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f-BIMINIMAL MAPS BETWEEN RIEMANNIAN MANIFOLDS

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Abstract. We give the definition of f-biminimal submanifolds and derive the equation for f-biminimal submanifolds. As an application, we give some examples of f-biminimal manifolds. Finally, we consider f-minimal hypersurfaces in the product space $\mathbb{R}^n \times \mathbb{S}^1(a)$ and derive two rigidity theorems.

Keywords: variational vector field; hypersurface; f-biminimal submanifold; mean curvature vector

MSC 2010: 53B25, 53C40

1. Introduction

All objects in this paper, including manifolds, tensor fields and maps, are assumed smooth unless stated otherwise.

Let $\varphi\colon (M^m, g) \to (N^n, \overline{g})$ be an isometric immersion between Riemannian manifolds and $f \in C^\infty(N)$ a positive function on the ambient space. From the variational formulas (see [2]), it follows that f-minimal submanifolds are the critical points for the weighted volume functional $V_f(x) = \int_M \mathrm{e}^{-f}\,\mathrm{d}\sigma$ where $\mathrm{d}\sigma$ is the volume element of the induced metric, generalizing the fact that minimal submanifolds are the critical points for the standard (i.e. nonweighted) volume functional $V(x) = \int_M \mathrm{d}\sigma$.

The f-mean curvature vector is $\vec{H}_f := -[\vec{H} + (\overline{\nabla} f)^\perp]$, where $\vec{H}$ is the mean curvature vector of φ, $\overline{\nabla} f$ denotes the gradient vector of f on N and $(\ldots)^\perp$ stands for the orthogonal projection of the vector $(\ldots)$ to the normal bundle $T^\perp M$ of φ in N. Further φ is called f-minimal if $H_f = 0$ or, equivalently

$$\vec{H}_f = -[\vec{H} + (\overline{\nabla} f)^\perp] = 0.$$

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In the context of biharmonic submanifolds, there has been a growing amount of study on biharmonic submanifolds in space forms [1], [3], [4], [5], [6], [9], [12], [14] in recent years. Some classifications of biharmonic submanifolds in conformally flat spaces $\mathbb{S}^m \times \mathbb{R}$ and $\mathbb{H}^m \times \mathbb{R}$ have been given in [11], [18]. Lu in [2] introduced f-biharmonic maps and calculated the first variation to obtain the f-biharmonic map equation and the equation for the f-biharmonic conformal maps between the same dimensional manifolds.

The f-biharmonic maps are critical points of the f-bienergy functional for maps $\varphi\colon (M^m, g) \to (N^n, \overline{g})$ between Riemannian manifolds:

$$E_{2,f}(\varphi) = \frac{1}{2}\int_M f|\tau(\varphi)|^2 V_g.$$

The Euler-Lagrange equation gives the f-biharmonic map equation (see [17])

$$\tau_{2,f}(\varphi) := f\tau_2(\varphi) + (\Delta f)\tau(\varphi) + 2\nabla^{\varphi}_{\nabla f}\tau(\varphi) = 0,$$

where τ is name of a map.

An immersion $\varphi\colon (M^m, g) \to (N^n, \overline{g})$ between Riemannian manifolds, or its image, is called biminimal, if it is a critical point of the bienergy functional E_2 for variations normal to the image $\varphi(M) \subset N$. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that φ is a critical point of the λ-bienergy $E_{2,\lambda} = E_2(\varphi) + \lambda E(\varphi)$ for any smooth variation of the map $\varphi_t\colon [-\varepsilon, \varepsilon]$, $\varphi_0 = \varphi$, such that $V = (\mathrm{d}/\mathrm{d}t)\varphi_t|_{t=0}$ is normal to $\varphi(M)$, where $E(\varphi) = \frac{1}{2}\int_M |\mathrm{d}\varphi|^2 V_g$, $E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 V_g$.

Although the research on f-minimal submanifolds has not exist for a very long time, it has been already approached by many authors, especially in the hypersurface case, see for example [7], [8], [15] and [16], and much efforts have been devoted to the study of the rigidity or classification problems, curvature estimates and the discussion of stability problems.

In this paper, we give the following definition:

Definition 1.1. Let (M^m, g) be an oriented compact Riemannian manifold with boundary and let $\varphi\colon (M^m, g) \to (N^n, \overline{g})$ be an isometric immersion. If

$$F\colon M \times (-\varepsilon, \varepsilon) \to N$$

are variations normal to the image $\varphi(M) \subset N$ with fixed energy and variational vector field W and $f \in C^\infty(N)$ is a positive function on the ambient space, then we say φ is f-biminimal if φ is the critical point of

$$V_2^f(t) = \int_M \mathrm{e}^{-f} H_t^2 \,\mathrm{d}V_t.$$

In Section 3, we derive the equation for f-biminimal submanifolds. As an application, in Section 4, we give some examples of f-biminimal submanifolds. Our examples include the hypersurfaces $\varphi\colon (M^m, g) \to (N^{m+1}, \overline{g})$, $\varphi\colon M^m \to \mathbb{R}^n$, $\varphi\colon \mathbb{S}^m(r) \to \mathbb{R}^{m+1}$ and $\varphi\colon \mathbb{S}^k(r_1) \times \mathbb{S}^{m-k}(r_2) \to \mathbb{R}^{m+2}$. In Section 5, we study f-minimal hypersurfaces in the product space $N = \mathbb{R}^n \times \mathbb{S}^1(a)$. By introducing a globally defined smooth function α, we derive two rigidity theorems.

2. Preliminary

First, we give some notation that will appear in our paper. We let $\varphi\colon (M^m, g) \to (N^n, \overline{g})$ be an isometric immersion with the induced metric $g = \varphi^*\overline{g}$ on M. Unless otherwise specified, we always use symbols with a bar to denote all quantities of the ambient space $(N^n, \overline{g})$. For instance, ∇, Δ and ∇f denote the Levi-Civita connection, the Laplacian and the gradient of the induced metric g respectively, while $\overline{\nabla}$, $\overline{\Delta}$ and $\overline{\nabla} f$ denote those of the metric $\overline{g}$, accordingly.

For any vector fields $X, Y \in \mathfrak{X}(M)$, the Gauss formula is given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

where h is the second fundamental form.

