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GENERALIZED SYMMETRY CLASSES OF TENSORS

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Abstract. Let V be a unitary space. For an arbitrary subgroup G of the full symmetric group S_m and an arbitrary irreducible unitary representation Λ of G, we study the generalized symmetry class of tensors over V associated with G and Λ. Some important properties of this vector space are investigated.

Keywords: irreducible character; generalized Schur function; orthogonal basis; symmetry class of tensors

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1. Introduction

Let S_m be the full symmetric group of degree m and G a subgroup of S_m. Let U be a unitary space and $\mathrm{End}(U)$ the set of all linear operators on U. Denote by $\mathbb{C}_{m\times m}$ the set of all $m\times m$ complex matrices. Suppose Λ is an irreducible unitary representation of G over U. The generalized Schur function $D_\Lambda\colon \mathbb{C}_{m\times m} \to \mathrm{End}(U)$ is defined by

$$D_\Lambda(A) = \sum_{\sigma\in G} \Lambda(\sigma) \prod_{i=1}^{m} a_{i\sigma(i)}$$

for $A = (a_{ij})_{m\times m} \in \mathbb{C}_{m\times m}$.

Let V be a unitary space of dimension n and denote by $V^{\otimes m}$ the mth tensor power of V. Then $U\otimes V^{\otimes m}$ is a unitary space with induced inner product that satisfies

$$(u\otimes x^{\otimes}, v\otimes y^{\otimes}) = (u,v)\prod_{i=1}^{m}(x_i,y_i),$$

where $u,v\in U$ and $x^{\otimes} = x_1\otimes\ldots\otimes x_m$, $y^{\otimes} = y_1\otimes\ldots\otimes y_m \in V^{\otimes m}$.

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For any $\sigma \in G$ there is a unique permutation operator

$$P(\sigma)\colon\ V^{\otimes m} \to V^{\otimes m}$$

satisfying $P(\sigma^{-1})(v^{\otimes}) = v^{\otimes}_{\sigma}$, where $v^{\otimes}_{\sigma} = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \ldots \otimes v_{\sigma(m)}$. The permutation operator yields a representation of G, i.e. $P\colon G \to GL(V^{\otimes m})$. It is well known that if $\dim V \geqslant 2$, then P is a faithful unitary reducible representation of G and $\operatorname{Tr} P(\sigma) = n^{c(\sigma)}$, where $c(\sigma)$ is the number of factors in the disjoint cycle factorization of σ, see [9].

The generalized symmetrizer associated with G and Λ is defined by

$$S_{\Lambda} = \frac{1}{|G|} \sum_{\sigma \in G} \Lambda(\sigma) \otimes P(\sigma) \in \operatorname{End}(U \otimes V^{\otimes m}).$$

In the following theorem we show that S_{Λ} is an orthogonal projection on $U \otimes V^{\otimes m}$.

Theorem 1.1. *Suppose Λ is an irreducible unitary representation of G over unitary space U. Then S_{Λ} is an orthogonal projection on $U \otimes V^{\otimes m}$.*

Proof. We first prove that S_{Λ} is Hermitian. We have

$$S_{\Lambda}^{*} = \Big(\frac{1}{|G|} \sum_{\sigma \in G} \Lambda(\sigma) \otimes P(\sigma)\Big)^{*} = \frac{1}{|G|} \sum_{\sigma \in G} \Lambda(\sigma)^{*} \otimes P(\sigma)^{*}$$
$$= \frac{1}{|G|} \sum_{\sigma \in G} \Lambda(\sigma^{-1}) \otimes P(\sigma^{-1}) = S_{\Lambda}.$$

Now we show that S_{Λ} is idempotent. We have

$$S_{\Lambda}^{2} = \Big(\frac{1}{|G|} \sum_{\sigma \in G} \Lambda(\sigma) \otimes P(\sigma)\Big)\Big(\frac{1}{|G|} \sum_{\pi \in G} \Lambda(\pi) \otimes P(\pi)\Big)$$
$$= \frac{1}{|G|^2} \sum_{\sigma \in G} \sum_{\pi \in G} \Lambda(\sigma)\Lambda(\pi) \otimes P(\sigma)P(\pi)$$
$$= \frac{1}{|G|^2} \sum_{\sigma \in G} \sum_{\pi \in G} \Lambda(\sigma\pi) \otimes P(\sigma\pi) \quad (\sigma\pi = \tau)$$
$$= \frac{1}{|G|^2} \sum_{\sigma \in G} \sum_{\tau \in G} \Lambda(\tau) \otimes P(\tau) = \frac{1}{|G|} \sum_{\sigma \in G} S_{\Lambda} = S_{\Lambda}.$$

□

Definition 1.1. The range of S_{Λ},

$$V_{\Lambda}(G) := S_{\Lambda}(U \otimes V^{\otimes m}),$$

is called the *generalized symmetry class of tensors over V associated with G and Λ*.

