Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 69 (2019), No. 4, 1133–1164

Persistent URL: `http://dml.cz/dmlcz/147921`

Czechoslovak Mathematical Journal, 69 (144) (2019), 1133–1164

THE WELLS MAP FOR ABELIAN EXTENSIONS OF 3-LIE ALGEBRAS

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Received February 23, 2018. Published online August 7, 2019.

Abstract. The Wells map relates automorphisms with cohomology in the setting of extensions of groups and Lie algebras. We construct the Wells map for some abelian extensions $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ of 3-Lie algebras to obtain obstruction classes in $H^1(B, A)$ for a pair of automorphisms in $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ to be inducible from an automorphism of L. Application to free nilpotent 3-Lie algebras is discussed.

Keywords: automorphisms of 3-Lie algebras; representations of 3-Lie algebras; abelian extensions; cohomology; free nilpotent 3-Lie algebras

MSC 2010: 16E40, 17A42, 17A36

1. Introduction

Given an extension $A \hookrightarrow B \to C$ of groups, the construction of automorphisms of B by extending automorphisms of A and lifting automorphisms of C is an important problem which goes back to Baer [1]. In [13] Wells defined a map, known as Wells map, from the set of compatible pairs to the second cohomology group of C in A, to obtain obstruction classes of inducibility of compatible pairs. Recent studies in this direction can be found in [6], [7], [9], [10]. In the setting of Lie algebras the Wells map was constructed by Bardakov and Singh in [2] for abelian extensions of Lie algebras.

In this paper we consider the analogous problem for abelian extensions of 3-Lie algebras. n-Lie algebras introduced by Filippov (see [4]) form an important example of n-ary algebras ($n \geqslant 3$), which has relations to generalized Nambu mechanics, see [11].

For any 3-Lie algebra $\mathfrak{g}$ denote by $\mathrm{Aut}(\mathfrak{g})$ the group of 3-Lie algebra automorphisms of $\mathfrak{g}$. Let $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ be an abelian extension of 3-Lie algebras.

The research has been supported by Grant number 11171233 of NSF of China.

Consider the subgroup $\mathrm{Aut}_A(L)$ of $\mathrm{Aut}(L)$ consisting of automorphisms γ such that $\gamma(A) = A$. As in the cases of groups and Lie algebras, there is a group homomorphism $\Phi\colon \mathrm{Aut}_A(L) \to \mathrm{Aut}(A) \times \mathrm{Aut}(B)$ which is defined via a section of π, but does not depend on choices of sections. (For details see Section 3.) A pair $(\sigma, \tau) \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)$ is called inducible if $(\sigma, \tau) \in \mathrm{im}\,\Phi$, that is, there is a $\gamma \in \mathrm{Aut}_A(L)$ extending σ and lifting τ. One may expect to construct the Wells map to obtain obstruction classes in some cohomology groups of B in A defined by Takhtajan, see [12]. However, drastically different from groups and Lie algebras, in general there are no representations of B on A. (For the definition of representations of 3-Lie algebras given by Kasymov see Definition 2.1 below.) Therefore we restrict ourselves to the so-called good abelian extensions of 3-Lie algebras in the following sense. (For a 3-Lie algebra $\mathfrak{g}$, its 3-Lie bracket is usually denoted by $[\cdot,\cdot,\cdot]_{\mathfrak{g}}$, and when there is no confusion $\mathfrak{g}$ may be suppressed.)

Definition 1.1. An abelian extension $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ of 3-Lie algebras is called good if the map $\varrho\colon \bigwedge^2 B \to \mathrm{End}(A)$ given by

$$\varrho(x_1, x_2)(a) = [s(x_1), s(x_2), a]_L, \quad x_i \in B,\ a \in A \tag{1.1}$$

does not depend on the choice of sections s of π and ϱ is a representation of B on A.

Note that for abelian extensions of Lie algebras the conditions in the above definition hold automatically (see, for example, [5], Chapter 7). We will give a sufficient and necessary condition on an abelian extension of 3-Lie algebras to be good in Proposition 2.1 below. We give some nontrivial examples of good abelian extensions in the next and the last sections. Note that a good abelian extension is not necessarily split, while a split abelian extension is not necessarily a good abelian extension either.

For any good abelian extension $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ of 3-Lie algebras we have the cochain complex due to Takhtajan [12] associated to the representation ϱ of B on A. First we show that $\ker\Phi \cong Z^0(B, A)$ as abelian groups (see Theorem 4.1). To define the Wells map (see (4.14)) we introduce a subgroup $C^L_{B,A}$ of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$ consists of the so-called compatible pairs (see Definition 4.1). Then we show that the Wells map fits into an exact sequence ending at $H^1(B, A)$. In particular, the image of the Wells map are obstruction classes for compatible pairs to be inducible (see Theorem 4.3). As a corollary, in the case of good split abelian extensions, compatiblity and inducibility are equivalent (see Corollary 4.3). We also show that in the case of good split abelian extensions the short exact sequence ending at $\mathrm{im}\,\Phi$ is also split (see Theorem 4.4). It turns out that there is a left action of the group $C^L_{B,A}$ on $H^1(B, A)$ such that the Wells map is an inner derivation (see Proposition 4.2). In general, as in the cases of groups and Lie algebras, the Wells map given by good abelian extensions of 3-Lie algebras is just a set map.

Application is given to free nilpotent 3-Lie algebras. In this case we obtain a criterion in terms of matrices and determinants for a pair of automorphisms to be inducible (see Theorem 5.1), and in the case of free 2-step 3-Lie algebra $B_{3,2}$ (see (5.1), where $n = 3$) we obtain a good abelian extension of 3-Lie algebras which is not split while the corresponding short exact sequence of automorphism groups is split (see Proposition 5.2 and Proposition 5.3). Note that in [2] Bardakov and Singh discussed inducible automorphisms of free nilpotent Lie algebras.

The paper is organized as follows. In Section 2 we review basic definitions of 3-Lie algebras, their representations and Takhtajan's cochain complexes. We give a description of good abelian extensions of 3-Lie algebras and low dimensional cocycles. In particular we obtain that ω given by (2.7) is a 2-cocycle. In Section 3 we give the definition of the group homomorphism $\Phi\colon \operatorname{Aut}_A(L) \to \operatorname{Aut}(A) \times \operatorname{Aut}(B)$ mentioned above following ideas in [2], [13] and give some auxiliary lemmas. In Section 4 we construct the Wells map for good abelian extensions of 3-Lie algebras and the corresponding four-term exact sequence. Then we give some applications to good split extensions of 3-Lie algebras. In this section we also study the action of the group $C^L_{B,A}$ on $H^1(B, A)$ with respect to which the Wells map becomes an inner derivation. In Section 5 we give an application to free nilpotent 3-Lie algebras and construct an example of nonsplit good abelian extensions of 3-Lie algebras, showing that the converse of Theorem 4.4 does not hold.

Throughout this paper all vector spaces are over a field $\mathbb{F}$ of characteristic 0. $\operatorname{Hom}(-,-)$ and $\operatorname{End}(-)$ consist of all $\mathbb{F}$-linear maps. We use c.p. to denote cyclic permutations, for example,

$$\theta(x,y)\alpha(z) + \text{c.p.} = \theta(x,y)\alpha(z) + \theta(y,z)\alpha(x) + \theta(z,x)\alpha(y).$$

$$\begin{aligned}\nu(\tau(x_1))(f(x_2),\sigma(a_3)) + \text{c.p.} &= \nu(\tau(x_1))(f(x_2),\sigma(a_3)) + \nu(\tau(x_2))(f(x_3),\sigma(a_1))\\ &\quad + \nu(\tau(x_3))(f(x_1),\sigma(a_2)).\end{aligned}$$

2. Preliminaries

Due to Filippov [4], a vector space $\mathfrak{g}$ with a 3-ary multilinear skew-symmetric multiplication $[\cdot,\cdot,\cdot]\colon \bigotimes^3 \mathfrak{g} \to \mathfrak{g}$ is a 3-Lie algebra if

$$\begin{aligned}&[x_1,x_2,[x_3,x_4,x_5]] - [[x_1,x_2,x_3],x_4,x_5]\\ &\qquad - [x_3,[x_1,x_2,x_4],x_5] - [x_3,x_4,[x_1,x_2,x_5]] = 0\end{aligned} \tag{2.1}$$

for all $x_1,x_2,x_3,x_4,x_5 \in \mathfrak{g}$. The identity (2.1) is called the fundamental identity (FI). Morphisms, subalgebras, ideals are defined in the usual way. A 3-Lie algebra $\mathfrak{g}$ is abelian if $[x,y,z] = 0$ for any $x,y,z \in \mathfrak{g}$.

Notation 2.1. As in [8], in the sequel we may denote the element $x_1 \wedge x_2 \wedge \ldots \wedge x_n \in \bigwedge^n \mathfrak{g}$ by $(x_1, x_2, \ldots, x_n)$ for simplicity.

Definition 2.1 ([8]). A representation of a 3-Lie algebra $\mathfrak{g}$ on a vector space V is a linear map $\theta\colon \bigwedge^2 \mathfrak{g} \to \mathrm{End}(V)$ such that for all $x_1, x_2, x_3, x_4 \in \mathfrak{g}$,

$$\theta(x_1,x_2)\theta(x_3,x_4)=\theta([x_1,x_2,x_3],x_4)+\theta(x_3,[x_1,x_2,x_4])+\theta(x_3,x_4)\theta(x_1,x_2),$$
$$\theta(x_1,[x_2,x_3,x_4])=\theta(x_3,x_4)\theta(x_1,x_2)-\theta(x_2,x_4)\theta(x_1,x_3)+\theta(x_2,x_3)\theta(x_1,x_4).$$

We denote a representation θ of $\mathfrak{g}$ on a vector space V by $(V;\theta)$.

By abuse of notation we will write, for any $X = x_1 \wedge x_2 \in \bigwedge^2 \mathfrak{g}$, $x_3 \in \mathfrak{g}$,

$$[X, x_3] := [x_1, x_2, x_3] \in \mathfrak{g}.$$

It is shown in [3] that $\bigwedge^2 \mathfrak{g}$ is a Leibniz algebra with the bilinear operation $[\cdot,\cdot]_F$ given by

$$[X,Y]_F = [X,y_1]\wedge y_2 + y_1 \wedge [X,y_2] \in \bigwedge^2 \mathfrak{g} \tag{2.2}$$

for any $X = x_1 \wedge x_2$, $Y = y_1 \wedge y_2$.

Let $(V;\theta)$ be a representation of $\mathfrak{g}$. Cohomogy groups of $\mathfrak{g}$ with coefficients in V are defined, see [12]. First, the space $C^{p-1}(\mathfrak{g}, V)$ of p-cochains is the set of multilinear maps of the form

$$\alpha\colon \underbrace{\bigwedge^2 \mathfrak{g} \otimes \ldots \otimes \bigwedge^2 \mathfrak{g}}_{(p-1)} \otimes \mathfrak{g} \to V,$$

while the coboundary operator $d_\theta\colon C^{p-1}(\mathfrak{g}, V) \to C^p(\mathfrak{g}, V)$ is given by

$$\begin{aligned}
(2.3)\quad & d_\theta(\alpha)(X_1,\ldots,X_p,z)\\
&= \sum_{1\leqslant j<k\leqslant p}(-1)^j\alpha(X_1,\ldots,\widehat{X_j},\ldots,X_{k-1},[X_j,X_k]_F,X_{k+1},\ldots,X_p,z)\\
&\quad+\sum_{j=1}^{p}(-1)^j\alpha(X_1,\ldots,\widehat{X_j},\ldots,X_p,[X_j,z])\\
&\quad+\sum_{j=1}^{p}(-1)^{j+1}\theta(X_j)\alpha(X_1,\ldots,\widehat{X_j},\ldots,X_p,z)\\
&\quad+(-1)^{p+1}(\theta(y_p,z)\alpha(X_1,\ldots,X_{p-1},x_p)+\theta(z,x_p)\alpha(X_1,\ldots,X_{p-1},y_p))
\end{aligned}$$

for all $X_i = x_i \wedge y_i \in \bigwedge^2 \mathfrak{g}$ and $z \in \mathfrak{g}$. Then the pth cohomology group is

$$H^p(\mathfrak{g}, V) = Z^p(\mathfrak{g}, V)/B^p(\mathfrak{g}, V), \tag{2.4}$$

where $Z^p(\mathfrak{g},V)$ or $B^p(\mathfrak{g},V)$ is the space of $(p+1)$-cocycles or $(p+1)$-coboundaries, respectively. One should notice that the upper indices in $Z^p(\mathfrak{g},V)$, $B^p(\mathfrak{g},V)$ and $H^p(\mathfrak{g},V)$ are different from those in the setting of Lie algebra cohomology.

By (2.3), for $\alpha \in C^0(\mathfrak{g},V)$, $X_1 = x\wedge y \in \bigwedge^2 \mathfrak{g}$ and $z\in\mathfrak{g}$,

$$\begin{aligned}d_\theta(\alpha)(X_1,z) &= -\alpha([X_1,z]) + \theta(X_1)\alpha(z) + \theta(y,z)\alpha(x) + \theta(z,x)\alpha(y)\\ &= -\alpha([x,y,z]) + \theta(x,y)\alpha(z) + \text{c.p.},\end{aligned}\tag{2.5}$$

and, for $\alpha\in C^1(\mathfrak{g},V)$, $X_1 = x_1\wedge x_2, X_2 = x_3\wedge x_4$, $x_5\in\mathfrak{g}$,

$$\begin{aligned}&d_\theta(\alpha)(X_1,X_2,x_5)\\ &\quad= -\alpha([X_1,X_2]_F,x_5) - \alpha(X_2,[X_1,x_5]) + \alpha(X_1,[X_2,x_5]) + \theta(X_1)\alpha(X_2,x_5)\\ &\qquad - \theta(X_2)\alpha(X_1,x_5) - \theta(x_4,x_5)\alpha(X_1,x_3) - \theta(x_5,x_3)\alpha(X_1,x_4)\\ &\quad= -\alpha([x_1,x_2,x_3],x_4,x_5) - \alpha(x_3,[x_1,x_2,x_4],x_5) - \alpha(x_3,x_4,[x_1,x_2,x_5])\\ &\qquad + \alpha(x_1,x_2,[x_3,x_4,x_5]) + \theta(x_1,x_2)\alpha(x_3,x_4,x_5) - \theta(x_3,x_4)\alpha(x_1,x_2,x_5)\\ &\qquad - \theta(x_4,x_5)\alpha(x_1,x_2,x_3) - \theta(x_5,x_3)\alpha(x_1,x_2,x_4),\end{aligned}\tag{2.6}$$

where $[\cdot,\cdot]_F$ is given by (2.2).

