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A NOTE ON THE DOUBLE ROMAN DOMINATION NUMBER OF GRAPHS

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Abstract. For a graph $G = (V, E)$, a double Roman dominating function is a function $f\colon V \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two neighbors assigned 2 under f or one neighbor with $f(w) = 3$, and if $f(v) = 1$, then the vertex v must have at least one neighbor with $f(w) \geqslant 2$. The weight of a double Roman dominating function f is the sum $f(V) = \sum_{v\in V} f(v)$. The minimum weight of a double Roman dominating function on G is called the double Roman domination number of G and is denoted by $\gamma_{\rm dR}(G)$. In this paper, we establish a new upper bound on the double Roman domination number of graphs. We prove that every connected graph G with minimum degree at least two and $G \neq C_5$ satisfies the inequality $\gamma_{\rm dR}(G) \leqslant \lfloor \frac{13}{11} n \rfloor$. One open question posed by R. A. Beeler et al. has been settled.

Keywords: double Roman domination number; domination number; minimum degree

MSC 2010: 05C69, 05C35

1. Introduction

Graph theory terminology not presented here can be found in [2]. Let $G = (V, E)$ be a graph with $|V| = n$. The degree, neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $d_G(v)$, $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. If the graph G is clear from context, we simply write $d(v)$, $N(v)$ and $N[v]$, respectively. The minimum degree and the maximum degree of the graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The graph induced by $S \subseteq V$ is denoted by $G[S]$. A *cycle* on n vertices is denoted by C_n.

A set $S \subseteq V$ in a graph G is called a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. A dominating set of G with cardinality $\gamma(G)$ is called a γ-set of G.

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Let $f\colon V \to \{0,1,2\}$ be a function having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbor $u \in N(v)$ with $f(u) = 2$. Such a function is called a *Roman dominating function*. The weight of a Roman dominating function is given by the sum $f(V) = \sum\limits_{v \in V} f(v)$. The minimum weight of a Roman dominating function on G is called the *Roman domination number* of G and is denoted $\gamma_R(G)$. Roman domination was defined and discussed by Stewart in [8]. It was developed by ReVelle and Rosing in [7] and Cockayne et al. in [3].

The original study of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great. In order to provide a level of defense that is both stronger and more flexible at a cheaper cost, Beeler et al. in [2] initiated the study of double Roman domination.

A function $f\colon V \to \{0,1,2,3\}$ is a *double Roman dominating function* on a graph G if the following conditions are met. Let V_i denote the set of vertices assigned i by the function f.

(i) If $f(v) = 0$, then the vertex v must have at least two neighbors in V_2 or one neighbor in V_3.

(ii) If $f(v) = 1$, then the vertex v must have at least one neighbor in $V_2 \cup V_3$.

The *double Roman domination number* $\gamma_{\mathrm{dR}}(G)$ equals the minimum weight of a double Roman dominating function on G, and a double Roman dominating function of G with weight $\gamma_{\mathrm{dR}}(G)$ is called a γ_{dR}-function of G.

Beeler et al. in [2] showed the relationship between domination and double Roman domination as follows.

Proposition 1.1 ([2])**.** *For any graph G, $2\gamma(G) \leqslant \gamma_{\mathrm{dR}}(G) \leqslant 3\gamma(G)$.*

A theorem of McQuaig and Shepherd in [4] proves that with the exception of seven graphs, every connected graph G having minimum degree at least two satisfies, $\gamma(G) \leqslant \frac{2}{5}n$. Beeler et al. in [2] posed the following open question.

Question 1.2 ([2])**.** *With the exception of seven graphs, every connected graph G having minimum degree at least two satisfies $\gamma_{\mathrm{dR}}(G) \leqslant \frac{6}{5}n$. Can this bound be improved?*

Similarly, a theorem of Reed in [6] proves that every connected graph G having minimum degree at least three satisfies the inequality $\gamma(G) \leqslant \frac{3}{8}n$. Beeler et al. in [2] posed the following open question.

Question 1.3 ([2])**.** *Every connected graph G having minimum degree at least three satisfies the inequality $\gamma_{\mathrm{dR}}(G) \leqslant \frac{9}{8}n$. Can this bound be improved?*

Ahangar Abdollahzadeh et al. in [1] gave the affirmative answer to Question 1.3. They proved that every connected graph G having minimum degree at least three satisfies the inequality $\gamma_{\rm dR}(G) \leqslant n$.