If $\dim U = 1$, then $V_\Lambda(G)$ reduces to $V_\lambda(G)$, the symmetry class of tensors associated with G and the irreducible character λ of G corresponding to the representation Λ (see [4], [5], [9], [10], [12], [13], [14]). Recently, the other types of symmetry classes have been studied by several authors (see [1], [2], [3], [7], [11], [15], [16]).

The elements in $V_\Lambda(G)$ of the form

$$u \circledast v^{\circledast} := S_\Lambda(u \otimes v^{\otimes})$$

are called *the generalized decomposable symmetrized tensors.* The equality of two generalized decomposable symmetrized tensors has been studied in [6], [8].

In this paper, we study some important properties of the vector space $V_\Lambda(G)$.

Lemma 1.1. *For any $\sigma \in G$, $u \in U$ and $x^{\otimes} \in V^{\otimes m}$ we have*

$$u \circledast x^{\circledast}_\sigma = \Lambda(\sigma) u \circledast x^{\circledast}.$$

Proof. The proof is straightforward. □

Theorem 1.2. *Suppose Λ is an irreducible unitary representation of G over unitary space U. If Λ affords the irreducible character λ of G, then*

$$\dim V_\Lambda(G) = \frac{1}{|G|} \sum_{\sigma \in G} \lambda(\sigma) n^{c(\sigma)}.$$

Proof. According to Theorem 1.1, S_Λ is an orthogonal projection, so we have

$$\begin{aligned}\dim V_\Lambda(G) &= \operatorname{rank} S_\Lambda = \operatorname{Tr} S_\Lambda = \frac{1}{|G|} \sum_{\sigma \in G} \operatorname{Tr}(\Lambda(\sigma) \otimes P(\sigma)) \\ &= \frac{1}{|G|} \sum_{\sigma \in G} \operatorname{Tr} \Lambda(\sigma) \operatorname{Tr} P(\sigma) = \frac{1}{|G|} \sum_{\sigma \in G} \lambda(\sigma) n^{c(\sigma)}.\end{aligned}$$

□

Notice that $\lambda(1) \dim V_\Lambda(G) = \dim V_\lambda(G)$.

Let $\Gamma_{m,n}$ be the set of all sequences $\alpha = (\alpha(1), \ldots, \alpha(m))$ with $1 \leqslant \alpha(i) \leqslant n$, $1 \leqslant i \leqslant m$. The group G acts on $\Gamma_{m,n}$ as

$$\alpha\sigma = (\alpha(\sigma(1)), \ldots, \alpha(\sigma(m))).$$

Two sequences α and β in $\Gamma_{m,n}$ are said to be equivalent modulo G, denoted by $\alpha \sim \beta \bmod G$, if there exists $\sigma \in G$ such that $\beta = \alpha\sigma$. For each $\alpha \in \Gamma_{m,n}$, the equivalence class $\Gamma_\alpha = \{\alpha\sigma\colon\ \sigma \in G\}$ is called the orbit containing α. So we have the following disjoint union $\Gamma_{m,n} = \bigcup_{\alpha \in \Delta} \Gamma_\alpha$. We know that $|\Gamma_\alpha| = [G : G_\alpha]$, in which G_α is the stabilizer subgroup of α. Let Δ be a system of representatives for the orbits such that each sequence in Δ is first in its orbit relative to the lexicographic order.

Definition 1.2. Suppose $\alpha \in \Gamma_{m,n}$. The linear map $T_\alpha \colon U \to U$ defined by

$$T_\alpha = \frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \Lambda(\sigma)$$

is called the *linear map corresponding to* α. If $\alpha \sim \beta \bmod G$, then we can easily see that T_α and T_β are similar.

Theorem 1.3. *For any $\alpha \in \Gamma_{m,n}$ the linear map T_α is an orthogonal projection on U.*

Proof. It is easy to see that T_α is Hermitian. Now we prove that T_α is idempotent. We have

$$\begin{aligned} T_\alpha^2 &= \left(\frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \Lambda(\sigma)\right)\left(\frac{1}{|G_\alpha|} \sum_{\pi \in G_\alpha} \Lambda(\pi)\right) = \frac{1}{|G_\alpha|^2} \sum_{\sigma \in G_\alpha} \sum_{\pi \in G_\alpha} \Lambda(\sigma)\Lambda(\pi) \\ &= \frac{1}{|G_\alpha|^2} \sum_{\sigma \in G_\alpha} \sum_{\pi \in G_\alpha} \Lambda(\sigma\pi) = \frac{1}{|G_\alpha|^2} \sum_{\sigma \in G_\alpha} \sum_{\tau \in G_\alpha} \Lambda(\tau) \quad (\sigma\pi = \tau) \\ &= \frac{1}{|G_\alpha|^2} \sum_{\sigma \in G_\alpha} |G_\alpha| T_\alpha = T_\alpha. \end{aligned}$$

□

According to Theorem 1.3, the linear map T_α is an orthogonal projection. So $\operatorname{rank} T_\alpha = \operatorname{Tr} T_\alpha$. Thus, we have the following result.