For any abelian extension $0\to A\hookrightarrow L\xrightarrow{\pi} B\to 0$ of 3-Lie algebras, recall that a section s of π is an $\mathbb{F}$-linear map from B to L such that $\pi s = 1_B$. Fix a section s of π. Define $\omega\colon \bigwedge^3 B\to A$ and $\nu\colon B\to \operatorname{Hom}(\bigwedge^2 A, A)$ by

$$\omega(x_1,x_2,x_3) = [s(x_1),s(x_2),s(x_3)]_L - s([x_1,x_2,x_3]_B),\tag{2.7}$$

$$\nu(x)(a_1,a_2) = [s(x),a_1,a_2]_L\tag{2.8}$$

for all $x,x_1,x_2,x_3\in B$, $a_1,a_2\in A$. Note that ν does not depend on s since A is abelian. We use ω, ν given by (2.7), (2.8) to characterize good abelian extensions of 3-Lie algebras (see Definition 1.1) in the following

Proposition 2.1. *Let $0\to A\hookrightarrow L\xrightarrow{\pi} B\to 0$ be an abelian extension of 3-Lie algebras. Then it is good if and only if the two identities*

$$\nu(x_1)(f(x_2),a) = \nu(x_2)(f(x_1),a),\tag{2.9}$$

$$\nu(x_1)(\omega(x_2,x_3,x_4),a) = 0\tag{2.10}$$

hold for all $f\in\operatorname{Hom}(B,A)$ and $x_1,x_2,x_3,x_4\in B$, $a\in A$.

Proof. Suppose that the abelian extension is good. For any $f\in\operatorname{Hom}(B,A)$, define $s'(x) = s(x)+f(x)$, $x\in B$. Note that $s'\colon B\to L$ is also a section of π. Let ϱ'

and ν' be defined respectively by (1.1) and (2.8) corresponding to s'. Then for any $x_1, x_2 \in B$, $a \in A$, we obtain

$$\begin{aligned}\varrho'(x_1,x_2)(a) &= [s'(x_1), s'(x_2), a]_L = [s(x_1)+f(x_1), s(x_2)+f(x_2), a]_L \\ &= \varrho(x_1,x_2)(a) + \nu(x_1)(f(x_2),a) - \nu(x_2)(f(x_1),a),\end{aligned}$$

and one has $\nu(x_1)(f(x_2),a) = \nu(x_2)(f(x_1),a)$ since ϱ does not depend on the choice of sections.

Moreover, for any $x_i \in B$, $i = 1,2,3,4$ and $a \in A$, by (FI) it follows that

$$\begin{aligned}0 &= [a, s(x_1), [s(x_2), s(x_3), s(x_4)]_L]_L - [[a, s(x_1), s(x_2)]_L, s(x_3), s(x_4)]_L \\ &\quad - [s(x_2), [a, s(x_1), s(x_3)]_L, s(x_4)]_L - [s(x_2), s(x_3), [a, s(x_1), s(x_4)]_L]_L,\end{aligned} \tag{2.11}$$

by (1.1), (2.7) and (2.8), we obtain that (2.11) is equivalent to

$$\begin{aligned}0 &= \nu(x_1)(\omega(x_2,x_3,x_4),a) + \varrho(x_1,[x_2,x_3,x_4]_B)(a) - \varrho(x_3,x_4)\varrho(x_1,x_2)(a) \\ &\quad + \varrho(x_2,x_4)\varrho(x_1,x_3)(a) - \varrho(x_2,x_3)\varrho(x_1,x_4)(a).\end{aligned} \tag{2.12}$$

Then we get $\nu(x_1)\ (\omega(x_2,x_3,x_4),a) = 0$ since ϱ is a representation of B.

Conversely, let $s'\colon B \to L$ be any other section of π and let ϱ' and ν' be defined, respectively by (1.1) and (2.8) corresponding to s'. Define $f\colon B \to A$ as $f(x) = s'(x) - s(x)$, $x \in B$. Then for any $x, x_1, x_2 \in B$, $a \in A$, we deduce

$$\begin{aligned}\varrho'(x_1,x_2)(a) &= [s'(x_1), s'(x_2), a]_L = [s(x_1)+f(x_1), s(x_2)+f(x_2), a]_L \\ &= [s(x_1), s(x_2), a]_L + \nu(x_1)(f(x_2),a) - \nu(x_2)(f(x_1),a) \\ &= \varrho(x_1,x_2)(a) \text{ (by (2.9))}\end{aligned}$$

which implies that ϱ does not depend on the choice of sections.

Furthermore, for any $x_i \in B$, $i = 1,2,3,4$ and $a \in A$, by (FI) we have

$$\begin{aligned}0 &= [a, s(x_1), [s(x_2), s(x_3), s(x_4)]_L]_L - [[a, s(x_1), s(x_2)]_L, s(x_3), s(x_4)]_L \\ &\quad - [s(x_2), [a, s(x_1), s(x_3)]_L, s(x_4)]_L - [s(x_2), s(x_3), [a, s(x_1), s(x_4)]_L]_L,\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}0 &= [s(x_1), s(x_2), [s(x_3), s(x_4), a]_L]_L - [[s(x_1), s(x_2), s(x_3)]_L, s(x_4), a]_L \\ &\quad - [s(x_3), [s(x_1), s(x_2), s(x_4)]_L, a]_L - [s(x_3), s(x_4), [s(x_1), s(x_2), a]_L]_L.\end{aligned} \tag{2.14}$$

By (1.1), (2.7) and (2.8) again we see that (2.13) and (2.14) are equivalent to

$$\begin{aligned}0 &= \nu(x_1)(\omega(x_2,x_3,x_4),a) + \varrho(x_1,[x_2,x_3,x_4]_B)(a) - \varrho(x_3,x_4)\varrho(x_1,x_2)(a) \\ &\quad + \varrho(x_2,x_4)\varrho(x_1,x_3)(a) - \varrho(x_2,x_3)\varrho(x_1,x_4)(a)\end{aligned} \tag{2.15}$$

and

$$
\begin{aligned}
(2.16)\qquad 0 &= \nu(x_4)(\omega(x_1,x_2,x_3),a) - \nu(x_3)(\omega(x_1,x_2,x_4),a)\\
&\quad + \varrho(x_1,x_2)\varrho(x_3,x_4)(a) - \varrho([x_1,x_2,x_3]_B,x_4)(a)\\
&\quad - \varrho(x_3,[x_1,x_2,x_4]_B)(a) - \varrho(x_3,x_4)\varrho(x_1,x_2)(a).
\end{aligned}
$$

Then by (2.10) we obtain that (2.15) and (2.16) become

$$
\begin{aligned}
0 &= \varrho(x_1,[x_2,x_3,x_4]_B)(a) - \varrho(x_3,x_4)\varrho(x_1,x_2)(a)\\
&\quad + \varrho(x_2,x_4)\varrho(x_1,x_3)(a) - \varrho(x_2,x_3)\varrho(x_1,x_4)(a)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \varrho(x_1,x_2)\varrho(x_3,x_4)(a) - \varrho([x_1,x_2,x_3]_B,x_4)(a)\\
&\quad - \varrho(x_3,[x_1,x_2,x_4]_B)(a) - \varrho(x_3,x_4)\varrho(x_1,x_2)(a).
\end{aligned}
$$

Therefore, ϱ is a representation of B. The proof is completed. □

Example 2.1. Let $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ be an extension of 3-Lie algebras with $[A,A,L]=0$. Then this extension is good abelian since (2.9) and (2.10) hold automatically for any section s of π.

We give two examples to show that the notion of good abelian extensions and the notion of split abelian extensions are not the same.

Example 2.2 ([14], Example 2.2)**.** Let B be the 3-dimensional 3-Lie algebra with a basis $\{x_1,x_2,x_3\}$ given by $[x_1,x_2,x_3]=x_1$, and A the 2-dimensional abelian 3-Lie algebra with a basis $\{a_1,a_2\}$. Set $L = B\oplus A$ and consider the following uniquely defined maps $\varrho\colon \bigwedge^2 B \to \mathrm{End}(A)$ and $\nu\colon B \to \mathrm{Hom}(\bigwedge^2 A, A)$ given respectively by (other actions are zero)

$$
\varrho(x_2,x_3)(a_1) = \varrho(x_2,x_3)(a_2) = \frac{1}{2}a_1 + \frac{1}{2}a_2, \tag{2.17}
$$

$$
\nu(x_2)(a_1,a_2) = \nu(x_3)(a_1,a_2) = a_1 + a_2. \tag{2.18}
$$

Define a trilinear bracket operation on L as

$$
[\xi_1+v_1,\xi_2+v_2,\xi_3+v_3]_L = [\xi_1,\xi_2,\xi_3]_B + \varrho(\xi_1,\xi_2)(v_3) + \text{c.p.} + \nu(\xi_1)(v_2,v_3) + \text{c.p.}
$$

for all $\xi_i \in B$ and $v_i \in A$. By a direct check we obtain that $(L,[\cdot,\cdot,\cdot]_L)$ is a 3-Lie algebra and ϱ is a representation of B on A. Hence we have an abelian extension $0 \to A \hookrightarrow L \xrightarrow{p} B \to 0$ of 3-Lie algebras, where $p\colon L \to B$ is the canonical projection. Choose the section $s\colon B \to L$ given by $s(x)=x$, $x\in B$. Note that s is a 3-Lie algebra homomorphism, hence this abelian extension is split.

On the other hand, let $f \in \mathrm{Hom}(B,A)$ be given by $f(x_i)=a_1$, $i=1,2,3$. Then we have $\nu(x_1)(f(x_2),a_2)=0$ whereas $\nu(x_2)(f(x_1),a_2)=a_1+a_2\neq 0$. Therefore, by Proposition 2.1, this abelian extension is not good.

Example 2.3 ([14], Example 4.1)**.** Let B and A be given by Example 2.2. Set $L = B \oplus A$ and consider the uniquely defined maps $\varrho\colon \bigwedge^2 B \to \mathrm{End}(A)$ and $\omega\colon \bigwedge^3 B \to A$ given respectively by (other actions are zero)

$$\varrho(x_2, x_3)(a_1) = \varrho(x_2, x_3)(a_2) = \frac{1}{2}a_1 + \frac{1}{2}a_2,$$
$$\omega(x_i, x_j, x_k) = \mathrm{sign}(ijk)(a_1 - 2a_2),\ (ijk) \in S_3,$$

where S_3 is the permutation group of $\{1, 2, 3\}$. Define a trilinear bracket operation on L as

$$[\xi_1 + v_1, \xi_2 + v_2, \xi_3 + v_3]_L = [\xi_1, \xi_2, \xi_3]_B + \omega(\xi_1, \xi_2, \xi_3) + \varrho(\xi_1, \xi_2)(v_3) + \text{c.p.}$$

for all $\xi_i \in B$ and $v_i \in A$. By a direct check we have that $(L, [\cdot, \cdot, \cdot]_L)$ is a 3-Lie algebra and ϱ is a representation of B on A. Moreover, we have an extension $0 \to A \hookrightarrow L \xrightarrow{p} B \to 0$ of 3-Lie algebras with $[A, A, L] = 0$, hence this abelian extension is good, where $p\colon L \to B$ is the canonical projection.

Next we show that this abelian extension is not split. Choose any section s of p. Set $s(x_i) = \sum\limits_{j=1}^{3} c_{ij}x_j + d_{i1}a_1 + d_{i2}a_2$, $i = 1, 2, 3$. Since $ps = 1_B$, we deduce that $s(x_i) = x_i + d_{i1}a_1 + d_{i2}a_2$. Thus we have

$$\begin{aligned}
&[s(x_1), s(x_2), s(x_3)]_L - s([x_1, x_2, x_3]_B)\\
&\quad = [x_1 + d_{11}a_1 + d_{12}a_2, x_2 + d_{21}a_1 + d_{22}a_2, x_3 + d_{31}a_1 + d_{32}a_2]_L - s(x_1)\\
&\quad = [x_1, x_2, x_3]_B + \omega(x_1, x_2, x_3) + \varrho(x_2, x_3)(d_{11}a_1 + d_{12}a_2) - (x_1 + d_{11}a_1 + d_{12}a_2)\\
&\quad = x_1 + (a_1 - 2a_2) + \Big(\frac{1}{2}a_1 + \frac{1}{2}a_2\Big)d_{11} + \Big(\frac{1}{2}a_1 + \frac{1}{2}a_2\Big)d_{12} - x_1 - d_{11}a_1 - d_{12}a_2\\
&\quad = \Big(1 - \frac{1}{2}d_{11} + \frac{1}{2}d_{12}\Big)a_1 + \Big(-2 + \frac{1}{2}d_{11} - \frac{1}{2}d_{12}\Big)a_2 \neq 0
\end{aligned}$$

as required.

From now on till the end of this section we assume that $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ is a good abelian extension of 3-Lie algebras and fix a section s of π. Then we have the representation $(\varrho; A)$ of B. We give a description of low dimensional cocycles associated to $(\varrho; A)$.

