.5) is of full row rank (see (CP3)), the only possibility is to (anti)diagonalize it. But with diagonal $(\widetilde{U}_3')^{\mathrm{T}} \widetilde{B}_1$, (5.5) in principle represents an SVD of $\begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$

[2] See for example the code included as supplementary material to [2]. MATLAB codes for verification (by numerical and symbolic calculation) for all three problems are on request freely available by the authors.

Calculation of this SVD therefore fixes the free parameters represented by G_L, G_R. Application of these matrices to the whole (5.4) then either reveals the chess-board structure, if the problem is composed, or not, if it is irreducible. Now it is easy to verify that neither of the three problem is composed.

There is no systematic method for the construction of irreducible core problems with the given number of right-hand sides in the given class. However, the examples above support the expectation that there exist irreducible core problems in all classes for any $d \geqslant 3$.

6. Conclusions

In this paper, we have investigated solvability classes of core problems within linear approximation problems with multiple observations. We have presented the full solvability classification revealing that, in particular, the core problem with two right-hand sides cannot be in the class $\mathcal{F}_3$. Then we have concentrated on the relations between solvability classes while core problems composing. It has been shown that any nongeneric (class $\mathcal{S}$) problem can be moved to generic (class $\mathcal{F}_3$) by employing a particular data correction represented by a composition with a single right-hand side core problem. However, the TLS solution of the corrected problem still does not exist. We have shown that the set of core problems without a TLS solution (i.e., $\mathcal{F}_3 \cup \mathcal{S}$) is closed with respect to composing its elements with components from other classes. Moreover, the set of core problems in the class $\mathcal{S}$ is closed with respect to composing its elements together. Finally, we have presented examples of irreducible core problems with two right-hand sides in all available classes.

The main results are summarized in Tables 1, 2, and 3. Results can be divided into four types of assertions ($\mathcal{C} \in \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{S}\}$):

Existential (based on examples)

$\exists(\mathsf{CP}, d_\alpha, \mathcal{F}_1)_\alpha$, $\exists(\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta$: $(\mathsf{CP}, d_\alpha, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta = (\mathsf{CP}, d_\alpha{+}d_\beta, \mathcal{F}_1)$.
$\exists(\mathsf{CP}, d_\alpha, \mathcal{F}_1)_\alpha$, $\exists(\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta$: $(\mathsf{CP}, d_\alpha, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta = (\mathsf{CP}, d_\alpha{+}d_\beta, \mathcal{F}_2)$.
$\exists(\mathsf{CP}, d_\alpha, \mathcal{F}_1)_\alpha$, $\exists(\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta$: $(\mathsf{CP}, d_\alpha, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta = (\mathsf{CP}, d_\alpha{+}d_\beta, \mathcal{F}_3)$.
$\exists(\mathsf{CP}, d_\alpha, \mathcal{F}_1)_\alpha$, $\exists(\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta$: $(\mathsf{CP}, d_\alpha, \mathcal{F}_1)_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta = (\mathsf{CP}, d_\alpha{+}d_\beta, \mathcal{S})$.

Semi-general

$\forall(\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha$, $\exists(\mathsf{CP}, 1, \mathcal{F}_1)_\beta$: $(\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha \boxplus (\mathsf{CP}, 1, \mathcal{F}_1)_\beta = (\mathsf{CP}, d_\alpha{+}1, \mathcal{C})$.
$\forall(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha$, $\exists(\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta$: $(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{F}_1)_\beta = (\mathsf{CP}, d_\alpha{+}d_\beta, \mathcal{F}_3)$.

General (positive)

$\forall(\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha$: $(\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha \boxplus (\mathsf{CP}, d_\alpha, \mathcal{C})_\alpha = (\mathsf{CP}, 2d_\alpha, \mathcal{C})$.
$\forall(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha$, $\forall(\mathsf{CP}, d_\beta, \mathcal{S})_\beta$: $(\mathsf{CP}, d_\alpha, \mathcal{S})_\alpha \boxplus (\mathsf{CP}, d_\beta, \mathcal{S})_\beta = (\mathsf{CP}, d_\alpha{+}d_\beta, \mathcal{S})$.

General (negative)

$\forall(\mathsf{CP},1,\mathcal{F}_1)_\alpha$, $\forall(\mathsf{CP},1,\mathcal{F}_1)_\beta$: $(\mathsf{CP},1,\mathcal{F}_1)_\alpha \boxplus (\mathsf{CP},1,\mathcal{F}_1)_\beta \neq (\mathsf{CP},2,\mathcal{F}_3)$.
$\forall(\mathsf{CP},d_\alpha,\mathcal{F}_3)_\alpha$, $\forall(\mathsf{CP},d_\beta,\mathcal{C})_\beta$: $(\mathsf{CP},d_\alpha,\mathcal{S})_\alpha \boxplus (\mathsf{CP},d_\beta,\mathcal{C})_\beta \neq (\mathsf{CP},d_\alpha{+}d_\beta,\mathcal{F}_1)$.
$\forall(\mathsf{CP},d_\alpha,\mathcal{F}_3)_\alpha$, $\forall(\mathsf{CP},d_\beta,\mathcal{C})_\beta$: $(\mathsf{CP},d_\alpha,\mathcal{S})_\alpha \boxplus (\mathsf{CP},d_\beta,\mathcal{C})_\beta \neq (\mathsf{CP},d_\alpha{+}d_\beta,\mathcal{F}_2)$.
$\forall(\mathsf{CP},d_\alpha,\mathcal{S})_\alpha$, $\forall(\mathsf{CP},d_\beta,\mathcal{C})_\beta$: $(\mathsf{CP},d_\alpha,\mathcal{S})_\alpha \boxplus (\mathsf{CP},d_\beta,\mathcal{C})_\beta \neq (\mathsf{CP},d_\alpha{+}d_\beta,\mathcal{F}_1)$.
$\forall(\mathsf{CP},d_\alpha,\mathcal{S})_\alpha$, $\forall(\mathsf{CP},d_\beta,\mathcal{C})_\beta$: $(\mathsf{CP},d_\alpha,\mathcal{S})_\alpha \boxplus (\mathsf{CP},d_\beta,\mathcal{C})_\beta \neq (\mathsf{CP},d_\alpha{+}d_\beta,\mathcal{F}_2)$.

We see that the TLS solvability of a core problem is strongly influenced by composing, and till now, it is not clear how to detect the possible (ir)reducibility in general. Therefore, understanding the properties of the composed problems is important for the analysis and solution of TLS problems in general.

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