Corollary 1.1. *Let Λ be an irreducible unitary representation of G over unitary space U. If Λ affords the irreducible character λ of G, then for each $\alpha \in \Gamma_{m,n}$ we have*

$$\operatorname{rank} T_\alpha = \frac{1}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \lambda(\sigma).$$

In particular, $T_\alpha \neq 0$ if and only if $\sum_{\sigma \in G_\alpha} \lambda(\sigma) \neq 0$.

In the following theorem we state the intimate relationship between generalized Schur functions and generalized decomposable symmetrized tensors.

Theorem 1.4. *For each $u, v \in U$ and $x^\otimes, y^\otimes \in V^{\otimes m}$ we have*

$$(u \circledast x^\circledast, v \circledast y^\circledast) = \frac{1}{|G|}(D_\Lambda(A)u, v),$$

where $A = ((x_i, y_j))_{m \times m}$.

Proof. According to Theorem 1.1 we have

$$\begin{aligned}(u \circledast x^{\circledast}, v \circledast y^{\circledast}) &= (S_\Lambda(u \otimes x^{\otimes}), S_\Lambda(v \otimes y^{\otimes})) = (S_\Lambda(u \otimes x^{\otimes}), v \otimes y^{\otimes}) \\ &= \left(\frac{1}{|G|}\sum_{\sigma\in G}\Lambda(\sigma)u \otimes P(\sigma)x^{\otimes}, v \otimes y^{\otimes}\right) \\ &= \frac{1}{|G|}\sum_{\sigma\in G}(\Lambda(\sigma)u, v)\prod_{i=1}^{m}(x_{\sigma^{-1}(i)}, y_i) \\ &= \frac{1}{|G|}\sum_{\sigma\in G}\left(\prod_{i=1}^{m}(x_{\sigma^{-1}(i)}, y_i)\Lambda(\sigma)u, v\right) \\ &= \frac{1}{|G|}\left(\sum_{\sigma\in G}\Lambda(\sigma)\prod_{i=1}^{m}(x_i, y_{\sigma(i)})u, v\right) = \frac{1}{|G|}(D_\Lambda(A)u, v).\end{aligned}$$

□

2. Bases of generalized symmetry classes of tensors

Suppose $\mathbb{F} = \{u_1, \ldots, u_r\}$ and $\mathbb{E} = \{e_1, \ldots, e_n\}$ are orthonormal bases for unitary spaces U and V, respectively. Then

$$\mathbb{E}_{\otimes} = \{u_i \otimes e_\alpha^{\otimes} \colon 1 \leqslant i \leqslant r, \ \alpha \in \Gamma_{m,n}\}$$

is an orthonormal basis of $U \otimes V^{\otimes m}$. Hence

$$V_\Lambda(G) = \langle u_i \circledast e_\alpha^{\circledast} \colon 1 \leqslant i \leqslant r, \ \alpha \in \Gamma_{m,n}\rangle.$$

For each $\alpha \in \Gamma_{m,n}$ the subspace

$$V_\alpha^{\circledast} = \langle u_i \circledast e_\alpha^{\circledast} \colon 1 \leqslant i \leqslant r\rangle$$

is called the *generalized orbital subspace* corresponding to α. By using Lemma 1.1, we deduce that

$$V_\Lambda(G) = \sum_{\alpha\in\Delta} V_\alpha^{\circledast}.$$

Since Λ is an irreducible representation of G over U,

$$U = \langle \Lambda(\sigma)u_1 \colon \sigma \in G\rangle.$$

Thus

$$V_\alpha^{\circledast} = \langle \Lambda(\sigma)u_1 \circledast e_\alpha^{\circledast} \colon \sigma \in G\rangle.$$

Again by Lemma 1.1 we have

$$V_\alpha^\circledast = \langle u_1 \circledast e_{\alpha\sigma}^\circledast \colon \ \sigma \in G\rangle.$$

For each $1 \leqslant i \leqslant r$ we define

$$V_\Lambda^i(G) = \langle u_i \circledast e_\alpha^\circledast \colon \ \alpha \in \Gamma_{m,n}\rangle.$$

Then $V_\Lambda(G) = \sum\limits_{i=1}^{r} V_\Lambda^i(G)$, but it is not necessary a direct sum. (This will be described more with an example.)