Lemma 2.1. *Let $f \in \mathrm{Hom}(B, A)$. Then $f \in Z^0(B, A)$ if and only if*

$$\varrho(x_1, x_2, f(x_3)) + \text{c.p.} - f([x_1, x_2, x_3]_B) = 0, \quad x_i \in B. \tag{2.19}$$

P r o o f. For any $x_1, x_2, x_3 \in B$, by (2.5) it follows that

$$d_\varrho(f)(x_1, x_2, x_3) = -f([x_1, x_2, x_3]_B) + \varrho(x_1, x_2, f(x_3)) + \text{c.p.}$$

as required. $\square$

Lemma 2.2. *Let* $f \in \operatorname{Hom}(\bigwedge^3 B, A) \subseteq \operatorname{Hom}(\bigwedge^2 B \otimes B, A) = C^1(B, A)$. *Then* $f \in Z^1(B; A)$ *if and only if*

$$\begin{aligned} 0 = & -\varrho(x_4, x_5)f(x_1, x_2, x_3) - \varrho(x_5, x_3)f(x_1, x_2, x_4) \\ & - \varrho(x_3, x_4)f(x_1, x_2, x_5) + \varrho(x_1, x_2)f(x_3, x_4, x_5) \\ & - f([x_1, x_2, x_3]_B, x_4, x_5) - f(x_3, [x_1, x_2, x_4]_B, x_5) \\ & - f(x_3, x_4, [x_1, x_2, x_5]_B) + f(x_1, x_2, [x_3, x_4, x_5]_B) \end{aligned} \tag{2.20}$$

for all $x_i \in B$.

P r o o f. It is straightforward due to (2.6). $\square$

As an application we have

Proposition 2.2. ω *is a 2-cocycle:* $\omega \in Z^1(B, A)$.

P r o o f. For any $x_i \in B$, $1 \leqslant i \leqslant 5$, by (FI) on 3-Lie algebra L we deduce that

$$\begin{aligned} 0 = & [s(x_1), s(x_2), [s(x_3), s(x_4), s(x_5)]_L]_L - [[s(x_1), s(x_2), s(x_3)]_L, s(x_4), s(x_5)]_L \\ & - [s(x_3), [s(x_1), s(x_2), s(x_4)]_L, s(x_5)]_L - [s(x_3), s(x_4), [s(x_1), s(x_2), s(x_5)]_L]_L. \end{aligned}$$

Expanding the above formula, then by (FI) on B it yields that

$$\begin{aligned} 0 = & -\varrho(x_4, x_5)\omega(x_1, x_2, x_3) - \varrho(x_5, x_3)\omega(x_1, x_2, x_4) \\ & - \varrho(x_3, x_4)\omega(x_1, x_2, x_5) + \varrho(x_1, x_2)\omega(x_3, x_4, x_5) \\ & - \omega([x_1, x_2, x_3]_B, x_4, x_5) - \omega(x_3, [x_1, x_2, x_4]_B, x_5) \\ & - \omega(x_3, x_4, [x_1, x_2, x_5]_B) + \omega(x_1, x_2, [x_3, x_4, x_5]_B). \end{aligned} \tag{2.21}$$

Then by Lemma 2.2 we deduce that ω is a 2-cocycle. $\square$

3. The group homomorphism Φ

In this section we fix an abelian extension $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ of 3-Lie algebras, which is not necessarily a good abelian extension. However, we still need to fix a section s of π. Set

$$\mathrm{Aut}_A(L) = \{\gamma \in \mathrm{Aut}(L)\colon \gamma(A) = A\}. \tag{3.1}$$

Note that $\mathrm{Aut}_A(L)$ is a subgroup of $\mathrm{Aut}(L)$. For any $\gamma \in \mathrm{Aut}_A(L)$, as in the cases of groups and Lie algebras (see [2], [13]), one may introduce $\mathbb{F}$-linear maps $\gamma|_A\colon A \to A$ and $\overline{\gamma}\colon B \to B$ given by

$$\gamma|_A(a) = \gamma(a),\ a \in A, \quad \overline{\gamma}(x) = \pi(\gamma(s(x))),\ x \in B, \tag{3.2}$$

respectively. We have

Lemma 3.1. *For any $\gamma \in \mathrm{Aut}_A(L)$,*
(1) *$\overline{\gamma}$ does not depend on the choice of sections of π;*
(2) *$\gamma|_A \in \mathrm{Aut}(A)$ and $\overline{\gamma} \in \mathrm{Aut}(B)$.*

Proof. (1) Let s_1, s_2 be two sections of π and $\overline{\gamma_i} = \pi\gamma s_i$, $i = 1, 2$. For any $x \in B$, since $\pi(s_1(x)) - \pi(s_2(x)) = 0$, we have $s_1(x) - s_2(x) \in A$, and hence $\gamma(s_1(x) - s_2(x)) \in A$. So

$$(\overline{\gamma_1} - \overline{\gamma_2})(x) = \pi(\gamma(s_1(x))) - \pi(\gamma(s_2(x))) = \pi(\gamma(s_1(x) - s_2(x))) = 0$$

as required.

(2) The statement $\gamma|_A \in \mathrm{Aut}(A)$ is direct by definition.

First we show that $\overline{\gamma}\colon B \to B$ is a bijection. For any $x \in B$, since $\gamma \in \mathrm{Aut}_A(L)$ and $L = s(B) \oplus A$, there exist $x_1 \in B$, $a_1 \in A$ such that $\gamma(s(x_1) + a_1) = s(x)$. Then we get

$$\overline{\gamma}(x_1) = \pi(\gamma(s(x_1))) = \pi(s(x) - \gamma(a_1)) = \pi(s(x)) - \pi(\gamma(a_1)) = x,$$

which means that $\overline{\gamma}$ is surjective.

Assume that $x \in B$ satisfies $\overline{\gamma}(x) = (\pi\gamma s)(x) = 0$. Then $\gamma(s(x)) = a \in A$. Since $\gamma(A) = A$ and $\gamma\colon L \to L$ is a bijection, $s(x) \in A$, and hence $s(x) \in s(B) \cap A = \{0\}$. Since $s\colon B \to L$ is injective, $x = 0$, which means that $\overline{\gamma}$ is injective.

Secondly, we show that $\overline{\gamma}$ is a 3-Lie algebra homomorphism. Let $x_1, x_2, x_3 \in B$. Recalling the definition of ω given by (2.7) we have

$$\begin{aligned}
\overline{\gamma}([x_1,x_2,x_3]_B) &= \pi(\gamma(s([x_1,x_2,x_3]_B))) \\
&= \pi(\gamma([s(x_1),s(x_2),s(x_3)]_L - \omega(x_1,x_2,x_3))) \\
&= \pi(\gamma([s(x_1),s(x_2),s(x_3)]_L)) \quad (\text{by } \omega(x_1,x_2,x_3) \in A) \\
&= [\pi(\gamma(s(x_1))),\pi(\gamma(s(x_2))),\pi(\gamma(s(x_3)))]_B \\
&= [\overline{\gamma}(x_1),\overline{\gamma}(x_2),\overline{\gamma}(x_3)]_B
\end{aligned}$$

as required. □

Thus there is a map $\Phi\colon \operatorname{Aut}_A(L) \to \operatorname{Aut}(A) \times \operatorname{Aut}(B)$ given by

$$\Phi(\gamma) = (\gamma|_A, \overline{\gamma}), \tag{3.3}$$

which is independent of the choice of sections of π. We have

Lemma 3.2.
(1) Φ *is a group homomorphism;*
(2) $\ker\Phi$ *is an abelian subgroup of* $\operatorname{Aut}_A(L)$.

Proof. (1) Let γ_1, $\gamma_2 \in \operatorname{Aut}_A(L)$. It suffices to show that

$$(\gamma_1\gamma_2)|_A = \gamma_1|_A\,\gamma_2|_A, \quad \overline{\gamma_1\gamma_2} = \overline{\gamma_1}\;\overline{\gamma_2}.$$

As $(\gamma_1\gamma_2)|_A = \gamma_1|_A\,\gamma_2|_A$ is clear, it remains to show that $\overline{\gamma_1\gamma_2} = \overline{\gamma_1}\;\overline{\gamma_2}$. For any $x \in B$, since $L = s(B) \oplus A$, there are $x_i \in B$, $a_i \in A$ such that $\gamma_i(s(x)) = s(x_i) + a_i$, $i = 1,2$. Since $\gamma_i \in \operatorname{Aut}_A(L)$, applying π to $\gamma_i(s(x)) = s(x_i) + a_i$ we deduce that $\overline{\gamma_i}(x) = x_i$. Then we have

$$\begin{aligned}
(\overline{\gamma_1\gamma_2})(x) &= \pi((\gamma_1\gamma_2)(s(x))) = \pi(\gamma_1(\gamma_2(s(x)))) = \pi(\gamma_1(s(x_2)+a_2)) \\
&= \pi(\gamma_1(s(x_2))) = \pi(\gamma_1(s(\overline{\gamma_2}(x)))) = (\pi\gamma_1 s)(\overline{\gamma_2}(x)) = \overline{\gamma_1}(\overline{\gamma_2}(x))
\end{aligned}$$

as required.

(2) By (1), $\ker\Phi$ is a subgroup of $\operatorname{Aut}(L)$. So it suffices to show that $\ker\Phi$ is abelian. Let $\gamma_1, \gamma_2 \in \ker\Phi$. For any $x \in B$, $a \in A$, by $L = s(B) \oplus A$, it suffices to check that

$$(\gamma_1\gamma_2)(s(x)+a) = (\gamma_2\gamma_1)(s(x)+a).$$

Since $\overline{\gamma_i} = 1_B$, it follows that $\pi(s(x)) = x = \overline{\gamma_i}(x) = \pi(\gamma_i(s(x)))$, which implies that $\gamma_i(s(x)) - s(x) = a_i \in A$, $i = 1,2$. Then

$$(\gamma_1\gamma_2)(s(x)+a) = \gamma_1(s(x)+a_2+a) = s(x)+a_1+a_2+a.$$

Similarly we deduce that $(\gamma_2\gamma_1)(s(x)+a) = s(x)+a_2+a_1+a$. □

As in [2] and [13] we make:

Definition 3.1. A pair of automorphisms $(\sigma, \tau) \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)$ is called inducible if there exists a $\gamma \in \mathrm{Aut}_A(L)$ such that $\Phi(\gamma) = (\sigma, \tau)$.

We will discuss inducibility in the setting of good abelian extensions in the next section. To close this section we give the following technical lemma for further use.

Lemma 3.3. *Let* $(\sigma, \tau) \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)$ *and* $g \in \mathrm{Hom}(B, A)$. *If* $f \in \mathrm{Hom}(B, A)$ *satisfies* (2.9) *then, for* $x_i \in B$, $a_i \in A$,

$$\nu(\tau(x_1))(f(x_2), g(x_3)) + \text{c.p.} + \nu(\tau(x_1))(g(x_2), f(x_3)) + \text{c.p.} = 0, \tag{3.4}$$

$$\nu(\tau(x_1))(f(x_2), \sigma(a_3)) + \text{c.p.} + \nu(\tau(x_1))(\sigma(a_2), f(x_3)) + \text{c.p.} = 0, \tag{3.5}$$

where ν *is given by* (2.8).

P r o o f. We show (3.4), while the proof of (3.5) is similar. For any $x_i \in B$ we have

$$\begin{aligned}
&\nu(\tau(x_1))(f(x_2), g(x_3)) + \text{c.p.} \\
&= \nu(\tau(x_1))(f(x_2), g(x_3)) + \nu(\tau(x_2))(f(x_3), g(x_1)) + \nu(\tau(x_3))(f(x_1), g(x_2)) \\
&= \nu(\tau(x_1))((f\tau^{-1})(\tau(x_2)), (g\tau^{-1})(\tau(x_3))) + \nu(\tau(x_2))((f\tau^{-1})(\tau(x_3)), (g\tau^{-1})(\tau(x_1))) \\
&\quad + \nu(\tau(x_3))((f\tau^{-1})(\tau(x_1)), (g\tau^{-1})(\tau(x_2))) \\
&= \nu(\tau(x_2))((f\tau^{-1})(\tau(x_1)), (g\tau^{-1})(\tau(x_3))) + \nu(\tau(x_3))((f\tau^{-1})(\tau(x_2)), (g\tau^{-1})(\tau(x_1))) \\
&\quad + \nu(\tau(x_1))((f\tau^{-1})(\tau(x_3)), (g\tau^{-1})(\tau(x_2))) \text{ (by (2.9))} \\
&= \nu(\tau(x_2))(f(x_1), g(x_3)) + \nu(\tau(x_3))(f(x_2), g(x_1)) + \nu(\tau(x_1))(f(x_3), g(x_2)) \\
&= -(\nu(\tau(x_1))(g(x_2), f(x_3)) + \text{c.p.})
\end{aligned}$$

as required. □

In particular, substituting $g = f$ and $(\sigma, \tau) = (1_A, 1_B)$ into (3.4) and (3.5) one has:

Corollary 3.1. *If* $f \in \mathrm{Hom}(B, A)$ *satisfies* (2.9) *then, for* $x_i \in B$, $a_i \in A$,

$$\nu(x_1)(f(x_2), f(x_3)) + \text{c.p.} = 0, \tag{3.6}$$

$$\nu(x_1)(f(x_2), a_3) + \text{c.p.} + \nu(x_1)(a_2, f(x_3)) + \text{c.p.} = 0. \tag{3.7}$$

4. The Wells map ϖ

In this section we fix a good abelian extension $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ of 3-Lie algebras with a fixed section s of π. First we have:

Theorem 4.1. *Let $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ be a good abelian extension of 3-Lie algebras. Then $Z^0(B, A) \cong \ker \Phi$ as groups.*

Proof. For any $f \in Z^0(B, A)$ define $\gamma_f \colon L \to L$ by

$$\gamma_f(s(x) + a) = s(x) + f(x) + a, \quad x \in B,\ a \in A.$$

Note that $L = s(B) \oplus A$. Clearly γ_f is $\mathbb{F}$-linear and $\gamma_f(A) = A$. Suppose that $\gamma_f(s(x) + a) = 0$. Then $s(x) = 0$ and $f(x) + a = 0$. Since s is injective, we obtain that $x = a = 0$ and hence γ_f is injective. For any $s(x) + a \in L$, $x \in B$, $a \in A$, we have $\gamma_f(s(x) + a - f(x)) = s(x) + f(x) + a - f(x) = s(x) + a$. Therefore, γ_f is bijective.