For any normal vector field ξ, the Weingarten map A_ξ with respect to ξ is given by

$$\overline{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

where $\nabla^\perp$ stands for the normal connection of the normal bundle of M in N.

It is well known that

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle,$$

and the mean curvature vector field $\vec{H}$ is defined by $\vec{H} = \operatorname{tr} h$.

Definition 2.1. Let (M^m, g) be a compact Riemannian manifold with boundary and let $\varphi\colon (M^m, g) \to (N^n, \overline{g})$ be a smooth immersion. Suppose

$$F\colon M \times (-\varepsilon, \varepsilon) \to N$$

satisfies the following conditions:

(1) $\varphi_0 = \varphi$;
(2) $\varphi_t|_{\partial M} = \varphi|_{\partial M}$ for all $t \in (-\varepsilon, \varepsilon)$;
(3) for every $\varphi_t\colon M \to N$ is an immersion.

Then F is a variation to the immersion submanifold $\varphi\colon (M^m, g) \to (N^n, \overline{g})$, with fixed boundary.

3. f-biminimal maps between Riemannian manifolds

In this section, we derive the equation for the f-biminimal submanifolds. Suppose that

$$F\colon\ M \times (-\varepsilon, \varepsilon) \to N$$

are variations normal to the image $\varphi(M) \subset N$, with fixed energy and variational vector field W. It is obvious that

$$W|_{\varphi(\partial M)} = 0.$$

Theorem 3.1. *Let (M^m, g) be an oriented compact Riemannian manifold with boundary and let $\varphi\colon\ (M^m, g) \to (N^n, \overline{g})$ be an isometric immersion submanifold. If*

$$F\colon\ M \times (-\varepsilon, \varepsilon) \to N$$

are variations normal to the image $\varphi(M) \subset N$ with fixed energy and variational vector field W and $f \in C^\infty(N)$ is a positive function on the ambient space, then φ is f-biminimal if and only if

$$H^2\vec{H}_f + 2[h \circ h^t(\vec{H}) + \mathrm{Ric}^\perp(\vec{H}) + \Delta_M^\perp \vec{H} + (|\nabla f|^2 - \Delta f)\vec{H} - 2\nabla^\perp_{\nabla f}\vec{H}] = 0. \tag{3.1}$$

Proof. Choose a local frame field $(U; u^i)$ in M, set

$$\widetilde{X}_i(p,t) = F_{*(p,t)}\Big(\frac{\partial}{\partial u^i}\Big) = (\varphi_t)_*\Big(\frac{\partial}{\partial u^i}\Big), \quad X_i = \widetilde{X}_i|_{t=0}, \quad 1 \leqslant i \leqslant m,$$
$$\widetilde{W}(p,t) = F_{*(p,t)}\Big(\frac{\partial}{\partial t}\Big),$$

then the variational vector field of φ is $W = \widetilde{W}|_{t=0}$.

For every t, the component of the Riemannian induced metric $g_t = (f_t)^*\overline{g}$ is

$$(g_t)_{ij} = \langle \widetilde{X}_i, \widetilde{X}_j \rangle.$$

$\vec{H}_t$ is the mean curvature vector field of $\varphi_t\colon\ M \to N$ and is defined as

$$\vec{H}_t = (g_t)^{ij} h_t\Big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\Big),$$

where $g_t = (\varphi_t)^*\overline{g}$.

Denote $G_t = \det((g_t)_{ij})$, then the volume element of (M, g) is

$$\mathrm{d}V_t = \sqrt{G_t}\,\mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^m.$$

On the other hand, we set

$$W_j = \Big\langle \widetilde{W}, (\varphi_t)_*\Big(\frac{\partial}{\partial u^j}\Big)\Big\rangle, \quad \tau_t = \sum_{i,j} g_t^{ij} W_j \frac{\partial}{\partial u^i}.$$

Therefore τ_t is a smooth vector field of M and we have $\tau_t|_{\partial M} = 0$.

Suppose that ∇^t is the Riemannian connection of the induced metric g_t. It follows from the fact that F is the normal variation that

$$\langle \widetilde{W}, \widetilde{X}_i\rangle|_{t=0} = 0, \quad 1 \leqslant i \leqslant m;$$
$$\Big\langle (\varphi_t)_*\Big(\nabla^t_{\partial/\partial u^i}\frac{\partial}{\partial u^j}\Big), \widetilde{W}\Big\rangle|_{t=0} = \Big\langle (\varphi_t)_*\Big(\nabla_{\partial/\partial u^i}\frac{\partial}{\partial u^j}\Big), W\Big\rangle = 0.$$

Now we compute $(V_2^f)'(t)$ as follows:

$$\begin{aligned}
(3.2)\quad (V_2^f)'(t) &= \int_M \frac{\partial(H_t^2 \mathrm{e}^{-f}\sqrt{G_t})}{\partial t}\,\mathrm{d}u^1 \ldots \wedge \mathrm{d}u^m \\
&= \int_M H_t^2 \mathrm{e}^{-f}\frac{\partial\sqrt{G_t}}{\partial t}\,\mathrm{d}u^1 \ldots \wedge \mathrm{d}u^m + \int_M \Big(\mathrm{e}^{-f}\frac{\partial H_t^2}{\partial t} + H_t^2\frac{\partial \mathrm{e}^{-f}}{\partial t}\Big)\,\mathrm{d}V_t \\
&= \int_M H_t^2 \mathrm{e}^{-f}\,\mathrm{Ric}\,\tau_t\,\mathrm{d}V_t - \int_M H_t^2\mathrm{e}^{-f}\sum_{i,j} g_t^{ij}\Big\langle \widetilde{W}, h_t\Big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\Big)\Big\rangle\,\mathrm{d}V_t \\
&\quad + \int_M 2\mathrm{e}^{-f}\langle \overline{\nabla}_{\partial/\partial t}\vec{H}_t, \vec{H}_t\rangle\,\mathrm{d}V_t - \int_M H_t^2\mathrm{e}^{-f}\frac{\partial f}{\partial t}\,\mathrm{d}V_t \\
&= \int_M H_t^2\,\mathrm{Ric}(\mathrm{e}^{-f}\tau_t)\,\mathrm{d}V_t - \int_M H_t^2\langle \nabla \mathrm{e}^{-f}, \tau_t\rangle\,\mathrm{d}V_t \\
&\quad - \int_M H_t^2\mathrm{e}^{-f}\langle \widetilde{W}, \vec{H}_t\rangle\,\mathrm{d}V_t + \int_M 2\mathrm{e}^{-f}\langle \overline{\nabla}_{\partial/\partial t}\vec{H}_t, \vec{H}_t\rangle\rangle\,\mathrm{d}V_t \\
&\quad - \int_M H_t^2\mathrm{e}^{-f}\langle \overline{\nabla} f, \widetilde{W}\rangle\,\mathrm{d}V_t = \int_M \mathrm{Ric}(H_t^2\mathrm{e}^{-f}\tau_t)\,\mathrm{d}V_t \\
&\quad - \int_M \langle \nabla H_t^2, \mathrm{e}^{-f}\tau_t\rangle\,\mathrm{d}V_t + \int_M H_t^2\langle \nabla f, \mathrm{e}^{-f}\tau_t\rangle\,\mathrm{d}V_t \\
&\quad - \int_M H_t^2\mathrm{e}^{-f}\langle \widetilde{W}, \vec{H}_t\rangle\,\mathrm{d}V_t + \int_M 2\mathrm{e}^{-f}\langle \overline{\nabla}_{\partial/\partial t}\vec{H}_t, \vec{H}_t\rangle\,\mathrm{d}V_t \\
&\quad - \int_M H_t^2\mathrm{e}^{-f}\langle \overline{\nabla} f, \widetilde{W}\rangle\,\mathrm{d}V_t = -\int_M \langle \nabla H_t^2, \mathrm{e}^{-f}\tau_t\rangle\,\mathrm{d}V_t \\
&\quad + \int_M H_t^2\langle \nabla f, \mathrm{e}^{-f}\tau_t\rangle\,\mathrm{d}V_t - \int_M H_t^2\mathrm{e}^{-f}\langle \widetilde{W}, \vec{H}_t\rangle\,\mathrm{d}V_t \\
&\quad + \int_M 2\mathrm{e}^{-f}\langle \overline{\nabla}_{\partial/\partial t}\vec{H}_t, \vec{H}_t\rangle\,\mathrm{d}V_t - \int_M H_t^2\mathrm{e}^{-f}\langle \overline{\nabla} f, \widetilde{W}\rangle\,\mathrm{d}V_t.
\end{aligned}$$

On the other hand, we deal with $\langle \overline{\nabla}_{\partial/\partial t}\vec{H}_t, \vec{H}_t\rangle$:

$$
\begin{aligned}
(3.3)\quad \langle \overline{\nabla}_{\partial/\partial t}\vec{H}_t, \vec{H}_t\rangle|_{t=0} &= \Big\langle \frac{\partial}{\partial t}(g_t)^{ij}\Big(\overline{\nabla}_{\partial/\partial u^i}\widetilde{X_j} - (f_t)_*\Big(\nabla^t_{\partial/\partial u^i}\frac{\partial}{\partial u^j}\Big)\Big), \vec{H}_t\Big\rangle\Big|_{t=0} \\
&\quad + \Big\langle (g_t)^{ij}\overline{\nabla}_{\partial/\partial t}\Big(\overline{\nabla}_{\partial/\partial u^i}\widetilde{X_j} - (f_t)_*\Big(\nabla^t_{\partial/\partial u^i}\frac{\partial}{\partial u^j}\Big)\Big), \vec{H}_t\Big\rangle\Big|_{t=0} \\
&= \Big\langle -(g_t)^{ik}(g_t)^{jl}\frac{\partial}{\partial t}(g_t)_{kl}\langle \overline{\nabla}_{\partial/\partial u^i}\widetilde{X_j}, \vec{H}_t\rangle|_{t=0} - g^{ij}\langle [\nabla^{\perp}_{\nabla_{\partial/\partial u^i}\partial/\partial u^j} W]^{\perp}\rangle|_{t=0}, \vec{H}\Big\rangle \\
&\quad + g^{ij}\Big\langle \Big[\overline{R}(W, X_i)X_j - h\Big(\frac{\partial}{\partial u^i}, A_W\Big(\frac{\partial}{\partial u^j}\Big)\Big) + \nabla^{\perp}_{\partial/\partial u^i}\nabla^{\perp}_{\partial/\partial u^j} W\Big]^{\perp}, \vec{H}\Big\rangle \\
&= \langle -(g_t)^{ik}(g_t)^{jl}(\langle \overline{\nabla}_{\partial/\partial t}\widetilde{X_k}, \widetilde{X_l}\rangle + \langle \widetilde{X_k}, \overline{\nabla}_{\partial/\partial t}\widetilde{X_l}\rangle)\langle \overline{\nabla}_{\partial/\partial u^i}\widetilde{X_j}, \vec{H}_t\rangle|_{t=0} \\
&\quad + g^{ij}\Big\langle \Big[\overline{R}(W, X_i)X_j - h\Big(\frac{\partial}{\partial u^i}, A_W\Big(\frac{\partial}{\partial u^j}\Big)\Big) + \nabla^{\perp}_{\partial/\partial u^i}\nabla^{\perp}_{\partial/\partial u^j} W\Big]^{\perp} \\
&\quad - [\nabla^{\perp}_{\nabla_{\partial/\partial u^i}\partial/\partial u^j} W]^{\perp}|_{t=0}, \vec{H}\Big\rangle \\
&= 2g^{ik}g^{jl}\Big\langle h_t\Big(\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^l}\Big), W\Big\rangle \langle \overline{\nabla}_{\partial/\partial u^i}\widetilde{X_j}, \vec{H}_t\rangle|_{t=0} \\
&\quad - g^{ij}\langle [\nabla^{\perp}_{\nabla_{\partial/\partial u^i}\partial/\partial u^j} W]^{\perp}|_{t=0}, \vec{H}\rangle \\
&\quad + g^{ij}\Big\langle \Big[\overline{R}(W, X_i)X_j - h\Big(\frac{\partial}{\partial u^i}, A_W\Big(\frac{\partial}{\partial u^j}\Big)\Big) + \nabla^{\perp}_{\partial/\partial u^i}\nabla^{\perp}_{\partial/\partial u^j} W\Big]^{\perp}, \vec{H}\Big\rangle \\
&= 2g^{ik}g^{jl}\Big\langle h\Big(\frac{\partial}{\partial u^k}, \frac{\partial}{\partial u^l}\Big), W\Big\rangle \Big\langle h\Big(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\Big), \vec{H}\Big\rangle \\
&\quad + \langle \mathrm{Ric}^{\perp}(W), \vec{H}\rangle - \langle h\circ h^t(\vec{H}), W\rangle + \langle \Delta^{\perp}_M W, \vec{H}\rangle \\
&= \langle h\circ h^t(\vec{H}), W\rangle + \langle \mathrm{Ric}^{\perp}(W), \vec{H}\rangle + \langle \Delta^{\perp}_M W, \vec{H}\rangle.
\end{aligned}
$$