Theorem 2.1. *For each* $1 \leqslant i, \ j \leqslant r$ *and* $\alpha, \beta \in \Gamma_{m,n}$ *we have*

$$(u_i \circledast e_\alpha^\circledast, u_j \circledast e_\beta^\circledast) = \begin{cases} 0, & \alpha \nsim \beta \mod G, \\ \dfrac{1}{[G : G_\alpha]}(T_\alpha u_i, u_j), & \alpha = \beta. \end{cases}$$

In particular,

$$\|u_i \circledast e_\alpha^\circledast\|^2 = \frac{1}{[G : G_\alpha]}\|T_\alpha u_i\|^2.$$

P r o o f. Let

$$A = (a_{ij})_{m\times m}, \ a_{ij} = (e_{\alpha(i)}, e_{\beta(j)}) = \delta_{\alpha(i),\beta(j)}.$$

Then by Theorem 1.4 we have

$$\begin{aligned}
(u_i \circledast e_\alpha^\circledast, u_j \circledast e_\beta^\circledast) &= \frac{1}{|G|}(D_\Lambda(A)u_i, u_j) = \frac{1}{|G|}\left(\sum_{\sigma\in G}\Lambda(\sigma)\prod_{k=1}^{m} a_{k\sigma(k)}u_i, u_j\right) \\
&= \frac{1}{|G|}\left(\sum_{\sigma\in G}\Lambda(\sigma)\prod_{k=1}^{m}\delta_{\alpha(k),\beta\sigma(k)}u_i, u_j\right) \\
&= \frac{1}{|G|}\left(\sum_{\sigma\in G}\Lambda(\sigma)\delta_{\alpha,\beta\sigma}u_i, u_j\right) \\
&= \begin{cases} 0, & \alpha \nsim \beta \mod G, \\ \dfrac{1}{|G|}\left(\sum\limits_{\sigma\in G_\alpha}\Lambda(\sigma)u_i, u_j\right), & \alpha = \beta, \end{cases} \\
&= \begin{cases} 0, & \alpha \nsim \beta \mod G, \\ \dfrac{1}{[G : G_\alpha]}(T_\alpha u_i, u_j), & \alpha = \beta. \end{cases}
\end{aligned}$$

In particular,

$$\begin{aligned}\|u_i \circledast e_\alpha^\circledast\|^2 = (u_i \circledast e_\alpha^\circledast, u_i \circledast e_\alpha^\circledast) &= \frac{1}{[G:G_\alpha]}(T_\alpha u_i, u_i)\\ &= \frac{1}{[G:G_\alpha]}(T_\alpha u_i, T_\alpha u_i) \quad \text{(by Theorem 1.3)}\\ &= \frac{1}{[G:G_\alpha]}\|T_\alpha u_i\|^2.\end{aligned}$$

□

Corollary 2.1. *For each $1 \leqslant i \leqslant r$ and $\alpha, \beta \in \Gamma_{m,n}$ we have $u_i \circledast e_\alpha^\circledast = 0$ if and only if $T_\alpha u_i = 0$.*

For any $1 \leqslant i \leqslant r$ let $\Omega_i = \{\alpha \in \Gamma_{m,n} \colon T_\alpha u_i \neq 0\}$. If we set $\bar{\Delta}_i = \Delta \cap \Omega_i$, then we can easily see that the set $\{u_i \circledast e_\alpha^\circledast \colon \alpha \in \bar{\Delta}_i\}$ is an orthogonal basis of $V_\Lambda^i(G)$. Let $\Omega = \bigcup\limits_{i=1}^{r} \Omega_i$ and $\bar{\Delta} = \Delta \cap \Omega$. Then by Corollary 1.1,

$$\bar{\Delta} = \{\alpha \in \Delta \colon T_\alpha \neq 0\} = \Big\{\alpha \in \Delta \colon \sum_{\sigma \in G_\alpha} \lambda(\sigma) \neq 0\Big\}.$$

Now we conclude the following corollary.