Next we show that γ_f is a 3-Lie algebra homomorphism. Indeed, for any $s(x_i) + a_i \in L$, $x_i \in B$, $a_i \in A$, we have

$$\begin{aligned}
&[\gamma_f(s(x_1) + a_1), \gamma_f(s(x_2) + a_2), \gamma_f(s(x_3) + a_3)]_L \\
&\quad = [s(x_1) + f(x_1) + a_1, s(x_2) + f(x_2) + a_2, s(x_3) + f(x_3) + a_3]_L \\
&\quad = [s(x_1), s(x_2), s(x_3)]_L + \varrho(x_1, x_2)(f(x_3)) + \text{c.p.} + \varrho(x_1, x_2)(a_3) + \text{c.p.} \\
&\qquad + \nu(x_1)(a_2, a_3) + \text{c.p.} \text{ (by (3.6) and (3.7))} \\
&\quad = s([x_1, x_2, x_3]_B) + \omega(x_1, x_2, x_3) + f([x_1, x_2, x_3]_B) + \varrho(x_1, x_2)(a_3) + \text{c.p.} \\
&\qquad + \nu(x_1)(a_2, a_3) + \text{c.p.} \text{ (by Lemma 2.1)} \\
&\quad = \gamma_f(s([x_1, x_2, x_3]_B) + \omega(x_1, x_2, x_3) + \varrho(x_1, x_2)(a_3) + \text{c.p.} + \nu(x_1)(a_2, a_3) + \text{c.p.}) \\
&\quad = \gamma_f([s(x_1) + a_1, s(x_2) + a_2, s(x_3) + a_3]_L)
\end{aligned}$$

as required. Thus we have a map $\eta \colon Z^0(B, A) \to \mathrm{Aut}(L)$ given by $f \mapsto \gamma_f$.

Since $\Phi(\gamma_f) = (1_A, 1_B)$, $\gamma_f \in \ker \Phi$ and hence η is a map from $Z^0(B, A)$ to $\ker \Phi$. We show that $\eta \colon Z^0(B, A) \to \ker \Phi$ is a group homomorphism. Indeed, for any $f_1, f_2 \in Z^0(B, A)$, $x \in B$ and $a \in A$, we have

$$\begin{aligned}
(\eta(f_1 + f_2))(s(x) + a) &= \gamma_{f_1 + f_2}(s(x) + a) = s(x) + (f_1 + f_2)(x) + a \\
&= \gamma_{f_1}(s(x) + f_2(x) + a) = \gamma_{f_1}(\gamma_{f_2}(s(x) + a)) = (\eta(f_1)\eta(f_2))(s(x) + a),
\end{aligned}$$

from which we get $\eta(f_1 + f_2) = \eta(f_1)\eta(f_2)$ since $L = s(B) \oplus A$. Suppose that $\eta(f) = \gamma_f = 0$. Then for all $x \in B$, $0 = \gamma_f(s(x)) = s(x) + f(x)$, and hence $f(x) = 0$, which implies that $f = 0$. So η is injective.

Now we show that η is surjective. For any $\gamma \in \ker \Phi$, by (3.2) it follows that for any $x \in B$, $x = \overline{\gamma}(x) = \pi(\gamma(s(x))) = \pi(s(x))$, so we can define a linear map $f\colon B \to A$ by $f(x) = \gamma(s(x)) - s(x) \in A$, $x \in B$. If $f \in Z^0(B, A)$ then

$$\eta(f)(s(x) + a) = \gamma_f(s(x) + a) = s(x) + f(x) + a = \gamma(s(x) + a)$$

and hence $\eta(f) = \gamma$. So it remains to show that $f \in Z^0(B, A)$. To this end we use Lemma 2.1 as follows. In view of (3.6), we get

$$\nu(x_1)(f(x_2), f(x_3)) + \text{c.p.} = 0.$$

Moreover, since $\gamma \in \ker \Phi$, $\gamma(a) = a$, $a \in A$. Then for any $x_1, x_2, x_3 \in B$ we obtain

$$\begin{aligned}
&\varrho(x_1, x_2)(f(x_3)) + c.p - f([x_1, x_2, x_3]_B) \\
&\quad = ([s(x_1), s(x_2), s(x_3)]_L + \varrho(x_1, x_2)(f(x_3)) + \text{c.p.} + \nu(x_1)(f(x_2), f(x_3)) + \text{c.p.}) \\
&\qquad - ([s(x_1), s(x_2), s(x_3)]_L + f([x_1, x_2, x_3]_B)) \\
&\quad = [s(x_1) + f(x_1), s(x_2) + f(x_2), s(x_3) + f(x_3)]_L \\
&\qquad - (s([x_1, x_2, x_3]_B) + \omega(x_1, x_2, x_3) + f([x_1, x_2, x_3]_B)) \\
&\quad = [\gamma(s(x_1)), \gamma(s(x_2)), \gamma(s(x_3))]_L - \gamma(s([x_1, x_2, x_3]_B) + \omega(x_1, x_2, x_3)) \\
&\quad = [\gamma(s(x_1)), \gamma(s(x_2)), \gamma(s(x_3))]_L - \gamma([s(x_1), s(x_2), s(x_3)]_L) = 0.
\end{aligned}$$

Therefore, we deduce that $f \in Z^0(B, A)$. $\square$

Assume that $\Phi(\gamma) = (\sigma, \tau)$, so $\overline{\gamma} = \pi\gamma s = \tau$. (Recall that s is a fixed section of π in the good abelian extension $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ of 3-Lie algebras.) It follows that, for any $x \in B$, $\overline{\gamma}(x) = \pi(\gamma(s(x))) = \tau(x) = \pi(s(\tau(x)))$, which means that $\gamma(s(x)) - s(\tau(x)) \in A$. So we get an $\mathbb{F}$-linear map $f \in \mathrm{Hom}(B, A)$ given by

$$f(x) = \gamma(s(x)) - s(\tau(x)), \quad x \in B. \tag{4.1}$$

To obtain a condition for (σ, τ) to be inducible we need the following:

Lemma 4.1. *If* $\Phi(\gamma) = (\sigma, \tau)$ *then*

$$(1)\ \sigma\varrho(x_1, x_2)\sigma^{-1} = \varrho(\tau(x_1), \tau(x_2)), \quad x_1, x_2 \in B, \tag{4.2}$$

$$(2)\ \sigma\nu(x)\sigma^{-1} = \nu(\tau(x)), \quad x \in B. \tag{4.3}$$

P r o o f. For any $x_1, x_2 \in B$, $a \in A$, by (4.1) it follows that

$$\begin{aligned}
&\sigma\varrho(x_1,x_2)\sigma^{-1}(a)\\
&\quad= \sigma([s(x_1), s(x_2), \sigma^{-1}(a)]_L) \text{ (by (1.1))}\\
&\quad= \gamma([s(x_1), s(x_2), \sigma^{-1}(a)]_L)\\
&\quad= [\gamma(s(x_1)), \gamma(s(x_2)), \gamma(\sigma^{-1}(a))]_L\\
&\quad= [s(\tau(x_1)) + f(x_1), s(\tau(x_2)) + f(x_2), a]_L\\
&\quad= \varrho(\tau(x_1), \tau(x_2))(a) + \nu(\tau(x_1))(f(x_2), a) - \nu(\tau(x_2))(f(x_1), a)\\
&\quad= \varrho(\tau(x_1), \tau(x_2))(a) + \nu(\tau(x_1))((f\tau^{-1})(\tau(x_2)), a) - \nu(\tau(x_2))((f\tau^{-1})(\tau(x_1)), a)\\
&\quad= \varrho(\tau(x_1), \tau(x_2))(a), \text{ (by (2.9))}
\end{aligned}$$

which means (4.2) holds. For any $x \in B$, $a_1, a_2 \in A$, we obtain

$$\begin{aligned}
\sigma\nu(x)\sigma^{-1}(a_1, a_2) &= \sigma\nu(x)(\sigma^{-1}(a_1), \sigma^{-1}(a_2))\\
&= \sigma([s(x), \sigma^{-1}(a_1), \sigma^{-1}(a_2)]_L) \text{ (by(2.8))}\\
&= \gamma([s(x), \sigma^{-1}(a_1), \sigma^{-1}(a_2)]_L)\\
&= [\gamma(s(x)), \gamma(\sigma^{-1}(a_1)), \gamma(\sigma^{-1}(a_2))]_L\\
&= [s(\tau(x)) + f(x), a_1, a_2]_L\\
&= \nu(\tau(x))(a_1, a_2) \text{ (by } f(x) \in A\text{)},
\end{aligned}$$

which means that (4.3) holds. The proof is finished. □

The following corollary is straightforward.

Corollary 4.1. *If* $\Phi(\gamma) = (\sigma, \tau)$ *then*

(1) $\sigma(\varrho(x_1, x_2)(a_3) + \text{c.p.}) = \varrho(\tau(x_1), \tau(x_2))(\sigma(a_3)) + \text{c.p.}$,

(2) $\sigma(\nu(x_1)(a_2, a_3) + \text{c.p.}) = \nu(\tau(x_1))(\sigma(a_2), \sigma(a_3)) + \text{c.p.}$

Motivated by Lemma 4.1 we introduce:

Definition 4.1. A pair of automorphisms $(\sigma, \tau) \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)$ is called compatible if it satisfies (4.2) and (4.3). Denote by $C^L_{B,A}$ the set of all compatible pairs.

Note that $C^L_{B,A}$ is a subgroup of $\mathrm{Aut}(A) \times \mathrm{Aut}(B)$, and independent of the choice of sections since $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ is a good abelian extension.

Similarly to the case of Lie algebras (see [2], Section 3), for any $(\sigma, \tau) \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)$ we introduce a trilinear map $\omega_{\sigma,\tau} \colon \bigwedge^3 B \to A$ defined by

$$\omega_{\sigma,\tau}(x_1, x_2, x_3) = \sigma\omega(\tau^{-1}(x_1), \tau^{-1}(x_2), \tau^{-1}(x_3)) - \omega(x_1, x_2, x_3), \quad x_i \in B. \tag{4.4}$$

We will show that $\omega_{\sigma,\tau}$ is a 2-cocycle if $(\sigma, \tau) \in C^L_{B,A}$. For this case we need:

Notation 4.1. For any $f \in \operatorname{Hom}(\bigwedge^3 B, A)$, we write the RHS of (2.20) as $S_{f,\varrho}(x_1,x_2,x_3,x_4,x_5)$. Hence if $f = \omega$, then by Proposition 2.2 one obtains

$$S_{\omega,\varrho}(x_1, x_2, x_3, x_4, x_5) = 0 \tag{4.5}$$

for all $x_i \in B$. Moreover, for any $(\sigma, \tau) \in \operatorname{Aut}(A) \times \operatorname{Aut}(B)$, we write

$$\begin{aligned}
S^{\sigma,\tau}_{\omega,\varrho}(x_1, x_2, x_3, x_4, x_5) = {} & \varrho(x_1, x_2)\sigma\omega(\tau^{-1}(x_3), \tau^{-1}(x_4), \tau^{-1}(x_5)) \\
& - \varrho(x_3, x_4)\sigma\omega(\tau^{-1}(x_1), \tau^{-1}(x_2), \tau^{-1}(x_5)) \\
& - \varrho(x_4, x_5)\sigma\omega(\tau^{-1}(x_1), \tau^{-1}(x_2), \tau^{-1}(x_3)) \\
& - \varrho(x_5, x_3)\sigma\omega(\tau^{-1}(x_1), \tau^{-1}(x_2), \tau^{-1}(x_4)) \\
& + \sigma\omega(\tau^{-1}(x_1), \tau^{-1}(x_2), [\tau^{-1}(x_3), \tau^{-1}(x_4), \tau^{-1}(x_5)]_B) \\
& - \sigma\omega(\tau^{-1}(x_3), \tau^{-1}(x_4), [\tau^{-1}(x_1), \tau^{-1}(x_2), \tau^{-1}(x_5)]_B) \\
& - \sigma\omega(\tau^{-1}(x_4), \tau^{-1}(x_5), [\tau^{-1}(x_1), \tau^{-1}(x_2), \tau^{-1}(x_3)]_B) \\
& - \sigma\omega(\tau^{-1}(x_5), \tau^{-1}(x_3), [\tau^{-1}(x_1), \tau^{-1}(x_2), \tau^{-1}(x_4)]_B).
\end{aligned}$$

With the above notation, we have

Lemma 4.2. *If $(\sigma, \tau) \in C^L_{B,A}$ then $S^{\sigma,\tau}_{\omega,\varrho}(x_1, x_2, x_3, x_4, x_5) = 0$.*

P r o o f. By (4.2) it follows that

$$\begin{aligned}
S^{\sigma,\tau}_{\omega,\varrho}(x_1, x_2, x_3, x_4, x_5) &= \sigma(S_{\omega,\varrho}(\tau^{-1}(x_1), \tau^{-1}(x_2), \tau^{-1}(x_3), \tau^{-1}(x_4), \tau^{-1}(x_5))) \\
&= \sigma(0) \text{ (by (4.5))} = 0
\end{aligned}$$

as required. □

Proposition 4.1. *If $(\sigma, \tau) \in C^L_{B,A}$ then $\omega_{\sigma,\tau}$ is a 2-cocycle.*

P r o o f. It suffices to show $d_\varrho(\omega_{\sigma,\tau}) = 0$. For any $x_1, x_2, x_3, x_4, x_5 \in B$, by a direct calculation it follows that

$$\begin{aligned}
d_\varrho(\omega_{\sigma,\tau})(x_1, x_2, x_3, x_4, x_5) &= S^{\sigma,\tau}_{\omega,\varrho}(x_1, x_2, x_3, x_4, x_5) - S_{\omega,\varrho}(x_1, x_2, x_3, x_4, x_5) \\
&= 0 - 0 \text{ (by Lemma 4.2 and (4.5))} = 0
\end{aligned}$$

as required. □

Now we give a criterion for $(\sigma, \tau) \in \operatorname{Aut}(A) \times \operatorname{Aut}(B)$ to be inducible in terms of maps ϱ, ω and ν (see (1.1), (2.7) and (2.8)).