In this paper, we establish a new upper bound on the double Roman domination number of graphs. We prove that every connected graph G with minimum degree at least two and $G \neq C_5$ satisfies the inequality $\gamma_{\rm dR}(G) \leqslant \lfloor \frac{13}{11} n \rfloor$. Question 1.2 has been settled.

2. Main results

A *cover of vertex disjoint paths* of G, or simply a vdp-cover, is a set of vertex disjoint paths $P_1, \ldots, P_k$ such that $V(G) = V(P_1) \cup \ldots \cup V(P_k)$. A path P is called a 0-, 1- or 2-path if $|V(P)|$ is congruent to $0, 1$ or 2 mod 3, respectively. For a vdp-cover S of G, let S_i $(i = 0, 1, 2)$ be the set of i-paths in S. If $P = P'xP''$, where P' is an i-path and P'' is a j-path (and x is on neither of those paths), then we say x is an (i, j)-vertex of P. Let $P \in S$ and x be an endpoint of P. We say that x is an *outendpoint* if it has a neighbor which is not on P. If P is a 2-path, we say that x is a $(2, 2)$-*endpoint* if it is not an outendpoint and is adjacent to some $(2, 2)$-vertex of P.

From now on, let G be a graph on n vertices with $\delta(G) \geqslant 2$. We may assume that G is connected (for otherwise we apply the result to each component of the graph). As in [6], choose a vdp-cover S of G such that

(1) $2|S_1| + |S_2|$ is minimized.
(2) Subject to (1), $|S_2|$ is minimized.
(3) Subject to (2), $\sum\limits_{P_i \in S_0} |V(P_i)|$ is minimized.
(4) Subject to (3), $\sum\limits_{P_i \in S_1} |V(P_i)|$ is minimized.

By the virtue of (1)–(4), the following assertion holds (for the proof, see [6], Observations 1–3).

Assertion 2.1. *Let x be an outendpoint of $P_i \in S_1 \cup S_2$, y a neighbor of x on some path P_j distinct from P_i. Let $P_j = P_j' y P_j''$. Then the following hold.*

(1) *P_j is not a 1-path.*
(2) *If P_j is a 0-path, then both P_j' and P_j'' are 1-paths.*
(3) *If P_j is a 2-path, then both P_j' and P_j'' are 2-paths.*

Having chosen the minimal vdp-cover $S = \{P_1, P_2, \ldots, P_k\}$, as in [6], rearrange the paths of S to obtain a new vdp-cover $S' = \{P_1', P_2', \ldots, P_k'\}$ such that P_i' is a Hamiltonian path on $V(P_i)$, and so as to maximize the number of outendpoints, and subject to this maximize the number of $(2,2)$-endpoints. Let $S_i^{'}$ be the set of i-paths in $S^{'}$ for $0 \leqslant i \leqslant 2$. Since $|V(P_i')| = |V(P_i)|$ for $1 \leqslant i \leqslant k$, it follows that $|S_i'| = |S_i|$ for $0 \leqslant i \leqslant 2$. Hence, S' is still minimal with respect to the above four conditions and Assertion 2.1 is still valid for the rearranged paths in S'. For convenience sake, we still denote by S the new vdp-cover of G.

For each 1-path P in S which has an outendpoint, choose some vertex $y \notin V(P)$ which is adjacent to an endpoint of P and call y the *acceptor* for P. For each 2-path P in S which has two outendpoints, for each of these endpoints choose a vertex of $G - V(P)$ which is adjacent to it and designate it as the *acceptor* corresponding to that endpoint. Call a path in S *accepting* if it contains an acceptor. In addition, for any $(2,2)$-endpoint x of any path P, choose a $(2,2)$-vertex y of P which is adjacent to x and designate it as an *inacceptor* for x.

For any accepting 2-path P, a partition $P = P_1P_2P_3$ such that both P_1 and P_3 are 1-paths which contain neither acceptors nor inacceptors, and are maximal with this property. We say that P_1 and P_3 are *tips* of P and P_2 is its *central path.* By the maximality of P_1, P_3 and Assertion 2.1, if $x \in P_2$ is adjacent in P_2 to an endpoint of P_2, then it is an acceptor or inacceptor.