Next we deal with $\int_M \mathrm{e}^{-f}\langle \Delta^{\perp}_M W, \vec{H}\rangle\,\mathrm{d}V_M$ denoting $X = g^{ij}\langle W, \nabla^{\perp}_{\partial/\partial u^j}\vec{H}\rangle \partial/\partial u^i$.

$$
\begin{aligned}
(3.4)\quad &\int_M \mathrm{e}^{-f}\langle \Delta^{\perp}_M W, \vec{H}\rangle\,\mathrm{d}V_M \\
&= \int_M \mathrm{e}^{-f}\langle \Delta^{\perp}_M \vec{H}, W\rangle\,\mathrm{d}V_M + \int_M \mathrm{e}^{-f}\,\triangle_M\,(\langle W, \vec{H}\rangle)\,\mathrm{d}V_M - 2\int_M \mathrm{e}^{-f}\,\mathrm{div}\,X\,\mathrm{d}V_M \\
&= \int_M \mathrm{e}^{-f}\langle \Delta^{\perp}_M \vec{H}, W\rangle\,\mathrm{d}V_M + \int_M \triangle_M(\mathrm{e}^{-f}\langle W, \vec{H}\rangle)\,\mathrm{d}V_M \\
&\quad - 2\int_M \langle \nabla \mathrm{e}^{-f}, \nabla(\langle W, \vec{H}\rangle)\rangle\,\mathrm{d}V_M - \int_M \langle W, \vec{H}\rangle \nabla \mathrm{e}^{-f}\,\mathrm{d}V_M \\
&\quad - 2\int_M \mathrm{div}(\mathrm{e}^{-f}X)\,\mathrm{d}V_M + 2\int_M \langle \nabla \mathrm{e}^{-f}, X\rangle\,\mathrm{d}V_M = \int_M \mathrm{e}^{-f}\langle \Delta^{\perp}_M \vec{H}, W\rangle\,\mathrm{d}V_M \\
&\quad - \int_{\partial M} \vec{n}(\mathrm{e}^{-f}\langle W, \vec{H}\rangle)\,\mathrm{d}V_M + \int_M 2\langle W, \vec{H}\rangle \Delta \mathrm{e}^{-f}\,\mathrm{d}V_M \\
&\quad + \int_{\partial M} 2\langle W, \vec{H}\rangle \vec{n}(\mathrm{e}^{-f})\,\mathrm{d}V_{\partial M} - \int_M \langle W, \vec{H}\rangle \Delta \mathrm{e}^{-f}\,\mathrm{d}V_M + 2\int_M \langle \nabla \mathrm{e}^{-f}, X\rangle\,\mathrm{d}V_M \\
&= \int_M \mathrm{e}^{-f}\langle \Delta^{\perp}_M \vec{H}, W\rangle\,\mathrm{d}V_M + \int_M \langle W, \vec{H}\rangle \Delta \mathrm{e}^{-f}\,\mathrm{d}V_M + 2\int_M \langle \nabla \mathrm{e}^{-f}, X\rangle\,\mathrm{d}V_M
\end{aligned}
$$

$$= \int_M \mathrm{e}^{-f} \langle \Delta_M^{\perp} \vec{H}, W \rangle \, \mathrm{d}V_M + \int_M \langle W, \vec{H} \rangle \Delta \mathrm{e}^{-f} \, \mathrm{d}V_M$$
$$+ 2 \int_M g^{ij} \Big\langle \nabla \mathrm{e}^{-f}, \frac{\partial}{\partial u^i} \Big\rangle \langle \nabla^{\perp}_{\partial/\partial u^j} \vec{H}, W \rangle \, \mathrm{d}V_M$$
$$= \int_M \mathrm{e}^{-f} \langle \Delta_M^{\perp} \vec{H}, W \rangle \, \mathrm{d}V_M + \int_M \langle W, \vec{H} \rangle \Delta \mathrm{e}^{-f} \, \mathrm{d}V_M - 2 \int_M \mathrm{e}^{-f} \langle \nabla^{\perp}_{\nabla f} \vec{H}, W \rangle \, \mathrm{d}V_M.$$

From (3.3) and (3.4), we get the identity

$$\begin{aligned}
(3.5) \qquad & \int_M 2\mathrm{e}^{-f} \langle \overline{\nabla}_{\partial/\partial t} \vec{H}_t, \vec{H}_t \rangle|_{t=0} \, \mathrm{d}V_M \\
&= \int_M 2\mathrm{e}^{-f} [\langle h \circ h^t(\vec{H}), W \rangle + \langle \mathrm{Ric}^{\perp}(W), \vec{H} \rangle + \langle \Delta_M^{\perp} W, \vec{H} \rangle] \, \mathrm{d}V_M \\
&= \int_M 2\mathrm{e}^{-f} [\langle h \circ h^t(\vec{H}), W \rangle + \langle \mathrm{Ric}^{\perp}(\vec{H}), W \rangle + \langle \Delta_M^{\perp} \vec{H}, W \rangle] \, \mathrm{d}V_M \\
&\quad + \int_M 2 \langle W, \vec{H} \rangle \Delta \mathrm{e}^{-f} \, \mathrm{d}V_M - 4 \int_M \mathrm{e}^{-f} \langle \nabla^{\perp}_{\nabla f} \vec{H}, W \rangle \, \mathrm{d}V_M.
\end{aligned}$$