Theorem 4.2. *Let $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ be a good abelian extension of 3-Lie algebras. Then (σ,τ) is inducible if and only if the following conditions (1)–(3) hold:*
(1) $\omega_{\sigma,\tau}$ *is a 2-coboundary,*
(2) $\sigma\varrho(x_1,x_2)\sigma^{-1} = \varrho(\tau(x_1),\tau(x_2))$ *for all* $x_1,x_2 \in B$,
(3) $\sigma\nu(x)\sigma^{-1} = \nu(\tau(x))$ *for all* $x \in B$.

Note that (2) and (3) are equivalent to saying that $(\sigma,\tau) \in C^L_{B,A}$.

Proof. Suppose that the pair (σ,τ) is inducible, that is, there is a $\gamma \in \mathrm{Aut}(L)$ such that $\Phi(\gamma) = (\sigma,\tau)$. Then by Lemma 4.1 we obtain that (2) and (3) hold. Now we show that (1) also holds.

Since $\Phi(\gamma) = (\sigma,\tau)$, we have a map $f \in \mathrm{Hom}(B,A)$ given by (4.1), that is,

$$f(x) = \gamma(s(x)) - s(\tau(x)), \quad x \in B.$$

For any $x_i \in B$, $a_i \in A$, $i = 1,2,3$, since $\gamma \in \mathrm{Aut}_A(L)$, we have

$$\begin{aligned}[]
[\gamma(s(x_1)+a_1),\gamma(s(x_2)+a_2),\gamma(s(x_3)+a_3)]_L \\
= \gamma([s(x_1)+a_1, s(x_2)+a_2, s(x_3)+a_3]_L).
\end{aligned} \tag{4.6}$$

By (4.1), the LHS of (4.6) becomes

$$\begin{aligned}
&[\gamma(s(x_1))+\sigma(a_1),\gamma(s(x_2))+\sigma(a_2),\gamma(s(x_3))+\sigma(a_3)]_L\\
&\quad = [s(\tau(x_1))+f(x_1)+\sigma(a_1), s(\tau(x_2))+f(x_2)+\sigma(a_2), s(\tau(x_3))+f(x_3)+\sigma(a_3)]_L\\
&\quad = [s(\tau(x_1)), s(\tau(x_2)), s(\tau(x_3))]_L + \varrho(\tau(x_1),\tau(x_2))(f(x_3)) + \text{c.p.}\\
&\qquad + \varrho(\tau(x_1),\tau(x_2))(\sigma(a_3)) + \text{c.p.} + \nu(\tau(x_1))(\sigma(a_2),\sigma(a_3)) + \text{c.p. (by Lemma 3.3)}.
\end{aligned}$$

By $\gamma|_A = \sigma$ and Corollary 4.1, the RHS of (4.6) equals

$$\begin{aligned}
&\gamma([s(x_1),s(x_2),s(x_3)]_L + \varrho(x_1,x_2)(a_3) + \text{c.p.} + \nu(x_1)(a_2,a_3) + \text{c.p.})\\
&\qquad = \gamma([s(x_1),s(x_2),s(x_3)]_L) + \sigma(\varrho(x_1,x_2)(a_3) + \text{c.p.}) + \sigma(\nu(x_1)(a_2,a_3) + \text{c.p.})\\
&\qquad = \gamma(s([x_1,x_2,x_3]_B)) + \sigma(\omega(x_1,x_2,x_3)) + \sigma(\varrho(x_1,x_2)(a_3)) + \text{c.p.}\\
&\qquad\quad + \sigma(\nu(x_1)(a_2,a_3) + \text{c.p.})\\
&\qquad = s(\tau([x_1,x_2,x_3]_B)) + f([x_1,x_2,x_3]_B) + \sigma(\omega(x_1,x_2,x_3))\\
&\qquad\quad + \varrho(\tau(x_1),\tau(x_2))(\sigma(a_3)) + \text{c.p.}\\
&\qquad\quad + \nu(\tau(x_1))(\sigma(a_2),\sigma(a_3)) + \text{c.p. (by Corollary 4.1)}.
\end{aligned}$$

By comparing the LHS and the RHS of (4.6) we obtain

$$\begin{aligned}
\sigma(\omega(x_1,x_2,x_3)) - \omega(\tau(x_1),\tau(x_2),\tau(x_3))\\
= -f([x_1,x_2,x_3]_B) + \varrho(\tau(x_1),\tau(x_2))(f(x_3)) + \text{c.p.}
\end{aligned} \tag{4.7}$$

Replacing x_i by $\tau^{-1}(x_i)$ in (4.7) we have that

$$\begin{aligned}(4.8)\qquad &\sigma(\omega(\tau^{-1}(x_1),\tau^{-1}(x_2),\tau^{-1}(x_3)))-\omega(x_1,x_2,x_3)\\ &=-f([\tau^{-1}(x_1),\tau^{-1}(x_2),\tau^{-1}(x_3)]_B)+\varrho(x_1,x_2)(f(\tau^{-1}(x_3)))+\text{c.p.}\end{aligned}$$

By (4.4) and (2.5) we deduce that (4.8) is equivalent to $\omega_{\sigma,\tau}=d_\varrho(f\tau^{-1})$, which implies that $\omega_{\sigma,\tau}$ is a 2-coboundary.

Conversely, since $\omega_{\sigma,\tau}$ is a 2-coboundary, there exists an $f\in\mathrm{Hom}(B,A)$ such that

$$(4.9)\qquad \omega_{\sigma,\tau}(x_1,x_2,x_3)=-f([x_1,x_2,x_3]_B)+\varrho(x_1,x_2)(f(x_3))+\text{c.p.}$$

Define $\gamma\colon L\to L$ by

$$(4.10)\qquad \gamma(s(x)+a)=s(\tau(x))+f(\tau(x))+\sigma(a),\quad x\in B,\ a\in A.$$

Recall that $L=s(B)\oplus A$. Note that γ is $\mathbb{F}$-linear and $\gamma(a)=\sigma(a)$, $a\in A$. Moreover, for any $x\in B$, $\overline{\gamma}(x)=\pi(\gamma(s(x)))=\pi(s(\tau(x))+f(\tau(x)))=\tau(x)$. So, it suffices to check that γ is a 3-Lie algebra automorphism of L.

Suppose that $\gamma(s(x)+a)=s(\tau(x))+f(\tau(x))+\sigma(a)=0$, $x\in B$, $a\in A$. Since $f(\tau(x))\in A$ and $s(B)\cap A=\{0\}$, we have $s(\tau(x))=0$ and $f(\tau(x))+\sigma(a)=0$. Since s is injective and $(\sigma,\tau)\in\mathrm{Aut}(A)\times\mathrm{Aut}(B)$, $x=0$ and hence $a=0$, which implies that γ is injective.

For any $s(x)+a\in L$, $x\in B$, $a\in A$, set $x_0=\tau^{-1}(x)\in B$ and $a_0=\sigma^{-1}(a-f(x))\in A$. By (4.10) it follows that

$$\begin{aligned}\gamma(s(x_0)+a_0)&=s(\tau(x_0))+f(\tau(x_0))+\sigma(a_0)\\ &=s(\tau(\tau^{-1}(x)))+f(\tau(\tau^{-1}(x)))+\sigma(\sigma^{-1}(a-f(x)))\\ &=s(x)+f(x)+a-f(x)=s(x)+a,\end{aligned}$$

which means that γ is surjective.

It remains to show that γ is a 3-Lie algebra homomorphism. For any $s(x_i)+a_i\in L$, $x_i\in B$, $a_i\in A$, $i=1,2,3$, by (4.10) one obtains

$$\begin{aligned}(4.11)\quad &[\gamma(s(x_1)+a_1),\gamma(s(x_2)+a_2),\gamma(s(x_3)+a_3)]_L\\ &=[s(\tau(x_1))+f(\tau(x_1))+\sigma(a_1),s(\tau(x_2))\\ &\quad+f(\tau(x_2))+\sigma(a_2),s(\tau(x_3))+f(\tau(x_3))+\sigma(a_3)]_L\\ &=[s(\tau(x_1)),s(\tau(x_2)),s(\tau(x_3))]_L+\varrho(\tau(x_1),\tau(x_2))(f(\tau(x_3)))+\text{c.p.}\\ &\quad+\varrho(\tau(x_1),\tau(x_2))(\sigma(a_3))+\text{c.p.}\\ &\quad+\nu(\tau(x_1))(\sigma(a_2),\sigma(a_3))+\text{c.p.}\ \text{(by Lemma 3.3)}\end{aligned}$$

$$
\begin{aligned}
&= s(\tau([x_1,x_2,x_3]_B)) + f(\tau([x_1,x_2,x_3]_B)) \\
&\quad + \sigma(\omega(x_1,x_2,x_3)) + \varrho(\tau(x_1),\tau(x_2))(\sigma(a_3)) + \text{c.p.} \\
&\quad + \nu(\tau(x_1))(\sigma(a_2),\sigma(a_3)) + \text{c.p. (by (4.9), replacing } x_i \text{ by } \tau(x_i)).
\end{aligned}
$$

By (2) it follows that

$$
\begin{aligned}
\sigma(\varrho(x_1,x_2)(a_3)+\text{c.p.}) &= \sigma(\varrho(x_1,x_2)(a_3) + \varrho(x_2,x_3)(a_1) + \varrho(x_3,x_1)(a_2)) \\
&= \varrho(\tau(x_1),\tau(x_2))(\sigma(a_3)) + \varrho(\tau(x_2),\tau(x_3))(\sigma(a_1)) \\
&\quad + \varrho(\tau(x_3),\tau(x_1))(\sigma(a_2)) \\
&= \varrho(\tau(x_1,\tau(x_2))(\sigma(a_3)) + \text{c.p.}
\end{aligned} \tag{4.12}
$$

and by (3) it follows that

$$
\begin{aligned}
\sigma(\nu(x_1)(a_2,a_3)+\text{c.p.}) &= \sigma(\nu(x_1)(a_2,a_3) + \nu(x_2)(a_3,a_1) + \nu(x_3)(a_1,a_2)) \\
&= \nu(\tau(x_1))(\sigma(a_2),\sigma(a_3)) + \nu(\tau(x_2))(\sigma(a_3),\sigma(a_1)) \\
&\quad + \nu(\tau(x_3))(\sigma(a_1),\sigma(a_2)) \\
&= \nu(\tau(x_1))(\sigma(a_2),\sigma(a_3)) + \text{c.p.}
\end{aligned} \tag{4.13}
$$

Substituting (4.12) and (4.13) into (4.11) we have that

$$
\begin{aligned}
&[\gamma(s(x_1)+a_1),\gamma(s(x_2)+a_2),\gamma(s(x_3)+a_3)]_L \\
&\quad = s(\tau([x_1,x_2,x_3]_B)) + f(\tau([x_1,x_2,x_3]_B)) \\
&\quad\quad + \sigma(\omega(x_1,x_2,x_3)) + \sigma(\varrho(x_1,x_2)(a_3)+\text{c.p.}) + \sigma(\nu(x_1)(a_2,a_3)+\text{c.p.}) \\
&\quad = \gamma(s([x_1,x_2,x_3]_B) + \omega(x_1,x_2,x_3) + \varrho(x_1,x_2)(a_3) + \text{c.p.} + \nu(x_1)(a_2,a_3) + \text{c.p.}) \\
&\quad = \gamma([s(x_1),s(x_2),s(x_3)]_B + \varrho(x_1,x_2)(a_3) + \text{c.p.} + \nu(x_1)(a_2,a_3) + \text{c.p.}) \\
&\quad = \gamma([s(x_1)+a_1,s(x_2)+a_2,s(x_3)+a_3]_L),
\end{aligned}
$$

which completes the proof. □

By Theorem 4.2 and Definition 4.1 we have

Corollary 4.2. *Let $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ be a good abelian extension of 3-Lie algebras. Suppose that $(\sigma,\tau) \in C^L_{B,A}$. Then (σ,τ) is inducible if and only if $\omega_{\sigma,\tau}$ is a 2-coboundary.*

By Proposition 4.1, we can define a map $\varpi\colon C^L_{B,A} \to H^1(B,A)$, called the Wells map, by

$$
\varpi(\sigma,\tau) = [\omega_{\sigma,\tau}]. \tag{4.14}
$$

Therefore, by Corollary 4.2, $[\omega_{\sigma,\tau}] \in H^1(B,A)$ is an obstruction to inducibility of the pair $(\sigma,\tau) \in C^L_{B,A}$.

Combining Theorem 4.1 and Corollary 4.2, we obtain the main theorem of this section as follows.