Corollary 2.2. *The generalized symmetry class of tensors $V_\Lambda(G)$ is the orthogonal direct sum of the generalized orbital subspaces $V_\alpha^\circledast$, as α ranges over $\bar{\Delta}$.*

Example 2.1. Let $G = S_3$. Consider the matrix representation $\Lambda \colon G \to GL(2, \mathbb{C})$ such that

$$\Lambda(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Lambda(1\ 2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Lambda(1\ 3) = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix},$$
$$\Lambda(2\ 3) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \quad \Lambda(1\ 2\ 3) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \quad \Lambda(1\ 3\ 2) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix},$$

where ω is a primitive third root of unity. It is easy to see that Λ is a unitary irreducible representation of G. Suppose that V is an two-dimensional vector space with an orthonormal basis $\mathbb{E} = \{e_1, e_2\}$. Let Δ be a system of distinct representatives for the equivalence classes of $\Gamma_{3,2}$ modulo G. Then

$$\Delta = \{\alpha = (1,1,1),\ \beta = (1,1,2),\ \gamma = (1,2,2),\ \delta = (2,2,2)\}.$$

It is obvious that $G_\alpha = G_\delta = G$. Since Λ is an irreducible representation of G, $\sum\limits_{\sigma \in G} \Lambda(\sigma) = 0$. Hence $T_\alpha = T_\delta = 0$. Similarly, we can see that

$$T_\beta = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad T_\gamma = \frac{1}{2}\begin{pmatrix} 1 & \omega \\ \omega^2 & 1 \end{pmatrix}.$$

Suppose $U = \mathbb{C}^2$ and $\mathbb{F} = \{u_1, u_2\}$ is the standard basis of U. Then

$$\bar{\Delta}_1 = \bar{\Delta}_2 = \{\beta, \gamma\}.$$

Thus, $\dim V_\Lambda^1(G) = |\bar{\Delta}_1| = 2$, $\dim V_\Lambda^2(G) = |\bar{\Delta}_2| = 2$. But

$$\dim V_\Lambda(G) = \frac{1}{|G|} \sum_{\sigma \in G} \lambda(\sigma) n^{c(\sigma)} = \frac{1}{6}[2(2)^3 + 2(-1)(2)] = 2.$$

Therefore $V_\Lambda(G) = V_\Lambda^1(G) + V_\Lambda^2(G)$ is not a direct sum.

The following theorem extends [10], Theorem 6.34 to the generalized symmetry classes of tensors.

Theorem 2.2 (Generalized Freese's Theorem)**.** *Let Λ be an irreducible unitary representation of G over unitary space U such that it affords character λ of G. If $\alpha \in \bar{\Delta}$, then*

$$\dim V_\alpha^{\circledast} = [\lambda, 1]_{G_\alpha},$$

where $[\,,\,]$ is the inner product of characters.

P r o o f. Let $G = \bigcup_{i=1}^{t} G_\alpha \sigma_i$, $\Gamma_\alpha = \{\alpha\sigma_1, \ldots, \alpha\sigma_t\}$ be the right coset decomposition of G_α in G. Notice that $V_\alpha^{\circledast} = S_\Lambda(W_\alpha)$, where

$$W_\alpha = \langle u_i \otimes e_{\alpha\sigma}^{\otimes} \colon 1 \leqslant i \leqslant r,\ \sigma \in G \rangle.$$

Then

$$\mathbb{E}_\alpha = \{u_i \otimes e_{\alpha\sigma_j}^{\otimes} \colon 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant t\}$$

is a basis of W_α, but the set

$$\{u_i \circledast e_{\alpha\sigma_j}^{\circledast} \colon 1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant t\}$$

may not be a basis for $V_\alpha^{\circledast}$. Since W_α is an invariant subspace of S_Λ, the restriction $S_\Lambda^\alpha = S_\Lambda|_{W_\alpha}$ is a linear operator on W_α. Let

$$C = (c_{(i,r),(j,s)}) = [S_\Lambda^\alpha]_{\mathbb{E}_\alpha}.$$

Now for each $\mu \in G$ we have

$$\begin{aligned}
S_\Lambda^\alpha(u_l \otimes e^{\otimes}_{\alpha\mu}) &= S_\Lambda(u_l \otimes e^{\otimes}_{\alpha\mu}) = S_\Lambda(\Lambda(\mu)u_l \otimes e^{\otimes}_{\alpha}) = \frac{1}{|G|}\sum_{\sigma\in G}\Lambda(\sigma^{-1})(\Lambda(\mu)u_l) \otimes e^{\otimes}_{\alpha\sigma} \\
&= \frac{1}{|G|}\sum_{i=1}^{t}\Bigg(\sum_{\sigma\in G_\alpha\sigma_i}\Lambda(\sigma^{-1}\mu)u_l \otimes e^{\otimes}_{\alpha\sigma}\Bigg) \\
&= \frac{1}{|G|}\sum_{i=1}^{t}\sum_{\tau\in G_\alpha}\Lambda(\sigma_i^{-1}\tau^{-1}\mu)u_l \otimes e^{\otimes}_{\alpha\tau\sigma_i} \\
&= \frac{1}{|G|}\sum_{i=1}^{t}\sum_{\tau\in G_\alpha}\Lambda(\sigma_i^{-1}\tau^{-1}\mu)u_l \otimes e^{\otimes}_{\alpha\sigma_i}.
\end{aligned}$$