Substituting (3.5) into (3.2), we have

$$\begin{aligned}
(3.6) \qquad (V_2^f)'(0) &= - \int_M \langle \nabla H_t^2, \mathrm{e}^{-f} \tau_t \rangle|_{t=0} \, \mathrm{d}V_M + \int_M H_t^2 \langle \nabla f, \mathrm{e}^{-f} \tau_t \rangle|_{t=0} \, \mathrm{d}V_M \\
&\quad - \int_M H_t^2 \mathrm{e}^{-f} \langle \widetilde{W}, \vec{H}_t \rangle|_{t=0} \, \mathrm{d}V_M - \int_M H_t^2 \mathrm{e}^{-f} \langle \overline{\nabla} f, \widetilde{W} \rangle|_{t=0} \, \mathrm{d}V_M \\
&\quad + \int_M 2\mathrm{e}^{-f} [\langle h \circ h^t(\vec{H}), W \rangle + \langle \mathrm{Ric}^{\perp}(\vec{H}), W \rangle + \langle \Delta_M^{\perp} \vec{H}, W \rangle] \, \mathrm{d}V_M \\
&\quad + \int_M 2 \langle W, \vec{H} \rangle \Delta \mathrm{e}^{-f} \, \mathrm{d}V_M - 4 \int_M \mathrm{e}^{-f} \langle \nabla^{\perp}_{\nabla f} \vec{H}, W \rangle \, \mathrm{d}V_M \\
&= \int_M H^2 \mathrm{e}^{-f} \langle -\overline{\nabla} f + \nabla f - \vec{H}, W \rangle \, \mathrm{d}V_M \\
&\quad + 2 \int_M \mathrm{e}^{-f} \langle W, \vec{H} \rangle (|\nabla f|^2 - \Delta f) \, \mathrm{d}V_M \\
&\quad + \int_M 2\mathrm{e}^{-f} [\langle h \circ h^t(\vec{H}) + \mathrm{Ric}^{\perp}(W) + \Delta_M^{\perp} \vec{H}, W \rangle] \, \mathrm{d}V_M \\
&\quad - 4 \int_M \mathrm{e}^{-f} \langle \nabla^{\perp}_{\nabla f} \vec{H}, W \rangle \, \mathrm{d}V_M \\
&= \int_M \mathrm{e}^{-f} \langle H^2 \vec{H}_f + 2[h \circ h^t(\vec{H}) + \mathrm{Ric}^{\perp}(\vec{H}) + \Delta_M^{\perp} \vec{H} \\
&\quad + (|\nabla f|^2 - \Delta f) \vec{H} - 2 \nabla^{\perp}_{\nabla f} \vec{H}], W \rangle \, \mathrm{d}V_M.
\end{aligned}$$

Then φ is f-biminimal if and only if

$$H^2 \vec{H}_f + 2[h \circ h^t(\vec{H}) + \mathrm{Ric}^{\perp}(\vec{H}) + \Delta_M^{\perp} \vec{H} + (|\nabla f|^2 - \Delta f) \vec{H} - 2 \nabla^{\perp}_{\nabla f} \vec{H}] = 0.$$

It is clear that if φ is minimal, then it is f-biminimal.

4. Some examples of f-biminimal submanifolds

In this section, we give some examples of f-biminimal submanifolds.

Example 4.1. Let us consider the hypersurface such that $\varphi\colon (M^m, g) \to (N^{m+1}, \overline{g})$ is an isometric immersion of codimension one.

Let $\{e_1, \dots, e_m, \xi\}$ be an orthonormal frame adapted to the hypersurface $\varphi\colon (M^m, g) \to (N^{m+1}, \overline{g})$ with ξ being the unit normal vector with respect to the metric $\overline{g}$. Now there exists $\lambda \in C^\infty(M)$ such that $W = \lambda\xi$ and $\lambda|_{\partial M} = 0$. Supposing $H = \mu\xi$ and $\mu \in C^\infty(M)$ we derive

$$\begin{aligned}
h \circ h^t(\vec{H}) &= h \circ h^t(\mu\xi) = \mu h \circ h^t(\xi) = \mu\Sigma_{i,j}(h_{ij})^2\xi = \mu|h|^2\xi,\\
\operatorname{Ric}^\perp(\vec{H}) &= \Sigma_{i,j}(\overline{R}(\mu\xi, f_*(e_i))f_*(e_j))^\perp = \mu\Sigma_{i,j}(\overline{R}(\xi, f_*(e_i))f_*(e_j))^\perp = \mu\,\overline{\operatorname{Ric}}(\xi)\xi,\\
\Delta_M^\perp \vec{H} &= \Delta_M^\perp \mu\xi = (\Delta_M \mu)\xi,\\
\nabla_{\nabla f}^\perp \vec{H} &= \nabla_{\nabla f}^\perp \mu\xi = (\nabla f)(\mu)\xi + \mu\nabla_{\nabla f}^\perp \xi = (\nabla f)(\mu)\xi.
\end{aligned}$$

It is obvious that

$$\begin{aligned}
-\mu^2[\mu\xi + \xi(f)\xi] &+ 2[\mu|h|^2\xi + \mu\,\overline{\operatorname{Ric}}(\xi)\xi + (\nabla f)(\mu)\xi + \mu(|\nabla f|^2 - \Delta f)\xi - 2(\nabla f)(\mu)\xi]\\
&= -\mu^2[\mu + \xi(f)]\xi + 2[\mu|h|^2 + \mu\,\overline{\operatorname{Ric}}(\xi)\\
&\quad + (\nabla f)(\mu) + \mu(|\nabla f|^2 - \Delta f) - 2(\nabla f)(\mu)]\xi = 0.
\end{aligned}$$

Consequently, we get the following Remark 4.1:

Remark 4.1. Let $\varphi\colon (M^m, g) \to (N^{m+1}, \overline{g})$ be an isometric immersion of codimension one. Let $f \in C^\infty(N)$ be a positive function on the ambient space and $\vec{H} = \mu\xi$, $\mu \in C^\infty(M)$. Then φ is f-biminimal if and only if

$$-\mu^2[\mu+\xi(f)]+2[\mu|h|^2+\mu\,\overline{\operatorname{Ric}}(\xi)+(\nabla f)(\mu)+\mu(|\nabla f|^2-\Delta f)-2(\nabla f)(\mu)] = 0. \tag{4.1}$$

Example 4.2. $\varphi\colon M^m \to \mathbb{R}^n$, $f\colon \mathbb{R}^{m+1} \to \mathbb{R}$ is defined by $f(x) = \frac{1}{2}x^2$.