Let E denote the set of such tips P_1 of an accepting 2-path P, which is in E if and only if the corresponding endpoint of P is neither an outendpoint nor a $(2,2)$-endpoint and we can not dominate P_1 using $\lfloor \frac{1}{3}|V(P_1)| \rfloor$.

Let W be the set of $(2,2)$-endpoints of accepting 2-paths for which we have chosen an inacceptor.

To any element T of E there corresponds an accepting 2-path P_T such that T is a tip of P_T. Define E' by saying that for each $T \in E$, T is in E' if the endpoint of P_T not in T is not an element of W. The following lemma was proved by Reed (for the proof, see [6], page 285, Fact 11.6).

Lemma 2.2 ([5]). *Let $T = a_1 \ldots a_k \in E'$. Let P be the accepted 2-path containing T and let $C = c_0 \ldots c_l$ be the central path of P. Assume that c_0 is adjacent to a_k on the path P. Then a_1 is adjacent only to the vertices of $V(T) \cup \{c_0\}$.*

Proposition 2.3 ([2]). *In a double Roman dominating function of weight $\gamma_{\mathrm{dR}}(G)$, no vertex needs to be assigned value* 1.

By Proposition 2.3, when determining the value $\gamma_{\mathrm{dR}}(G)$ for any graph G, we can assume that $V_1 = \emptyset$ for all double Roman dominating functions under consideration.

Lemma 2.4. $\gamma_{\mathrm{dR}}(C_4) = 4$, $\gamma_{\mathrm{dR}}(C_5) = 6$.

Theorem 2.5. *Let G be a connected graph with order n and minimum degree at least two. If $G \neq C_5$, then $\gamma_{\mathrm{dR}}(G) \leqslant \lfloor \frac{13}{11} n \rfloor$.*

P r o o f. Let S be the minimal vdp-cover of G. Then $S = S_0 \cup S_1 \cup S_2$. For any path $P \in S$, let G_P denote the subgraph induced by $V(P)$.

Claim 2.6. *For each* 0*-path* $P \in S_0$, $\gamma_{\rm dR}(G_P) \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$.

P r o o f. Let $D = \{x \in V(P) | x$ is a $(1,1)$-vertex of $P\}$. Then D is a dominating set of P. Let f_0^P be a function assigning 3 to every vertex in D and 0 to all other vertices in $V(P) \setminus D$. It is obvious that f_0^P is a double Roman dominating function of G_P. Hence, $\gamma_{\rm dR}(G_P) \leqslant 3|D| = 3|V(P)|/3 = |V(P)| \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$. □

Claim 2.7. *For each* 1*-path* $P \in S_1$ *with* $|V(P)| \geqslant 7$, $\gamma_{\rm dR}(G_P) \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$.

P r o o f. Assume that $P = a_1 a_2 \ldots a_{3k+1}$. Then $k \geqslant 2$. Let $D = \{a_{3i}\colon i = 1, 2, \ldots, k\}$. Let f_{11}^P be a function assigning 3 to every vertex in D, 2 to a_1 and 0 to all other vertices in $V(P) \setminus (D \cup \{a_1\})$. It is obvious that f_{11}^P is a double Roman dominating function of G_P. Hence, $\gamma_{\rm dR}(G_P) \leqslant 3|D| + 2 = 3k + 2 = |V(P)| + 1 \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$. □

Claim 2.8. *Let* $P = a_1 a_2 a_3 a_4$ *be a path in* S_1. *If* a_1 *is an outendpoint, then* $\gamma_{\rm dR}(G[V(P) \setminus \{a_1\}]) \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$.

P r o o f. Let f_{12}^P be a function assigning 3 to vertex a_3 and 0 to all other vertices in $V(P) \setminus \{a_1, a_3\}$. It is obvious that f_{12}^P is a double Roman dominating function of $G[V(P) \setminus \{a_1\}]$. Hence, $\gamma_{\rm dR}(G[V(P) \setminus \{a_1\}]) \leqslant 3 < \lfloor \frac{13}{11}|V(P)| \rfloor$. □

Claim 2.9. *Let* $P = a_1 a_2 a_3 a_4$ *be a* 1*-path with no outendpoint. Then* $\gamma_{\rm dR}(G_P) \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$.