In particular,

$$\begin{aligned}
S_\Lambda^\alpha(u_l \otimes e^{\otimes}_{\alpha\sigma_j}) &= \frac{1}{|G|}\sum_{i=1}^{t}\sum_{\tau\in G_\alpha}\Lambda(\sigma_i^{-1}\tau^{-1}\sigma_j)u_l \otimes e^{\otimes}_{\alpha\sigma_i} \\
&= \frac{1}{|G|}\sum_{i=1}^{t}\sum_{\tau\in G_\alpha}\sum_{k=1}^{r} m_{kl}(\sigma_i^{-1}\tau^{-1}\sigma_j)u_k \otimes e^{\otimes}_{\alpha\sigma_i} \\
&= \sum_{i=1}^{t}\sum_{k=1}^{r}\Bigg[\frac{1}{|G|}\sum_{\tau\in G_\alpha} m_{kl}(\sigma_i^{-1}\tau\sigma_j)\Bigg]u_k \otimes e^{\otimes}_{\alpha\sigma_i}.
\end{aligned}$$

So

$$c_{(k,i),(l,j)} = \frac{1}{|G|}\sum_{\tau\in G_\alpha} m_{kl}(\sigma_i^{-1}\tau\sigma_j), \quad k,l = 1,\ldots,r,\ i,j = 1,\ldots,t.$$

We prove that C is an idempotent matrix. We have

$$\begin{aligned}
(C^2)_{(k,i),(l,j)} &= \sum_{p=1}^{r}\sum_{q=1}^{t} c_{(k,i),(p,q)}\ c_{(p,q),(l,j)} \\
&= \sum_{p=1}^{r}\sum_{q=1}^{t}\Bigg(\frac{1}{|G|}\sum_{\sigma\in G_\alpha} m_{kp}(\sigma_i^{-1}\sigma\sigma_q)\Bigg)\Bigg(\frac{1}{|G|}\sum_{\tau\in G_\alpha} m_{pl}(\sigma_q^{-1}\tau\sigma_j)\Bigg) \\
&= \frac{1}{|G|^2}\sum_{p=1}^{r}\sum_{q=1}^{t}\sum_{\sigma\in G_\alpha}\sum_{\tau\in G_\alpha} m_{kp}(\sigma_i^{-1}\sigma\sigma_q)m_{pl}(\sigma_q^{-1}\tau\sigma_j) \\
&= \frac{1}{|G|^2}\sum_{\sigma,\tau\in G_\alpha}\sum_{q=1}^{t} m_{kl}(\sigma_i^{-1}\sigma\tau\sigma_j) = \frac{t}{|G|^2}\sum_{g,\tau\in G_\alpha} m_{kl}(\sigma_i^{-1}g\sigma_j) \\
&= \frac{t|G_\alpha|}{|G|}\frac{1}{|G|}\sum_{g\in G_\alpha} m_{kl}(\sigma_i^{-1}g\sigma_j) = c_{(k,i),(l,j)}.
\end{aligned}$$

Thus,

$$\dim V_\alpha^{\circledast} = \operatorname{rank}(S_\Lambda^\alpha) = \operatorname{rank} C = \operatorname{Tr} C.$$

Now we calculate $\operatorname{Tr} C$. We have

$$\begin{aligned}
\operatorname{Tr} C &= \sum_{k=1}^{r}\sum_{i=1}^{t} c_{(k,i),(k,i)} = \sum_{k=1}^{r}\sum_{i=1}^{t}\Big(\frac{1}{|G|}\sum_{\sigma\in G_\alpha} m_{kk}(\sigma_i^{-1}\sigma\sigma_i)\Big)\\
&= \frac{1}{|G|}\sum_{\sigma\in G_\alpha}\sum_{i=1}^{t}\sum_{k=1}^{r} m_{kk}(\sigma_i^{-1}\sigma\sigma_i) = \frac{1}{|G|}\sum_{\sigma\in G_\alpha}\sum_{i=1}^{t}\operatorname{Tr}\Lambda(\sigma_i^{-1}\sigma\sigma_i)\\
&= \frac{1}{|G|}\sum_{\sigma\in G_\alpha}\sum_{i=1}^{t}\lambda(\sigma_i^{-1}\sigma\sigma_i) = \frac{1}{|G|}\sum_{\sigma\in G_\alpha}\sum_{i=1}^{t}\lambda(\sigma)\\
&= \frac{t}{|G|}\sum_{\sigma\in G_\alpha}\lambda(\sigma) \quad ([G:G_\alpha]=t)\\
&= \frac{1}{|G_\alpha|}\sum_{\sigma\in G_\alpha}\lambda(\sigma) = [\lambda,1]_{G_\alpha}.
\end{aligned}$$

□

We now construct a basis of $V_\Lambda(G)$. By Corollary 2.2, $V_\Lambda(G) = \bigoplus_{\alpha\in\bar\Delta} V_\alpha^{\circledast}$. In order to find a basis for $V_\Lambda(G)$, it suffices to find bases of the generalized orbital subspaces $V_\alpha^{\circledast}$, $\alpha\in\bar\Delta$.