$$\overline{\nabla} f = \overline{\nabla}\frac{x^2}{2} = x\overline{\nabla} x = \sum_{\alpha=m+1}^{n} x_\alpha \frac{\partial x_\alpha}{\partial x_i}\frac{\partial}{\partial x_i} = x,$$

$$\nabla f = x^\top,$$

$$\Delta f = \Delta\frac{x^2}{2} = \langle x, x_i\rangle_{,i} = \langle x_i, x_i\rangle + \langle x, x_{ii}\rangle = m + \langle x, H\rangle.$$

Substituting these identities into (3.1), we derive

$$\begin{aligned}
H^2\vec{H}_f &+ 2[h \circ h^t(\vec{H}) + \operatorname{Ric}^\perp(\vec{H}) + \Delta_M^\perp\vec{H} + (|\nabla f|^2 - \Delta f)\vec{H} - 2\nabla_{\nabla f}^\perp\vec{H}]\\
&= H^2[-x + x^\top - \vec{H}] + 2[h \circ h^t(\vec{H}) + \Delta_M^\perp\vec{H}\\
&\quad + (|x^\top|^2 - m - \langle x, \vec{H}\rangle)\vec{H} - 2\nabla_{x^\top}^\perp\vec{H}] = 0.
\end{aligned}$$

Consequently, we get:

Remark 4.2. Suppose $\varphi\colon M^m \to \mathbb{R}^n$ and $f\colon \mathbb{R}^{m+1} \to \mathbb{R}$ is defined by $f(x) = \frac{1}{2}x^2$. Then φ is f-biminimal if and only if

$$H^2[-x+x^\top-\vec{H}]+2[h\circ h^t(\vec{H})+\Delta_M^\perp \vec{H}+(|x^\top|^2-m-\langle x,\vec{H}\rangle)\vec{H}-2\nabla^\perp_{x^\top}\vec{H}] = 0. \tag{4.2}$$

Example 4.3. $\varphi\colon \mathbb{S}^m(r) \to \mathbb{R}^{m+1}$, $f\colon \mathbb{R}^{m+1} \to \mathbb{R}$ is defined by $f(x) = \frac{1}{2}x^2$.

Suppose $\vec{H} = \mu\xi$ and $\mu \in C^\infty(M)$. Then we have $\mu = m/r$, $x = -r\xi$, $\vec{H} = m/r\xi$ and $|h|^2 = m/r^2$. Substituting these identities into (4.1), we derive

$$\begin{aligned}
-\mu^2[\mu+\xi(f)] &+ 2[\mu|h|^2+\mu\,\overline{\mathrm{Ric}}(\xi)+(\nabla f)(\mu)+\mu(|\nabla f|^2-\Delta f)-2(\nabla f)(\mu)] \\
&= -\frac{m^2}{r^2}\Big(\frac{m}{r}-r\Big)+2\frac{m}{r}\frac{m}{r^2} = \frac{m^2(2-m+r^2)}{r^3} = 0.
\end{aligned}$$

Then

$$2-m+r^2 = 0.$$

That is to say that r has the only value $r = \sqrt{m-2}$ for $m > 2$. Therefore, we have:

Remark 4.3. Supposing $\varphi\colon \mathbb{S}^m(r) \to \mathbb{R}^{m+1}$ and $f\colon \mathbb{R}^{m+1} \to \mathbb{R}$ is defined by $f(x) = \frac{1}{2}x^2$, then φ is f-biminimal if and only if $r = \sqrt{m-2}$ for $m > 2$.

Example 4.4. $\varphi\colon \mathbb{S}^k(r_1) \times \mathbb{S}^{m-k}(r_2) \to \mathbb{R}^{m+2}$, $f\colon \mathbb{R}^{m+2} \to \mathbb{R}$ is defined by $f(x) = \frac{1}{2}x^2$.

Let ξ_1 and ξ_2 be the unit normal vectors. We have

$$\begin{gathered}
x = -r_1\xi_1 - r_2\xi_2, \quad \vec{H} = \frac{k}{r_1}\xi_1 + \frac{m-k}{r_2}\xi_2, \\
h\circ h^t(\vec{H}) = \frac{k}{r_1}h\circ h^t(\xi_1) + \frac{m-k}{r_2}h\circ h^t(\xi_2) = \frac{k^2}{r_1^3}\xi_1 + \frac{(m-k)^2}{r_2^3}\xi_2.
\end{gathered}$$

Substituting these identities into (3.1), we derive:

$$\begin{aligned}
H^2\vec{H}_f &+ 2[h\circ h^t(\vec{H})+\mathrm{Ric}^\perp(\vec{H})+\Delta_M^\perp\vec{H}+(|\nabla f|^2-\Delta f)\vec{H}-2\nabla^\perp_{\nabla f}\vec{H}] \\
&= \Big(\frac{k^2}{r_1^2}+\frac{(m-k)^2}{r_2^2}\Big)\Big(r_1\xi_1+r_2\xi_2-\frac{k}{r_1}\xi_1-\frac{m-k}{r_2}\xi_2\Big) \\
&\quad + 2\Big[\frac{k^2}{r_1^3}\xi_1+\frac{(m-k)^2}{r_2^3}\xi_2\Big] = \Big\{\Big[\frac{k^2}{r_1^2}+\frac{(m-k)^2}{r_2^2}\Big]\frac{r_1^2-k}{r_1}+\frac{2k^2}{r_1^3}\Big\}\xi_1 \\
&\quad + \Big\{\Big[\frac{k^2}{r_1^2}+\frac{(m-k)^2}{r_2^2}\Big]\frac{r_2^2-(m-k)}{r_2}+\frac{2(m-k)^2}{r_2^3}\Big\}\xi_2.
\end{aligned}$$

By virtue of (3.1), the following formulas hold:

$$\Big[\frac{k^2}{r_1^2}+\frac{(m-k)^2}{r_2^2}\Big]\frac{r_1^2-k}{r_1}+\frac{2k^2}{r_1^3}=0, \tag{4.3}$$

$$\Big[\frac{k^2}{r_1^2}+\frac{(m-k)^2}{r_2^2}\Big]\frac{r_2^2-(m-k)}{r_2}+\frac{2(m-k)^2}{r_2^3}=0. \tag{4.4}$$

According to (4.3) and (4.4), it follows that

$$[k^2r_2^2+(m-k)^2r_1^2](2-m+r_1^2+r_2^2)=0,$$

or equivalently

$$2-m+r_1^2+r_2^2=0.$$

This is to say that r_1 and r_2 satisfy

$$r_1^2+r_2^2=m-2$$

for $m>2$. Consequently, we get:

Remark 4.4. Let $\varphi\colon \mathbb{S}^k(r_1)\times\mathbb{S}^{m-k}(r_2)\to\mathbb{R}^{m+2}$ and let $f\colon \mathbb{R}^{m+2}\to\mathbb{R}$ be defined by $f(x)=\frac{1}{2}x^2$, then φ is f-biminimal if and only if $r_1^2+r_2^2=m-2$ for $m>2$.