Theorem 4.3. *Let $0 \to A \hookrightarrow L \overset{\pi}{\to} B \to 0$ be a good abelian extension of 3-Lie algebras. Then there is an exact sequence*

$$0 \to Z^0(B,A) \to \mathrm{Aut}_A(L) \overset{\Phi}{\to} C^L_{B,A} \overset{\varpi}{\to} H^1(B,A). \tag{4.15}$$

Applying this result to good split abelian extensions of 3-Lie algebras we get

Corollary 4.3. *Let $0 \to A \hookrightarrow L \overset{\pi}{\to} B \to 0$ be a good split abelian extension of 3-Lie algebras. Then every compatible pair is inducible.*

Proof. It suffices to show that ϖ is trivial. Since the extension is split, there exists a section $s_0\colon B \to L$ which is a 3-Lie algebra homomorphism. Let ϱ_0, ω_0, ν_0 be given, respectively by (1.1), (2.7), (2.8) corresponding to s_0. Define a linear map $f\colon B \to A$ as

$$f(x) = s(x) - s_0(x), \quad x \in B.$$

Since s_0 is a 3-Lie algebra homomorphism, $\omega_0(x_1,x_2,x_3) = 0$, $x_i \in B$. Then by a direct computation we have

$$\begin{aligned}
&\omega(x_1,x_2,x_3)\\
&= [s(x_1),s(x_2),s(x_3)]_L - s([x_1,x_2,x_3]_B)\\
&= [s_0(x_1)+f(x_1), s_0(x_2)+f(x_2), s_0(x_3)+f(x_3)]_L\\
&\quad - (s_0([x_1,x_2,x_3]_B) + f([x_1,x_2,x_3]_B))\\
&= \varrho_0(x_1,x_2)(f(x_3)) + \text{c.p.} - f([x_1,x_2,x_3]_B) + \nu_0(x_1)(f(x_2),f(x_3)) + \text{c.p.}\\
&= d_{\varrho_0}(f)(x_1,x_2,x_3) \ \text{(by (2.5) and (3.6))}\\
&= d_{\varrho}(f)(x_1,x_2,x_3) \ \text{(by Definition 1.1)}
\end{aligned} \tag{4.16}$$

and similarly we obtain that

$$\begin{aligned}
&\omega(\tau^{-1}(x_1),\tau^{-1}(x_2),\tau^{-1}(x_3))\\
&= [s(\tau^{-1}(x_1)), s(\tau^{-1}(x_2)), s(\tau^{-1}(x_3))]_L - s([\tau^{-1}(x_1),\tau^{-1}(x_2),\tau^{-1}(x_3)]_B)\\
&= \varrho(\tau^{-1}(x_1),\tau^{-1}(x_2))(f(\tau^{-1}(x_3))) + \text{c.p.} - f(\tau^{-1}([x_1,x_2,x_3]_B)).
\end{aligned}$$

Since (σ,τ) is compatible, in view of (4.3) one has

$$\begin{aligned}(4.17)\qquad \sigma(\omega(\tau^{-1}(x_1),\tau^{-1}(x_2),\tau^{-1}(x_3)))\\ &= \varrho(x_1,x_2)\sigma(f(\tau^{-1}(x_3))) + \text{c.p.} - \sigma(f(\tau^{-1}([x_1,x_2,x_3]_B)))\\ &= d_\varrho(\sigma f\tau^{-1})(x_1,x_2,x_3).\end{aligned}$$

Combining (4.16) and (4.17), we deduce

$$\begin{aligned}\omega_{\sigma,\tau}(x_1,x_2,x_3) &= \sigma\omega(\tau^{-1}(x_1),\tau^{-1}(x_2),\tau^{-1}(x_3)) - \omega(x_1,x_2,x_3)\\ &= d_\varrho(\sigma f\tau^{-1} - f)(x_1,x_2,x_3),\end{aligned}$$

which completes the proof. $\square$

We give another application of Theorem 4.3. By Theorem 4.1 we have the short exact sequence

$$(4.18)\qquad 0 \to Z^0(B,A) \to \mathrm{Aut}_A(L) \xrightarrow{\Phi} \operatorname{im}\Phi \to 0.$$

Theorem 4.4. *Let $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ be a good split abelian extension of 3-Lie algebras. Then the short exact sequence* (4.18) *is also split.*

Proof. Since the good abelian extension $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ is split, there is a section $s_0\colon B \to L$ of π which is also a 3-Lie algebra homomorphism, and, in view of Corollary 4.3, we obtain that ϖ is trivial and hence $\operatorname{im}\Phi = C^L_{B,A}$ by (4.15). Define $f\colon B \to A$ as

$$f(x) = s(x) - s_0(x), \quad x \in B.$$

Define $\Psi\colon \operatorname{im}\Phi \to \mathrm{End}(L)$ as $\Psi(\sigma,\tau) = \gamma$, where $\gamma \in \mathrm{End}(L)$ is defined as

$$(4.19)\qquad \gamma(s(x)+a) = s(\tau(x)) + \sigma(f(x)) - f(\tau(x)) + \sigma(a), \quad x \in B,\ a \in A.$$

Note that $\gamma(A) = A$.

Since $(\sigma,\tau) \in \operatorname{im}\Phi$, $\omega_{\sigma,\tau}$ is a 2-coboundary. By the proof of Corollary 4.3 one obtains

$$\begin{aligned}(4.20)\qquad \omega_{\sigma,\tau}(x_1,x_2,x_3) &= d_\varrho(\sigma f\tau^{-1} - f)(x_1,x_2,x_3)\\ &= d_\varrho(\sigma f\tau^{-1})(x_1,x_2,x_3) - d_\varrho(f)(x_1,x_2,x_3)\\ &= \varrho(x_1,x_2)((\sigma f\tau^{-1})(x_3)) + \text{c.p.} - (\sigma f\tau^{-1})([x_1,x_2,x_3]_B)\\ &\quad - (\varrho(x_1,x_2)(f(x_3)) + \text{c.p.} - f([x_1,x_2,x_3]_B))\end{aligned}$$

for all $x_1, x_2, x_3 \in B$. Replacing x_i by $\tau(x_i)$ in (4.20) we have that

$$\begin{aligned}(4.21)\qquad &\sigma(\omega(x_1,x_2,x_3)) - \omega(\tau(x_1),\tau(x_2),\tau(x_3))\\ &= \varrho(\tau(x_1),\tau(x_2))((\sigma f)(x_3)) + \text{c.p.} - (\sigma f)([x_1,x_2,x_3]_B)\\ &\quad - (\varrho(\tau(x_1),\tau(x_2))((f\tau)(x_3)) + \text{c.p.} - (f\tau)([x_1,x_2,x_3]_B)).\end{aligned}$$

Next we show that γ is bijective. Suppose that $\gamma(s(x)+a) = 0$, namely, $s(\tau(x)) + \sigma(f(x)) - f(\tau(x)) + \sigma(a) = 0$. Then one has $s(\tau(x)) = 0$, $\sigma(f(x)) - f(\tau(x)) + \sigma(a) = 0$. Since s is injective and $(\sigma,\tau) \in \mathrm{Aut}(A) \times \mathrm{Aut}(B)$, we deduce $x = 0$ and hence $a = 0$. To show γ is surjective, let $s(x) + a \in L$. Set $x_0 = \tau^{-1}(x)$ and $a_0 = \sigma^{-1}(a - \sigma(f(\tau^{-1}(x))) + f(x))$. Then we obtain

$$\begin{aligned}\gamma(s(x_0)+a_0) &= s(\tau(x_0)) + \sigma(f(x_0)) - f(\tau(x_0)) + \sigma(a_0)\\ &= s(\tau(\tau^{-1}(x))) + \sigma(f(\tau^{-1}(x))) - f(\tau(\tau^{-1}(x)))\\ &\quad + \sigma(\sigma^{-1}(a - \sigma(f(\tau^{-1}(x))) + f(x)))\\ &= s(x) + \sigma(f(\tau^{-1}(x))) - f(x) + a - \sigma(f(\tau^{-1}(x))) + f(x)\\ &= s(x) + a.\end{aligned}$$

Therefore, to show Ψ is a map from $\operatorname{im}\Phi$ to $\mathrm{Aut}_A(L)$, it remains to show that γ is a 3-Lie algebra homomorphism. For any $x_1, x_2, x_3 \in B$, $a_1, a_2, a_3 \in A$, we obtain

$$\begin{aligned}&[\gamma(s(x_1)+a_1), \gamma(s(x_2)+a_2), \gamma(s(x_3)+a_3)]_L\\ &\quad= [s(\tau(x_1)) + \sigma(f(x_1)) - f(\tau(x_1)) + \sigma(a_1), s(\tau(x_2)) + \sigma(f(x_2))\\ &\qquad - f(\tau(x_2)) + \sigma(a_2), s(\tau(x_3)) + \sigma(f(x_3)) - f(\tau(x_3)) + \sigma(a_3)]_L\\ &\quad= s([\tau(x_1),\tau(x_2),\tau(x_3)]_B) + \omega(\tau(x_1),\tau(x_2),\tau(x_3)) + \varrho(\tau(x_1),\tau(x_2))(\sigma(a_3)) + \text{c.p.}\\ &\qquad + \varrho(\tau(x_1),\tau(x_2))(\sigma f(x_3)) + \text{c.p.} - (\varrho(\tau(x_1),\tau(x_2))(f\tau(x_3)) + \text{c.p.})\\ &\qquad + \nu(\tau(x_1))(\sigma(a_2),\sigma(a_3)) + \text{c.p.}\ \text{(by Lemma 3.3)}\\ &\quad= s(\tau([x_1,x_2,x_3]_B)) + \sigma(\omega(x_1,x_2,x_3)) + (\sigma f)([x_1,x_2,x_3]_B) - (f\tau)([x_1,x_2,x_3]_B)\\ &\qquad + \varrho(\tau(x_1),\tau(x_2))(\sigma(a_3)) + \text{c.p.} + \nu(\tau(x_1))(\sigma(a_2),\sigma(a_3)) + \text{c.p.}\ \text{(by (4.21))}\\ &\quad= \gamma(s([x_1,x_2,x_3]_B) + \omega(x_1,x_2,x_3) + \varrho(x_1,x_2)(a_3) + \text{c.p.}\\ &\qquad + \nu(x_1)(a_2,a_3) + \text{c.p.})\ \text{(by (4.19) and } (\sigma,\tau) \in C^L_{B,A})\\ &\quad= \gamma([s(x_1)+a_1, s(x_2)+a_2, s(x_3)+a_3]_L).\end{aligned}$$

Moreover, we show that Ψ is a section of Φ. For any $(\sigma,\tau) \in \operatorname{im}\Phi$, set $\Psi(\sigma,\tau) = \gamma$, where γ is given by (4.19) associated to (σ,τ) and hence γ is bijective. Then for all $x \in B$ and $a \in A$ we have

$$\Phi(\Psi(\sigma,\tau)) = \Phi(\gamma) = (\sigma,\tau)$$

since $\gamma(a) = \sigma(a)$ and $\overline{\gamma}(x) = \pi(\gamma(s(x))) = \pi(s(\tau(x)) + \sigma(f(x)) - f(\tau(x))) = \tau(x)$.

Finally, we show that Ψ is a group homomorphism and hence

$$0 \to Z^0(B,A) \to \mathrm{Aut}_A(L) \xrightarrow{\Phi} \mathrm{im}\,\Phi \to 0$$

is split. For any $(\sigma_1,\tau_1),(\sigma_2,\tau_2) \in \mathrm{im}\,\Phi$ and $x \in B$, $a \in A$, we deduce that

$$\begin{aligned}
&\Psi(\sigma_1,\tau_1)(\Psi(\sigma_2,\tau_2))(s(x)+a)\\
&\quad= \Psi(\sigma_1,\tau_1)(s(\tau_2(x)) + \sigma_2(f(x)) - f(\tau_2(x)) + \sigma_2(a))\\
&\quad= s(\tau_1(\tau_2(x))) + \sigma_1(f(\tau_2(x))) - f(\tau_1(\tau_2(x))) + \sigma_1(\sigma_2(f(x)) - f(\tau_2(x)) + \sigma_2(a))\\
&\quad= s(\tau_1(\tau_2(x))) + \sigma_1(\sigma_2(f(x))) - f(\tau_1(\tau_2(x))) + \sigma_1(\sigma_2(a))\\
&\quad= \Psi(\sigma_1\sigma_2,\tau_1\tau_2)(s(x)+a)
\end{aligned}$$

as required. $\square$

Let $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ be a good abelian extension of 3-Lie algebras. Similarly to the case of Lie algebras (see [2]), we show that there is a left action of $C^L_{B,A}$ on $H^1(B,A)$ such that ϖ is an inner derivation.

Let $(\sigma,\tau) \in C^L_{B,A}$ and $f \in Z^1(B,A)$. Define ${}^{(\sigma,\tau)}f \in \mathrm{Hom}(\bigwedge^3 B, A)$ as

$${}^{(\sigma,\tau)}f(x_1,x_2,x_3) = \sigma(f(\tau^{-1}(x_1),\tau^{-1}(x_2),\tau^{-1}(x_3))), \quad x_1,x_2,x_3 \in B.$$

By the proof of Lemma 4.2 we obtain that ${}^{(\sigma,\tau)}f \in Z^1(B,A)$. On the other hand, if $f \in B^1(B,A)$, i.e., $f = d_\varrho(g)$ for some $g \in \mathrm{Hom}(B,A)$, then by the proof of Corollary 4.3, one has ${}^{(\sigma,\tau)}f = d_\varrho(\sigma f \tau^{-1})$. And hence $[{}^{(\sigma,\tau)}f] \in H^1(B,A)$. Note that this defines a left action of $C^L_{B,A}$ on $H^1(B,A)$ given by

$${}^{(\sigma,\tau)}[f] = [{}^{(\sigma,\tau)}f].$$

Definition 4.2. Keep the notation as above. $\xi\colon C^L_{B,A} \to H^1(B,A)$ is called an inner derivation if $\xi(\sigma,\tau) = [{}^{(\sigma,\tau)}f] - [f]$ for some $[f] \in H^1(B,A)$.