Choose a lexicographically ordered set $\{\alpha_1=\alpha,\alpha_2,\ldots,\alpha_{s_\alpha}\}$ from $\{\alpha\sigma\colon \sigma\in G\}$ such that

$$\{u_1\circledast e_{\alpha_1}^{\circledast}, u_1\circledast e_{\alpha_2}^{\circledast},\ldots,u_1\circledast e_{\alpha_{s_\alpha}}^{\circledast}\}$$

is a basis of $V_\alpha^{\circledast}$. The same is done for any $\alpha\in\bar\Delta$. If $\{\alpha,\beta,\gamma,\ldots\}$ is the lexicographically ordered set $\bar\Delta$, take $\widehat\Delta = \{\alpha_1,\ldots,\alpha_{s_\alpha},\beta_1,\ldots,\beta_{s_\beta},\ldots\}$ to be ordered as indicated. Then $\{u_1\circledast e_\alpha^{\circledast}\colon \alpha\in\widehat\Delta\}$ is a basis of $V_\Lambda(G)$. Obviously, $\bar\Delta=\{\alpha_1,\beta_1,\ldots\}$ is lexicographically ordered, but note that $\widehat\Delta$ is not lexicographically ordered; it is possible that $\alpha_2>\beta_1$. Such order in $\widehat\Delta$ is called an *orbital order*. If λ is a linear character, then $\dim V_\alpha^{\circledast}=1$ and in this case, the set $\{u_1\circledast e_\alpha^{\circledast}\colon \alpha\in\bar\Delta\}$ is an orthogonal basis of $V_\Lambda(G)$. We call a basis consisting of generalized decomposable symmetrized tensors $u_1\circledast e_\alpha^{\circledast}$, an *orthogonal* $\circledast$*-basis*. If λ is not linear, it is possible that $V_\Lambda(G)$ has no orthogonal $\circledast$-basis.

Corollary 2.3. *Suppose* $\dim V_\alpha^{\circledast}=s_\alpha$. *Then*

$$\dim V_\Lambda(G) = |\widehat\Delta| = \sum_{\alpha\in\bar\Delta} s_\alpha = \sum_{\sigma\in\bar\Delta}[\lambda,1]_{G_\alpha}.$$

Now we give a necessary condition for the existence of orthogonal $\circledast$-basis.

Theorem 2.3. *Let Λ be an irreducible unitary representation of G over a unitary space U such that it affords the character λ of G. If there is $\alpha \in \Gamma_{m,n}$ such that*

$$\lambda(1) < [G : G_\alpha] < 2[\lambda, 1]_{G_\alpha},$$

then $V_\Lambda(G)$ has no orthogonal $\circledast$-basis.

P r o o f. Let $G = \bigcup_{i=1}^{s} G_\alpha t_i, [G : G_\alpha] = s$ be the right coset decomposition of G_α in G. Then

$$V_\alpha^{\circledast} = \langle u_1 \circledast e_{\alpha t_i}^{\circledast} : \ 1 \leqslant i \leqslant s\rangle.$$

For any i and j we have

$$\begin{aligned}(u_1 \circledast e_{\alpha t_i}^{\circledast}, u_1 \circledast e_{\alpha t_j}^{\circledast}) &= (S_\Lambda(u_1 \otimes e_{\alpha t_i}^{\otimes}), S_\Lambda(u_1 \otimes e_{\alpha t_j}^{\otimes})) = (S_\Lambda(u_1 \otimes e_{\alpha t_i}^{\otimes}), u_1 \otimes e_{\alpha t_j}^{\otimes}) \\ &= \frac{1}{|G|}\Big(\sum_{\sigma\in G} \Lambda(\sigma)u_1 \otimes e_{\alpha t_i \sigma^{-1}}^{\otimes}, u_1 \otimes e_{\alpha t_j}^{\otimes}\Big) \\ &= \frac{1}{|G|}\sum_{\sigma\in G}(\Lambda(\sigma)u_1, u_1)\delta_{\alpha t_i\sigma^{-1},\alpha t_j} = \frac{1}{|G|}\sum_{\sigma\in t_i^{-1}G_\alpha t_j}(\Lambda(\sigma)u_1, u_1) \\ &= \frac{1}{|G|}\sum_{\sigma\in t_i^{-1}G_\alpha t_j} m_{11}(\sigma) = \frac{1}{|G|}\sum_{\sigma\in G_\alpha} m_{11}(t_i^{-1}\sigma t_j).\end{aligned}$$