5. f-minimal hypersurfaces in the product space $N=\mathbb{R}^n\times\mathbb{S}^1(a)$

In this section, we aim at studying f-minimal hypersurfaces in the product space $N=\mathbb{R}^n\times\mathbb{S}^1(a)$ for some positive function $f\in C^\infty(\mathbb{R}^n\times\mathbb{S}^1(a))$. Let $x\colon M^n\to\mathbb{R}^n\times\mathbb{S}^1(a)$ be an immersed hypersurface with the induced metric $g=x^*\overline{g}$ on M. The function on $\mathbb{R}^n\times\mathbb{S}^1(a)$ is defined by $f(x)=\frac{1}{2}|x_1|^2$ for any $x=(x_1,x_2)\in\mathbb{R}^n\times\mathbb{S}^1(a)$. By choosing the natural frame $\{\partial/\partial x^i\}_{i=1}^n$ on $\mathbb{R}^n$, we obtain a local natural frame field $\{\partial/\partial x^i,\partial/\partial s\}_{i=1}^n$ on $N=\mathbb{R}^n\times\mathbb{S}^1(a)$. Then the metric $\overline{g}$ has the form

$$\overline{g}=\sum_{i=1}^n(\mathrm{d}x^i)^2+\mathrm{d}s^2.$$

Let $x\colon M^n\to\mathbb{R}^n\times\mathbb{S}^1(a)$ be f-minimal, using the local orthonormal frame $\{e_i\}_{i=1}^n$ on M^n we have a frame field $\{\overline{e}_A\}_{A=1}^{n+1}$ along x with $\overline{e}_i=x_*e_i$ and $\overline{e}_{n+1}=v$, the unit normal vector of x. The angle function is defined by $\alpha=\langle v,\partial/\partial s\rangle$ and $T=(\partial/\partial s)^\top$. In the case of $f(x)=\frac{1}{2}|x_1|^2=f(x)=\frac{1}{2}(|x|^2-a^2)$, we have

$$\overline{\nabla}f=\Sigma_i\langle x,e_i\rangle e_i+\langle x,v\rangle v,\quad \vec{H}=-(\overline{\nabla}f)^\perp.$$

or equivalently,

$$f_i = \langle x, e_i\rangle, \quad i = 1, 2, \ldots, n, \quad f_v = \langle x, v\rangle = -H.$$

Then we state and prove our theorems.

Theorem 5.1. *Let M^n be an oriented compact Riemannian manifold with boundary and let $x\colon M^n \to \mathbb{R}^n \times \mathbb{S}^1(a)$ be an isometric immersion submanifold. If*

$$F\colon M^n \times (-\varepsilon, \varepsilon) \to \mathbb{R}^n \times \mathbb{S}^1(a)$$

are variations normal to the image $x(M^n) \subset \mathbb{R}^n \times \mathbb{S}^1(a)$, with fixed energy and variational vector field W, $x\colon M^n \to \mathbb{R}^n \times \mathbb{S}^1(a)$ is f-minimal and $f \in C^\infty(N)$ is defined by $f(x) = \frac{1}{2}|x_1|^2$, then x is f-biminimal if and only if

$$(2f - n + 2)H - 2h(T, T) - f_k\Big\langle e_k, \frac{\partial}{\partial s}\Big\rangle\alpha - f_k f_l h_{lk} = 0.$$

P r o o f. Similarly to the calculation of Theorem 3.1 in [15], we have the following equations

$$\text{(5.1)}\quad h \circ h^t(\vec{H}) = -\langle x, v\rangle h \circ h^t(v) = -\langle x, v\rangle |h|^2 v = |h|^2\vec{H}.$$

$$\begin{aligned}\text{(5.2)}\qquad \Delta_M^{\perp}\vec{H} &= (\Delta_M \vec{H})(v) = H_{,ii}v \\ &= \Big[h_{ii}(1+\alpha^2) - 2\sum_k h_{ik}\Big\langle e_i, \frac{\partial}{\partial s}\Big\rangle\Big\langle e_i, \frac{\partial}{\partial s}\Big\rangle + \sum_k f_v h_{ki}^2 \\ &\qquad + \sum_k f_k h_{ii,k}\Big]v \\ &= [H(1+\alpha^2) - 2h(T,T) - H|h|^2 + f_k H_{,k}]v \\ &= \Big[H(1+\alpha^2) - 2h(T,T) - H|h|^2 + f_k\Big\langle e_k, \frac{\partial}{\partial s}\Big\rangle\alpha + f_k f_l h_{lk}\Big]v.\end{aligned}$$

$$\text{(5.3)}\qquad |\nabla f|^2 = 2f - H^2.$$

$$\text{(5.4)}\qquad \Delta f = (e_ie_i - \nabla_{e_i}e_i)f = [e_ie_i - \overline{\nabla}_{e_i}e_i + h(e_i, e_i)]f = n - 1 + \alpha^2 - H^2.$$

$$\begin{aligned}\text{(5.5)}\ \nabla^{\perp}_{\nabla f}\vec{H} &= \nabla^{\perp}_{\langle x, e_i\rangle e_i}(-\langle x, v\rangle v) = -\langle x, e_i\rangle e_i(\langle x, v\rangle)v = \langle x, e_i\rangle e_i(H)v \\ &= \langle x, e_i\rangle\Big(\Big\langle e_i, \frac{\partial}{\partial s}\Big\rangle\alpha + f_k h_{ki}\Big)v = f_i\Big(\Big\langle e_i, \frac{\partial}{\partial s}\Big\rangle\alpha + f_k h_{ki}\Big)v.\end{aligned}$$

Substituting (5.1)–(5.5) into (3.1), we get

$$\begin{aligned}\text{(5.6)}\quad |h|^2\vec{H} &+ \Big[H(1+\alpha^2) - 2h(T,T) - H|h|^2 + f_k\Big\langle e_k, \frac{\partial}{\partial s}\Big\rangle\alpha + f_k f_l h_{lk}\Big]v \\ &+ [2f - H^2 - (n - 1 + \alpha^2 - H^2)]H - 2f_i\Big(\Big\langle e_i, \frac{\partial}{\partial s}\Big\rangle\alpha + f_k h_{ki}\Big)v = 0,\end{aligned}$$

which implies

$$(2f - n + 2)H - 2h(T, T) - f_k\Big\langle e_k, \frac{\partial}{\partial s}\Big\rangle\alpha - f_k f_l h_{lk} = 0.$$

□

Now, from [15], Theorems 4.3–4.7, we have our rigidity theorems.