We have:

Proposition 4.2. *Let $0 \to A \hookrightarrow L \xrightarrow{\pi} B \to 0$ be a good abelian extension of 3-Lie algebras. Then ϖ given by* (4.14) *is an inner derivation with respect to the action of $C^L_{B,A}$ on $H^1(B,A)$.*

Proof. For any $(\sigma,\tau) \in C^L_{B,A}$, we deduce

$$\begin{aligned}
\omega_{\sigma,\tau}(x_1,x_2,x_3) &= \sigma\omega(\tau^{-1}(x_1),\tau^{-1}(x_2),\tau^{-1}(x_3)) - \omega(x_1,x_2,x_3)\\
&= {}^{(\sigma,\tau)}\omega(x_1,x_2,x_3) - \omega(x_1,x_2,x_3)
\end{aligned}$$

for all $x_1, x_2, x_3 \in B$. Thus

$$\varpi(\sigma,\tau) = [\omega_{\sigma,\tau}] = [{}^{(\sigma,\tau)}\omega - \omega] = {}^{(\sigma,\tau)}[\omega] - [\omega]$$

as required. $\square$

5. Inducible automorphisms of free nilpotent 3-Lie algebras

In this section we apply the results of the last section to free nilpotent 3-Lie algebras, and give an example to show that the converse of Theorem 4.4 does not hold. Let B be the free 3-Lie algebra of rank n with generators $\{x_1, \ldots, x_n\}$. A free 2-step nilpotent 3-Lie algebra $B_{n,2}$ of rank n is defined by

$$B_{n,2} := B/\langle[[x_i, x_j, x_k], x_l, x_m]\rangle \tag{5.1}$$

for all $x_i, x_j, x_k, x_l, x_m \in \{x_1, \ldots, x_n\}$. Since free 3-Lie algebras of rank $n \leqslant 2$ are abelian, we assume that $n \geqslant 3$.

Let $B_{n,2}^{(1)} = \langle[x_i, x_j, x_k]\colon 1 \leqslant k < j < i \leqslant n\rangle$ be the derived subalgebra of $B_{n,2}$ and $Y = \{y_{ijk}\colon y_{ijk} = [x_i, x_j, x_k] \text{ for } 1 \leqslant k < j < i \leqslant n\}$. Then we have:

Lemma 5.1. *$B_{n,2}^{(1)}$ is an abelian 3-Lie algebra with a basis $Y = \{y_{ijk}\colon 1 \leqslant k < j < i \leqslant n\}$.*

Proof. It suffices to show that $B_{n,2}^{(1)}$ is abelian. For any i_1, i_2, i_3, j_1, j_2, j_3, k_1, k_2, $k_3 \in \{1, \ldots, n\}$, set

$$I = [x_{i_1}, x_{i_2}, x_{i_3}], \quad J = [x_{j_1}, x_{j_2}, x_{j_3}], \quad K = [x_{k_1}, x_{k_2}, x_{k_3}].$$

By (FI) this yields that

$$\begin{aligned}[I, J, x_{k_1}] &= [x_{k_1}, I, J]\\ &= [[x_{k_1}, I, x_{j_1}], x_{j_2}, x_{j_3}] + [x_{j_1}, [x_{k_1}, I, x_{j_2}], x_{j_3}] + [x_{j_1}, x_{j_2}, [x_{k_1}, I, x_{j_3}]]\\ &= 0 \text{ (by 5.1)}.\end{aligned}$$

Since $x_{k_1}, x_{k_2}, x_{k_3}$ are arbitrary, $[I, J, x_{k_1}] = [I, J, x_{k_2}] = [I, J, x_{k_3}] = 0$. Thus we have

$$\begin{aligned}[I, J, K] &= [I, J, [x_{k_1}, x_{k_2}, x_{k_3}]]\\ &= [[I, J, x_{k_1}], x_{k_2}, x_{k_3}] + [x_{k_1}, [I, J, x_{k_2}], x_{k_3}] + [x_{k_1}, x_{k_2}, [I, J, x_{k_3}]] = 0\end{aligned}$$

as required. $\square$

Taking the lexicographic order on the basis Y given by

$$y_{321} < y_{421} < y_{431} < \ldots < y_{n,n-1,n-2},$$

we see that

$$\mathrm{Aut}(B_{n,2}^{(1)}) \cong \mathrm{GL}\Big(\frac{n(n-1)(n-2)}{6}, \mathbb{F}\Big).$$

With respect to the basis Y any $\sigma \in \mathrm{Aut}(B_{n,2}^{(1)})$ can be written as

$$\sigma(y_{ijk}) = \sum_{1\leqslant l_3<l_2<l_1\leqslant n} b_{ijk;l_1l_2l_3} y_{l_1l_2l_3}, \quad 1 \leqslant k < j < i \leqslant n, \tag{5.2}$$

and we denote its matrix by

$$[\sigma] \in \mathrm{GL}\Big(\frac{n(n-1)(n-2)}{6}, \mathbb{F}\Big).$$

Similarly, $B_{n,2}^{ab} := B_{n,2}/B_{n,2}^{(1)} = \langle \overline{x}_1, \ldots, \overline{x}_n\rangle$ is also an abelian 3-Lie algebra of rank n with a basis $\{\overline{x}_1, \ldots, \overline{x}_n\}$, and $\mathrm{Aut}(B_{n,2}^{ab}) \cong \mathrm{GL}(n, \mathbb{F})$. With respect to this basis any $\tau \in \mathrm{Aut}(B_{n,2}^{ab})$ can be written as

$$\tau(\overline{x}_i) = \sum_{j=1}^{n} a_{ij}\overline{x}_j, \quad 1 \leqslant i \leqslant n, \tag{5.3}$$

and we denote its matrix by $[\tau] \in \mathrm{GL}(n, \mathbb{F})$.

By the definition of $B_{n,2}^{(1)}$ and $B_{n,2}^{ab}$, we have the abelian extension of 3-Lie algebras

$$0 \to B_{n,2}^{(1)} \hookrightarrow B_{n,2} \xrightarrow{p} B_{n,2}^{ab} \to 0,$$

where p is the canonical projection. Note that the set

$$\{x_1, x_2, \ldots, x_n, y_{321}, y_{421}, \ldots, y_{n,n-1,n-2}\}$$

is a basis of $B_{n,2}$.

Keep the notation as above. By the proof of Lemma 5.1 we have

$$[B_{n,2}^{(1)}, B_{n,2}^{(1)}, B_{n,2}] = 0,$$

and hence Example 2.1 yields:

Proposition 5.1. $0 \to B_{n,2}^{(1)} \hookrightarrow B_{n,2} \xrightarrow{p} B_{n,2}^{ab} \to 0$ *is a good abelian extension of 3-Lie algebras.*

We will see that this good abelian extension is not split. Since $[B_{n,2}^{(1)}, B_{n,2}^{(1)}, B_{n,2}] = 0$ for any section s of p, by the definition of ϱ, ν given by (1.1), (2.8), respectively, it follows that $\varrho = 0$, $\nu = 0$. So, to derive a criterion for $(\sigma, \tau) \in \mathrm{Aut}(B_{n,2}^{(1)}) \times \mathrm{Aut}(B_{n,2}^{ab})$ to be inducible it remains to compute the map ω given by (2.7) for a section s of p. We begin with:

Lemma 5.2. *For any $s \in \mathrm{Hom}(B_{n,2}^{ab}, B_{n,2})$, s is a section of p if and only if s has the form*

$$s(\overline{x}_i) = x_i + \sum_{1\leqslant l_3<l_2<l_1\leqslant n} d_{i;l_1l_2l_3} y_{l_1l_2l_3}, \quad 1 \leqslant i \leqslant n \tag{5.4}$$

for some $d_{i;l_1l_2l_3} \in \mathbb{F}$.

P r o o f. The "if" part is clear. Conversely, since $\{\overline{x}_1, \ldots, \overline{x}_n\}$ is a basis of $B_{n,2}^{ab}$ and

$$\{x_1, x_2, \ldots, x_n, y_{321}, y_{421}, \ldots, y_{n,n-1,n-2}\}$$

is a basis of $B_{n,2}$, any $s \in \mathrm{Hom}(B_{n,2}^{ab}, B_{n,2})$ has the form

$$s(\overline{x}_i) = \sum_{j=1}^{n} c_{ij} x_j + \sum_{1\leqslant l_3<l_2<l_1\leqslant n} d_{i;l_1l_2l_3} y_{l_1l_2l_3}.$$

So, if s is a section of p then

$$\overline{x}_i = p(s(\overline{x}_i)) = p\Bigl(\sum_{j=1}^{n} c_{ij} x_j + \sum_{1\leqslant l_3<l_2<l_1\leqslant n} d_{i;l_1l_2l_3} y_{l_1l_2l_3}\Bigr) = \sum_{j=1}^{3} c_{ij}\overline{x}_j,$$

from which we deduce that $c_{ij} = \delta_{ij}$, that is,

$$s(\overline{x}_i) = x_i + \sum_{1\leqslant l_3<l_2<l_1\leqslant n} d_{i;l_1l_2l_3} y_{l_1l_2l_3}$$

as required. □

The next result on ω follows essentially by an exterior algebra argument.

Lemma 5.3. *Fix any section $s\colon B_{n,2}^{ab} \to B_{n,2}$ of p. Let $\overline{z}_i = \sum\limits_{r=1}^{n} a_{ir}\overline{x}_r$, $1 \leqslant i \leqslant n$. Then*

$$\omega(\overline{z}_i, \overline{z}_j, \overline{z}_k) = \sum_{1\leqslant l_3<l_2<l_1\leqslant n} \det\begin{bmatrix} a_{il_1} & a_{il_2} & a_{il_3} \\ a_{jl_1} & a_{jl_2} & a_{jl_3} \\ a_{kl_1} & a_{kl_2} & a_{kl_3} \end{bmatrix} y_{l_1l_2l_3} \tag{5.5}$$

for all $1 \leqslant i, j, k \leqslant n$.

P r o o f. By linearity of ω we have

$$\begin{aligned}\omega(\overline{z}_i,\overline{z}_j,\overline{z}_k)&=\omega\Bigg(\sum_{l_1=1}^{n}a_{il_1}\overline{x}_{l_1},\sum_{l_2=1}^{n}a_{jl_2}\overline{x}_{l_2},\sum_{l_3=1}^{n}a_{kl_3}\overline{x}_{l_3}\Bigg)\\&=\sum_{1\leqslant l_1,l_2,l_3\leqslant n}a_{il_1}a_{jl_2}a_{kl_3}\omega(\overline{x}_{l_1},\overline{x}_{l_2},\overline{x}_{l_3}).\end{aligned}$$

Set $I=\{(l_1,l_2,l_3)\colon\ 1\leqslant l_1,l_2,l_3\leqslant n\}$. Then I is the disjoint union of I_i, $1\leqslant i\leqslant 6$, where

$$\begin{aligned}&I_1=\{(l_1,l_2,l_3)\colon\ 1\leqslant l_3<l_2<l_1\leqslant n\},\quad I_2=\{(l_1,l_2,l_3)\colon\ 1\leqslant l_3<l_1<l_2\leqslant n\},\\&I_3=\{(l_1,l_2,l_3)\colon\ 1\leqslant l_2<l_1<l_3\leqslant n\},\quad I_4=\{(l_1,l_2,l_3)\colon\ 1\leqslant l_2<l_3<l_1\leqslant n\},\\&I_5=\{(l_1,l_2,l_3)\colon\ 1\leqslant l_1<l_2<l_3\leqslant n\},\quad I_6=\{(l_1,l_2,l_3)\colon\ 1\leqslant l_1<l_3<l_2\leqslant n\},\end{aligned}$$

and hence $\omega(\overline{z}_i,\overline{z}_j,\overline{z}_k)$ is the sum of six summands, each of which is summed over I_i. We only consider the summand corresponding to I_2, since the other summands can be discussed by similar arguments. Let $l_1'=l_2$, $l_2'=l_1$, $l_3'=l_3$. Then by anti-symmetry of ω one obtains

$$\begin{aligned}\sum_{1\leqslant l_3<l_1<l_2\leqslant n}a_{il_1}a_{jl_2}a_{kl_3}\omega(\overline{x}_{l_1},\overline{x}_{l_2},\overline{x}_{l_3})&=\sum_{1\leqslant l_3'<l_2'<l_1'\leqslant n}a_{il_2'}a_{jl_1'}a_{kl_3'}\omega(\overline{x}_{l_2'},\overline{x}_{l_1'},\overline{x}_{l_3'})\\&=\sum_{1\leqslant l_3<l_2<l_1\leqslant n}a_{il_2}a_{jl_1}a_{kl_3}\omega(\overline{x}_{l_2},\overline{x}_{l_1},\overline{x}_{l_3})\\&=-\sum_{1\leqslant l_3<l_2<l_1\leqslant n}a_{il_2}a_{jl_1}a_{kl_3}\omega(\overline{x}_{l_1},\overline{x}_{l_2},\overline{x}_{l_3}).\end{aligned}$$

By Lemma 5.2 and the definition of Y it follows that

$$[s(\overline{x}_i),s(\overline{x}_j),s(\overline{x}_k)]_{B_{n,2}}=[x_i,x_j,x_k]_{B_{n,2}}=y_{ijk}.$$

Moreover, since $B_{3,2}^{ab}$ is abelian, one has $[\overline{x}_3,\overline{x}_2,\overline{x}_1]_{B_{3,2}^{ab}}=0$. Thus we have

$$\begin{aligned}\omega(\overline{x}_i,\overline{x}_j,\overline{x}_k)&=[s(\overline{x}_i),s(\overline{x}_j),s(\overline{x}_k)]_{B_{n,2}}-s([\overline{x}_3,\overline{x}_2,\overline{x}_1]_{B_{n,2}^{ab}})\\&=[s(\overline{x}_i),s(\overline{x}_j),s(\overline{x}_k)]_{B_{n,2}}=y_{ijk}.\end{aligned}$$

Therefore,

$$\begin{aligned}\omega(\overline{z}_i,\overline{z}_j,\overline{z}_k)=&\sum_{1\leqslant l_3<l_2<l_1\leqslant n}a_{il_1}a_{jl_2}a_{kl_3}y_{l_1l_2l_3}+a_{il_2}a_{jl_3}a_{kl_1}y_{l_1l_2l_3}+a_{il_3}a_{jl_1}a_{kl_2}y_{l_1l_2l_3}\\&\qquad-a_{il_1}a_{jl_3}a_{kl_2}y_{l_1l_2l_3}-a_{il_3}a_{jl_2}a_{kl_1}y_{l_1l_2l_3}-a_{il_2}a_{jl_1}a_{kl_3}y_{l_1l_2l_3}\\=&\sum_{1\leqslant l_3<l_2<l_1\leqslant n}\det\begin{bmatrix}a_{il_1}&a_{il_2}&a_{il_3}\\a_{jl_1}&a_{jl_2}&a_{jl_3}\\a_{kl_1}&a_{kl_2}&a_{kl_3}\end{bmatrix}y_{l_1l_2l_3}\end{aligned}$$

as required. $\square$

By Theorem 4.2 and Lemma 5.3, we are now ready to prove the main result of this section.