Now we define an $s \times s$ matrix D as

$$d_{ij} = \frac{1}{|G|}\sum_{\sigma\in G_\alpha} m_{11}(t_i^{-1}\sigma t_j).$$

Observe that

$$\begin{aligned}D_{ij}^2 &= \sum_{p=1}^{s} d_{ip}d_{pj} = \sum_{p=1}^{s}\Big(\frac{1}{|G|}\sum_{x\in G_\alpha} m_{11}(t_i^{-1}xt_p)\Big)\Big(\frac{1}{|G|}\sum_{y\in G_\alpha} m_{11}(t_p^{-1}yt_j)\Big) \\ &= \frac{1}{|G|^2}\sum_{p=1}^{s}\sum_{x\in G_\alpha t_p}\sum_{y\in t_p^{-1}G_\alpha} m_{11}(t_i^{-1}x)m_{11}(yt_j) \\ &= \frac{1}{|G|^2}\sum_{p=1}^{s}\sum_{h\in G_\alpha}\sum_{x\in G_\alpha t_p} m_{11}(t_i^{-1}x)m_{11}(x^{-1}ht_j) \quad (xy = h) \\ &= \frac{1}{|G|^2}\sum_{h\in G_\alpha}\sum_{z\in G} m_{11}(z)m_{11}(z^{-1}t_i^{-1}ht_j) \quad (t_i^{-1}x = z) \\ &= \frac{1}{\lambda(1)|G|}\sum_{h\in G_\alpha} m_{11}(t_i^{-1}ht_j) \ \ \text{(by Schur Relations)} \\ &= \frac{1}{\lambda(1)}D_{ij}.\end{aligned}$$

Therefore $\lambda(1)D^2 = D$.

Let $V_\Lambda(G)$ have an orthogonal $\circledast$-basis. Then $V_\alpha^\circledast$ has an orthogonal basis. Now suppose $\dim V_\alpha^\circledast = k$ and consider that $\mathbb{B} = \{u_1 \circledast e_{\alpha t_1}^\circledast, \ldots, u_1 \circledast e_{\alpha t_k}^\circledast\}$ is an orthogonal basis of $V_\alpha^\circledast$. Thus, the matrix D has the block partition form

$$\begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix},$$

where

$$E_1 = \begin{pmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{kk} \end{pmatrix}.$$

It follows that

$$D^2 = \begin{pmatrix} E_1^2 + E_2E_3 & E_1E_2 + E_2E_4 \\ E_3E_1 + E_4E_3 & E_3E_2 + E_4^2 \end{pmatrix}.$$

Using $\lambda(1)D^2 = D$ we obtain

$$E_1^2 + E_2E_3 = \frac{1}{\lambda(1)}E_1.$$

So

$$E_2E_3 = \begin{pmatrix} \frac{d_{11}}{\lambda(1)} - d_{11}^2 & & 0 \\ & \ddots & \\ 0 & & \frac{d_{kk}}{\lambda(1)} - d_{kk}^2 \end{pmatrix}.$$

We know that $d_{ii} \neq 0$ for any $1 \leqslant i \leqslant k$ because $d_{ii} = \|u_1 \circledast e_{\alpha t_i}^\circledast\|^2$. If

$$\frac{d_{ii}}{\lambda(1)} - d_{ii}^2 = 0$$

for some $1 \leqslant i \leqslant k$, then

$$\begin{aligned} \frac{1}{\lambda(1)} = d_{ii} &= \left| \frac{1}{|G|} \sum_{x \in G_\alpha} m_{11}(t_i^{-1}xt_i) \right| \leqslant \frac{1}{|G|} \sum_{x \in G_\alpha} |m_{11}(t_i^{-1}xt_i)| \\ &\leqslant \frac{1}{|G|} \sum_{x \in G_\alpha} 1 = \frac{1}{[G : G_\alpha]}, \end{aligned}$$

and this contradicts the assumption $\lambda(1) < [G : G_\alpha]$ of the theorem. Thus, E_2E_3 is an invertible $k \times k$ matrix. This implies that $k \leqslant s - k$. Therefore

$$[\lambda, 1]_{G_\alpha} \leqslant \frac{[G : G_\alpha]}{2},$$

and the result holds. $\square$

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