P r o o f. If $a_1 a_3 \in E(G)$, then let f_{13}^P be a function assigning 3 to vertex a_3 and 0 to all other vertices in $V(P) \setminus \{a_3\}$. It is obvious that f_{13}^P is a double Roman dominating function of G_P. Hence, $\gamma_{\rm dR}(G_P) \leqslant 3 < \lfloor \frac{13}{11}|V(P)| \rfloor$. We may assume that $a_1 a_3 \notin E(G)$. Since $\delta(G) \geqslant 2$, $a_1 a_4 \in E(G)$. Hence C_4 is a spanning subgraph of G_P. By Lemma 2.4, $\gamma_{\rm dR}(G_P) \leqslant \gamma_{\rm dR}(C_4) = 4 \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$. □

Claim 2.10. *For each* 2*-path* $P \in S_2$ *with* $|V(P)| \geqslant 11$, $\gamma_{\rm dR}(G_P) \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$.

P r o o f. Assume that $P = a_1 a_2 \ldots a_{3k+2}$. Then $k \geqslant 3$. Let $D = \{x \in V(P) | x$ is an $(2,2)$-vertex of $P\}$. Let f_{21}^P be a function assigning 3 to every vertex in D, 2 to every vertex in $\{a_1, a_{3k+2}\}$, and 0 to all other vertices in $V(P) \setminus (D \cup \{a_1, a_{3k+2}\})$. It is obvious that f_{21}^P is a double Roman dominating function of G_P. Hence, $\gamma_{\rm dR}(G_P) \leqslant 3|D| + 4 = 3k + 4 = |V(P)| + 2 \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$. □

Claim 2.11. *Let $P = a_1a_2\dots a_{3k+2}$ be a path in S_2 with $0 \leqslant k \leqslant 2$. If a_1 is an outendpoint or a $(2,2)$-endpoint, then $\gamma_{\rm dR}(G[V(P) \setminus \{a_1\}]) \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$.*

Proof. Let $D = \{x \in V(P) | x \text{ is a } (2,2)\text{-vertex of } P\}$. Let f_{22}^P be a function assigning 3 to every vertex in D, 2 to vertex a_{3k+2} and 0 to all other vertices in $V(P) \setminus (D \cup \{a_1, a_{3k+2}\})$. It is obvious that f_{22}^P is a double Roman dominating function of $G[V(P) \setminus \{a_1\}]$. Hence, $\gamma_{\rm dR}(G[V(P) \setminus \{a_1\}]) \leqslant 3|D| + 2 = 3k + 2 = |V(P)| \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$. □

Claim 2.12. *Let $P = a_1a_2\dots a_{3k+2}$ be an accepting 2-path which has neither an outendpoint nor a $(2,2)$-endpoint. Then $k \geqslant 3$.*

Proof. Since a_1 has degree at least two in G and a_1 is neither an outendpoint nor a $(2,2)$-endpoint, it has at least two neighbors in $V(P)$. By Lemma 2.2, a_3 is not an acceptor. Similarly, a_{3k} is not an acceptor. Hence, $k \geqslant 3$. □

By Claim 2.12, if a path $P \in S_2$ with $|V(P)| \in \{5, 8\}$ has neither an outendpoint nor a $(2,2)$-endpoint, then the path P is a nonaccepting 2-path.

Claim 2.13. *Let $P = a_1a_2a_3a_4a_5$ be a nonaccepting 2-path which has neither an outendpoint nor a $(2,2)$-endpoint. Then $\gamma_{\rm dR}(G_P) \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$.*