Theorem 5.2. *Let $x\colon M^n \to \mathbb{R}^n \times \mathbb{S}^1(a)$ be a complete and properly immersed oriented f-minimal hypersurface. If one of the following four conditions is satisfied,*
(1) *the angle function α is constant, or*
(2) *the angle function α does not change its sign and $|h|^2 \in L^2_f$, or*
(3) *$|h|^2 \leqslant 1 + \alpha^2$ and there is a constant c, $0 < c < 1$ such that $2|\nabla\alpha|^2 \leqslant c|\nabla h|^2$ or*
(4) *$\frac{1}{2}(1 - \alpha^2 - c) \leqslant |h|^2 \leqslant \frac{1}{2}(1 - \alpha^2 + c)$, $c = \sqrt{(1-\alpha^2)(1-9\alpha^2)}$, then we have either*
(1) *$\alpha = 1$ and x is f-biminimal, or*
(2) *$\alpha = 0$ and x is f-biminimal if and only if*

$$(2f - n + 2)H - 2h\Big(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\Big) - f_k f_l h_{lk} = 0.$$

Theorem 5.3. *Let $x\colon M^n \to \mathbb{R}^n \times \mathbb{S}^1(a)$ be a complete and properly immersed oriented f-minimal hypersurface. If one of the following conditions is satisfied,*
(1) *$|h|^2| \leqslant 2\alpha^2$ and there is a constant c, $0 < c < 1$ such that $2|\nabla\alpha|^2 \leqslant c|\nabla h|^2$ or*
(2) *$|h|^2| \leqslant 3\alpha^2 - 1$, then we have $h = 0$ and x is f-biminimal.*

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References

[1] *A. Balmuş, S. Montaldo, C. Oniciuc*: Classification results for biharmonic submanifolds in spheres. Isr. J. Math. *168* (2008), 201–220. zbl MR doi
[2] *V. Bayle*: Propriétés de concavité du profil isopérimétrique et applications. These de Doctorat, Université Joseph-Fourier, Grenoble, 2003. (In French.)
[3] *R. Caddeo, S. Montaldo, C. Oniciuc*: Biharmonic submanifolds of $\mathbb{S}^3$. Int. J. Math. *12* (2001), 867–876. zbl MR doi
[4] *R. Caddeo, S. Montaldo, C. Oniciuc*: Biharmonic submanifolds in spheres. Isr. J. Math. *130* (2002), 109–123. zbl MR doi
[5] *B.-Y. Chen*: Some open problems and conjectures on submanifolds of finite type. Soochow J. Math. *17* (1991), 169–188. zbl MR
[6] *B.-Y. Chen, S. Ishikawa*: Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces. Kyushu J. Math. *52* (1998), 167–185. zbl MR doi

[7] *X. Cheng, T. Mejia, D. Zhou*: Eigenvalue estimate and compactness for closed f-minimal surfaces. Pac. J. Math. *271* (2014), 347–367. zbl MR doi
[8] *X. Cheng, T. Mejia, D. Zhou*: Stability and compactness for complete f-minimal surfaces. Trans. Am. Math. Soc. *367* (2015), 4041–4059. zbl MR doi
[9] *I. Dimitrić*: Submanifolds of E^m with harmonic mean curvature vector. Bull. Inst. Math., Acad. Sin. *20* (1992), 53–65. zbl MR
[10] *J. Eells*, Jr., *J. H. Sampson*: Harmonic mappings of Riemannian manifolds. Am. J. Math. *86* (1964), 109–160. zbl MR doi
[11] *D. Fetcu, C. Oniciuc, H. Rosenberg*: Biharmonic submanifolds with parallel mean curvature in $\mathbb{S}^n \times \mathbb{R}$. J. Geom. Anal. *23* (2013), 2158–2176. zbl MR doi
[12] *T. Hasanis, T. Vlachos*: Hypersurfaces in E^4 with harmonic mean curvature vector field. Math. Nachr. *172* (1995), 145–169. zbl MR doi
[13] *G. Jiang*: 2-harmonic maps and their first and second variational formulas. Chin. Ann. Math., Ser. A *7* (1986), 389–402. (In Chinese.) zbl MR doi
[14] *G. Jiang*: Some nonexistence theorems on 2-harmonic and isometric immersions in Euclidean space. Chin. Ann. Math., Ser. A *8* (1987), 377–383. (In Chinese.) zbl MR
[15] *X. X. Li, J. T. Li*: The rigidity and stability of complete f-minimal hypersurfaces in $\mathbb{R} \times \mathbb{S}^1(a)$. To appear in Proc. Am. Math. Soc..
[16] *G. Liu*: Stable weighted minimal surfaces in manifolds with non-negative Bakry-Emery Ricci tensor. Commun. Anal. Geom. *21* (2013), 1061–1079. zbl MR doi
[17] *W. J. Lu*: On f-bi-harmonic maps and bi-f-harmonic maps between Riemannian manifolds. Sci. China, Math. *58* (2015), 1483–1498. zbl MR doi
[18] *Y.-L. Ou, Z.-P. Wang*: Constant mean curvature and totally umbilical biharmonic surfaces in 3-dimensional geometries. J. Geom. Phys. *61* (2011), 1845–1853. zbl MR doi
[19] *S. Ouakkas, R. Nasri, M. Djaa*: On the f-harmonic and f-biharmonic maps. JP J. Geom. Topol. *10* (2010), 11–27. zbl MR

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