Theorem 5.1. *Let* $(\sigma,\tau)\in \mathrm{Aut}(B_{n,2}^{(1)})\times \mathrm{Aut}(B_{n,2}^{ab})$. *Then* (σ,τ) *is inducible if and only if*

$$\det\begin{bmatrix} a_{il_1} & a_{il_2} & a_{il_3}\\ a_{jl_1} & a_{jl_2} & a_{jl_3}\\ a_{kl_1} & a_{kl_2} & a_{kl_3}\end{bmatrix} = b_{ijk;l_1l_2l_3} \tag{5.6}$$

for all $1\leqslant k<j<i\leqslant n$ *and* $1\leqslant l_3<l_2<l_1\leqslant n$, *where* $[\sigma]=(b_{i_1i_2i_3;j_1j_2j_3})$ *and* $[\tau]=(a_{ij})$ *are matrices of* σ *and* τ, *respectively.*

P r o o f. We may use the section $s\colon B_{n,2}^{ab}\to B_{n,2}$ of π given by $s(\overline{x}_i)=x_i$. Then we have $\omega(\overline{x}_i,\overline{x}_j,\overline{x}_k)=y_{ijk}$. By Theorem 4.2, (σ,τ) is inducible if and only if

$$\sigma(\omega(\overline{x}_i,\overline{x}_j,\overline{x}_k))=\omega(\tau(\overline{x}_i),\tau(\overline{x}_j),\tau(\overline{x}_k)) \tag{5.7}$$

for all $1\leqslant i,j,k\leqslant n$. We calculate the LHS and the RHS of (5.7) to deduce (5.6).

For any $1\leqslant i,j,k\leqslant n$, by (5.2) we have

$$\sigma(\omega(\overline{x}_i,\overline{x}_j,\overline{x}_k))=\sigma(y_{ijk})=\sum_{1\leqslant l_3<l_2<l_1\leqslant n} b_{ijk;l_1l_2l_3}y_{l_1l_2l_3}, \tag{5.8}$$

and

$$\begin{aligned}\omega(\tau(\overline{x}_i),\tau(\overline{x}_j),\tau(\overline{x}_k)) &= \omega\Bigl(\sum_{l_1=1}^{n} a_{il_1}\overline{x}_{l_1},\sum_{l_2=1}^{n} a_{jl_2}\overline{x}_{l_2},\sum_{l_3=1}^{n} a_{kl_3}\overline{x}_{l_3}\Bigr) \ \text{(by (5.3))}\\ &= \sum_{1\leqslant l_3<l_2<l_1\leqslant n}\det\begin{bmatrix} a_{il_1} & a_{il_2} & a_{il_3}\\ a_{jl_1} & a_{jl_2} & a_{jl_3}\\ a_{kl_1} & a_{kl_2} & a_{kl_3}\end{bmatrix} y_{l_1l_2l_3}\ \text{(by Lemma 5.3).}\end{aligned} \tag{5.9}$$

By comparing (5.8) and (5.9) we obtain that

$$\det\begin{bmatrix} a_{il_1} & a_{il_2} & a_{il_3}\\ a_{jl_1} & a_{jl_2} & a_{jl_3}\\ a_{kl_1} & a_{kl_2} & a_{kl_3}\end{bmatrix} = b_{ijk;l_1l_2l_3} \tag{5.10}$$

holds for all $1\leqslant k<j<i\leqslant n$ and $1\leqslant l_3<l_2<l_1\leqslant n$. □

In the remaining part we consider the case $n=3$. Note that $B_{3,2}^{(1)}$ is 1-dimensional and any automorphism of $B_{3,2}^{(1)}$ is just a nonzero element t in $\mathbb{F}$. By Theorem 5.1 we have:

Corollary 5.1. *Let* $(t,\tau)\in \mathrm{Aut}(B_{3,2}^{(1)})\times \mathrm{Aut}(B_{3,2}^{ab})$. *Then* (t,τ) *is inducible if and only if* $t=\det[\tau]$.

We also have:

Proposition 5.2. *The good abelian extension of 3-Lie algebras* $0 \to B_{3,2}^{(1)} \hookrightarrow B_{3,2} \xrightarrow{p} B_{3,2}^{ab} \to 0$ *is not split.*

Proof. Fix any section $s'\colon B_{3,2}^{ab} \to B_{3,2}$. Then by Lemma 5.2, $s'(\overline{x}_i) = x_i + k_i y_{321}$ for $1 \leqslant i \leqslant 3$ and some $k_i \in \mathbb{F}$. It follows that

$$\begin{aligned}\omega(\overline{x}_3, \overline{x}_2, \overline{x}_1) &= [s'(\overline{x}_3), s'(\overline{x}_2), s'(\overline{x}_1)]_{B_{3,2}} - s'([\overline{x}_3, \overline{x}_2, \overline{x}_1]_{B_{3,2}^{ab}}) \\ &= [x_3 + k_3 y_{321}, x_2 + k_2 y_{321}, x_1 + k_1 y_{321}]_{B_{3,2}} - 0 \\ &= [x_3, x_2, x_1]_{B_{3,2}} = y_{321} \neq 0,\end{aligned}$$

which means that s' is not a 3-Lie algebra homomorphism, and hence the above abelian extension is not split. $\square$

For the good abelian extension $0 \to B_{3,2}^{(1)} \hookrightarrow B_{3,2} \xrightarrow{p} B_{3,2}^{ab} \to 0$ of 3-Lie algebras, the short exact sequence given by (4.18) becomes

$$0 \to Z^0(B_{3,2}^{ab}, B_{3,2}^{(1)}) \to \mathrm{Aut}_{B_{3,2}^{(1)}}(B_{3,2}) \xrightarrow{\Phi} \mathrm{im}\,\Phi \to 0. \tag{5.11}$$

Proposition 5.3. *The short exact sequence* (5.11) *is split.*

Proof. Let $s\colon B_{3,2}^{ab} \to B_{3,2}$ be the section of p given by $s(\overline{x}_i) = x_i$. Define $\Psi\colon \mathrm{im}\,\Phi \to \mathrm{End}(B_{3,2})$ as

$$\Psi(\sigma, \tau) = \gamma\colon \begin{cases} x_1 \mapsto s(\tau(\overline{x}_1)) \\ x_2 \mapsto s(\tau(\overline{x}_2)) \\ x_3 \mapsto s(\tau(\overline{x}_3)) \\ y_{321} \mapsto \sigma(y_{321}) := t y_{321}. \end{cases} \tag{5.12}$$

First we show that $\mathrm{im}\,\Psi \subseteq \mathrm{Aut}_{B_{3,2}^{(1)}}(B_{3,2})$. Then we show Ψ is a section of Φ and is also a group homomorphism, which implies that (5.11) is split.

Note that $t \neq 0$ and we obtain $\gamma(B_{3,2}^{(1)}) = B_{3,2}^{(1)}$ since $B_{3,2}^{(1)}$ is 1-dimensional spanned by y_{321}. We show that $\gamma \in \mathrm{Aut}_{B_{3,2}^{(1)}}(B_{3,2})$ as follows.

Suppose that $\gamma\Big(\sum\limits_{j=1}^{3} c_{ij} x_j + k_i y_{321}\Big) = 0$, i.e., $\sum\limits_{j=1}^{3} c_{ij} s(\tau(\overline{x}_j)) + k_i t y_{321} = 0$. Set $\tau(\overline{x}_j) = \sum\limits_{l=1}^{3} a_{jl} \overline{x}_l$ for $j = 1, 2, 3$. Then $\sum\limits_{j,l=1}^{3} c_{ij} a_{jl} x_l + k_i t y_{321} = 0$. Since $[\tau] = (a_{jl}) \in \mathrm{GL}(3, \mathbb{F})$, we obtain $c_{ij} = 0$ and $k_i = 0$, which implies that γ is injective. To show γ is surjective, let $\sum\limits_{j=1}^{3} c_{ij} x_j + k_i y_{321} \in B_{3,2}$. Define

$$(d_{i1}, d_{i2}, d_{i3}) := (c_{i1}, c_{i2}, c_{i3})[\tau]^{-1} \quad \text{and} \quad r_i := k_i / t.$$

Then we have

$$\begin{aligned}
\gamma\Big(\sum_{j=1}^{3} d_{ij}x_j + r_i y_{321}\Big) &= \sum_{j,l=1}^{3} d_{ij}a_{jl}x_l + r_i t y_{321}\\
&= (d_{i1}, d_{i2}, d_{i3})[\tau][x_1, x_2, x_3]^T + k_i y_{321}\\
&= (c_{i1}, c_{i2}, c_{i3})[\tau]^{-1}[\tau][x_1, x_2, x_3]^T + k_i y_{321}\\
&= \sum_{j=1}^{3} c_{ij}x_j + k_i y_{321}.
\end{aligned}$$

Therefore, γ is bijective. We show that γ is a 3-Lie algebra homomorphism. Indeed,

$$\begin{aligned}
[\gamma(x_3), \gamma(x_2), \gamma(x_1)]_{B_{3,2}} &= [s(\tau(\overline{x}_3)), s(\tau(\overline{x}_2)), s(\tau(\overline{x}_1))]_{B_{3,2}}\\
&= \Big[s\Big(\sum_{l_1=1}^{3} a_{3l_1}\overline{x}_{l_1}\Big), s\Big(\sum_{l_2=1}^{3} a_{2l_2}\overline{x}_{l_2}\Big), s\Big(\sum_{l_3=1}^{3} a_{1l_3}\overline{x}_{l_3}\Big)\Big]_{B_{3,2}}\\
&= \omega\Big(\sum_{l_1=1}^{3} a_{3,l_1}\overline{x}_{l_1}, \sum_{l_2=1}^{3} a_{2,l_2}\overline{x}_{l_2}, \sum_{l_3=1}^{3} a_{1,l_3}\overline{x}_{l_3}\Big)\\
&= \det\begin{bmatrix} a_{33} & a_{32} & a_{31}\\ a_{23} & a_{22} & a_{21}\\ a_{13} & a_{12} & a_{11}\end{bmatrix} y_{321} \text{ (by Lemma 5.3)}\\
&= \det\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} y_{321} = t y_{321} \text{ (by Corollary 5.1)}\\
&= \gamma([x_3, x_2, x_1]_{B_{3,2}}).
\end{aligned}$$

Then we have $\gamma \in \mathrm{Aut}_{B_{3,2}^{(1)}}(B_{3,2})$. By the definition of γ, we obtain $\gamma(y_{321}) = \sigma(y_{321})$. And for $i = 1, 2, 3$, we deduce that

$$\overline{\gamma}(\overline{x_i}) = p(\gamma(s(\overline{x}_i))) = p(\gamma(x_i)) = p(s(\tau(\overline{x}_i)) = \tau(\overline{x}_i),$$

therefore, $\Phi(\gamma) = (\sigma, \tau)$. Then for any $(\sigma, \tau) \in \operatorname{im}\Phi$, one obtains

$$\Phi(\Psi(\sigma, \tau)) = \Phi(\gamma) = (\sigma, \tau),$$

which implies that Ψ given by (5.12) is a section of Φ. Finally, we show that Ψ is a group homomorphism. For any $(\sigma_1, \tau_1), (\sigma_2, \tau_2) \in \operatorname{im}\Phi$, set $[\sigma_1] = t_1$, $[\sigma_2] = t_2$ and

$[\tau_1] = (a_{ij})$, $[\tau_2] = (a'_{ij})$, respectively. Then for any $k = 1, 2, 3$ we have

$$\begin{aligned}\Psi(\sigma_1,\tau_1)\Psi(\sigma_2,\tau_2)(x_k) &= \Psi(\sigma_1,\tau_1)(s(\tau_2(\overline{x}_k))) = \Psi(\sigma_1,\tau_1)\biggl(s\Bigl(\sum_{l=1}^{3} a'_{kl}\overline{x}_l\Bigr)\biggr)\\ &= \Psi(\sigma_1,\tau_1)\Bigl(\sum_{l=1}^{3} a'_{kl}x_l\Bigr) = s(\tau_1(\sum_{l=1}^{3} a'_{kl}\overline{x}_l))\\ &= s\Bigl(\sum_{l,m=1}^{3} a_{lm}a'_{kl}\overline{x}_m\Bigr) = \sum_{l,m=1}^{3} a_{lm}a'_{kl}x_m = s(\tau_1(\tau_2(\overline{x}_k)))\\ &= \Psi(\sigma_1\sigma_2,\tau_1\tau_2)(x_k),\end{aligned}$$

and we also have

$$\begin{aligned}\Psi(\sigma_1,\tau_1)\Psi(\sigma_2,\tau_2)(y_{321}) &= \Psi(\sigma_1,\tau_1)(\sigma_2(y_{321})) = \sigma_1(t_2y_{321})\\ &= t_1t_2y_{321} = \sigma_1(\sigma_2(y_{321})) = \Psi(\sigma_1\sigma_2,\tau_1\tau_2)(y_{321}).\end{aligned}$$

Since $\{x_1, x_2, x_3, y_{321}\}$ is a basis of $B_{3,2}$, we obtain that

$$\Psi(\sigma_1,\tau_1)\Psi(\sigma_2,\tau_2) = \Psi(\sigma_1\sigma_2,\tau_1\tau_2)$$

as required. □

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