Proof. If $a_1a_3 \in E(G)$, then let f_{23}^P be a function assigning 3 to vertex a_3, 2 to vertex a_5 and 0 to all other vertices in $V(P)\setminus\{a_3, a_5\}$. It is obvious that f_{23}^P is a double Roman dominating function of G_P. Hence, $\gamma_{\rm dR}(G_P) \leqslant 5 \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$. We may assume that $a_1a_3 \notin E(G)$. If $a_1a_4 \in E(G)$, then let f_{23}^P be a function assigning 3 to vertex a_4, 2 to vertex a_2 and 0 to all other vertices in $V(P) \setminus \{a_2, a_4\}$. It is obvious that f_{23}^P is a double Roman dominating function of G_P. Hence, $\gamma_{\rm dR}(G_P) \leqslant 5 \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$. We may assume that $a_1a_3 \notin E(G)$ and $a_1a_4 \notin E(G)$. Since $\delta(G) \geqslant 2$, $a_1a_5 \in E(G)$. Then, the subgraph induced by $V(P)$ has a hamiltonian cycle. As we choose S so as to maximize the number of the outendpoints, $|V(G)| = |V(P)| = 5$. Since $G \neq C_5$, $\{a_2a_4, a_2a_5, a_3a_5\} \cap E(G) \neq \emptyset$. In a way similar to the above, it follows that $\gamma_{\rm dR}(G_P) \leqslant 5 \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$. □

Claim 2.14. *Let $P = a_1a_2\dots a_8$ be a nonaccepting 2-path which has neither an outendpoint nor a $(2,2)$-endpoint. Then $\gamma_{\rm dR}(G_P) \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$.*

Proof. It is obvious that $\gamma(G_P) \leqslant 3$. Let D be a γ-set of G_P. Let f_{24}^P be a function assigning 3 to all vertices in D and 0 to all other vertices in $V(P) \setminus D$. It is obvious that f_{24}^P is a double Roman dominating function of G_P. Hence, $\gamma_{\rm dR}(G_P) \leqslant 3|D| = 9 \leqslant \lfloor \frac{13}{11}|V(P)| \rfloor$. □

Now, we define a double Roman dominating function f of G as follows: Let P be a path in S.

(1) If $P \in S_0$, then let $f = f_0^P$.

(2) Suppose that $P \in S_1$. If $|V(P)| \geqslant 7$, then let $f = f_{11}^P$. If $|V(P)| = 4$ and P has an outendpoint, then let $f = f_{12}^P$. If $|V(P)| = 4$ and P has no outendpoint, then let $f = f_{13}^P$.

(3) Suppose that $P \in S_2$. If $|V(P)| \geqslant 11$, then let $f = f_{21}^P$. If $|V(P)| = 2, 5, 8$ and P has an outendpoint or a $(2,2)$-endpoint, then let $f = f_{22}^P$. If P is a nonaccepting 2-path with $|V(P)| = 5$ and P has neither an outendpoint nor a $(2,2)$-endpoint, then let $f = f_{23}^P$. If P is a nonaccepting 2-path with $|V(P)| = 8$ and P has neither an outendpoint nor a $(2,2)$-endpoint, then let $f = f_{24}^P$.

(4) For any outendpoint or $(2,2)$-endpoint v, define $f(v) = 0$.

For any $(1,1)$-vertex v, it follows that $f(v) = 3$ by Claim 2.6. For any $(2,2)$-vertex v, if v belongs to an accepting path, it follows that $f(v) = 3$ by Claims 2.10, 2.11 and 2.12. Hence, f assigns 3 to every acceptor. By Claims 2.10 and 2.11, f assigns 3 to every inacceptor. So any outendpoint or $(2,2)$-endpoint is adjacent to a vertex w with $f(w) = 3$. Since $\delta(G) \geqslant 2$, if $P \in S_1$ with $|V(P)| = 1$, say $V(P) = \{v\}$, then v is an outendpoint. By Claims 2.6–2.14, f is a double Roman dominating function of G. Hence, $\gamma_{\rm dR}(G) \leqslant \sum\limits_{P\in S} \gamma_{\rm dR}(G_P) \leqslant \sum\limits_{P\in S} \lfloor \frac{13}{11}|V(P)| \rfloor \leqslant \lfloor \frac{13}{11} n \rfloor$. $\square$

Remark 2.15. Let $C_5 = v_1v_2v_3v_4v_5v_1$. Let H be the graph obtained from C_5 by adding an edge v_2v_5. It is obvious that if $G \in \{C_3, C_4, H\}$, then $\gamma_{\rm dR}(G) = \lfloor \frac{13}{11} n \rfloor$. Hence, the upper bound in Theorem 2.5 is tight.

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