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# COMPRESSION OF SLANT TOEPLITZ OPERATORS ON THE HARDY SPACE OF $n$-DIMENSIONAL TORUS

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*Abstract.* This paper studies the compression of a $k$th-order slant Toeplitz operator on the Hardy space $H^2(\mathbb{T}^n)$ for integers $k \geqslant 2$ and $n \geqslant 1$. It also provides a characterization of the compression of a $k$th-order slant Toeplitz operator on $H^2(\mathbb{T}^n)$. Finally, the paper highlights certain properties, namely isometry, eigenvalues, eigenvectors, spectrum and spectral radius of the compression of $k$th-order slant Toeplitz operator on the Hardy space $H^2(\mathbb{T}^n)$ of $n$-dimensional torus $\mathbb{T}^n$.

*Keywords*: Toeplitz operator; compression of slant Toeplitz operator; $n$-dimensional torus; Hardy space

*MSC 2020*: 47B35

## 1. Introduction

Throughout the paper, the set of all complex numbers, the open unit disc and the unit circle in the complex plane are denoted by $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{T}$, respectively. The theory of slant Toeplitz operators on $L^2(\mathbb{T})$ was developed by Ho (see [5], [7]), who investigated several features of the slant Toeplitz operators on $L^2(\mathbb{T})$, such as norms, spectrum and eigen spaces etc. Arora and Batra in [1] and [2] extended this concept to the $k$th-order slant Toeplitz operators on $L^2(\mathbb{T})$ and its compression on $H^2(\mathbb{T})$. Ding, Sun and Zheng studied Toeplitz operators and their commutativity on the bi-disk in [4]. Lu and Zhang discussed the notion of commuting Hankel and Toeplitz operators on the Hardy space of the bi-disk, see [8]. The study of the Toeplitz operator is generalized to a $n$-dimensional structure in [9]. For the fundamental

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terminologies and concepts of Toeplitz and Hankel operators, one is referred to [10]. Enlightened from the work of Ho (see [5], [7]), slant Toeplitz operators are considered on $L^2(\mathbb{T}^n)$ in [3]. This paper extends the study of the compression of $k$th-order slant Toeplitz operators to $H^2(\mathbb{T}^n)$, where the set $\mathbb{T}^n \subset \mathbb{C}^n$, the distinguished boundary of open unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$, denotes the Cartesian product of $n$ copies of the unit circle $\mathbb{T} \subset \mathbb{C}$.

Throughout the paper, the space of all Lebesgue measurable complex valued functions defined on $\mathbb{T}^n$, which satisfies

$$\int_{\mathbb{T}^n} |f|^2 \, \mathrm{d}\sigma < \infty,$$

where $\mathrm{d}\sigma$ is a normalized Lebesgue Haar measure, is denoted by $L^2(\mathbb{T}^n)$. The space $L^\infty(\mathbb{T}^n)$ represents the space of all essentially bounded measurable functions on $\mathbb{T}^n$. By the use of multiple Fourier series on $\mathbb{T}^n$ from the Chapter VII of [11], the space $L^2(\mathbb{T}^n)$ can be expressed as

$$L^2(\mathbb{T}^n) = \Bigg\{ f\colon\ f(z_1, z_2, \ldots, z_n) = \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n} f_{m_1,m_2,\ldots,m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}, \\ \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n} |f_{m_1,m_2,\ldots,m_n}|^2 < \infty \Bigg\}.$$

In the similar way, the space $H^2(\mathbb{T}^n)$ of $n$-dimensional torus $\mathbb{T}^n$ is given by

$$H^2(\mathbb{T}^n) = \Bigg\{ f\colon\ f(z_1, z_2, \ldots, z_n) = \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}_+^n} f_{m_1,m_2,\ldots,m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}, \\ \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}_+^n} |f_{m_1,m_2,\ldots,m_n}|^2 < \infty \Bigg\},$$

where $\mathbb{Z}$ and $\mathbb{Z}_+$ indicate the set of all integers and the set of all non-negative integers, respectively. The space $H^2(\mathbb{T}^n)$ is the Hilbert space with the norm induced by the inner product given by

$$\langle f, g\rangle = \frac{1}{(2\pi)^n} \underbrace{\int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi}}_{n\text{-times}} f(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \ldots, \mathrm{e}^{\mathrm{i}\theta_n}) \overline{g(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \ldots, \mathrm{e}^{\mathrm{i}\theta_n})} \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 \ldots \mathrm{d}\theta_n.$$

The collection

$$\{e_{m_1,m_2,\ldots,m_n}\colon\ (m_1, m_2, \ldots, m_n) \in \mathbb{Z}_+^n\},$$

where

$$e_{m_1,m_2,\dots,m_n}(z_1,z_2,\dots,z_n)=z_1^{m_1}z_2^{m_2}\dots z_n^{m_n},$$

forms an orthonormal basis for the space $H^2(\mathbb{T}^n)$. The basis elements are usually written as $z_1^{m_1}z_2^{m_2}\dots z_n^{m_n}$ instead of $e_{m_1,m_2,\dots,m_n}$ whenever there is no confusion. This space can also be viewed as the closed subspace of $L^2(\mathbb{T}^n)$ consisting of all those elements $f$ of $L^2(\mathbb{T}^n)$ for which $\langle f, e_{m_1,m_2,\dots,m_n}\rangle = 0$, whenever $m_j<0$ for at least one $j=1,2,\dots,n$ (see [6]).

For $n\geqslant 1$, let $\mathbb{D}^n$ denote the open unit polydisc in $\mathbb{C}^n$. The Hardy space $H^2(\mathbb{D}^n)$ over $\mathbb{D}^n$ is the Hilbert space of all holomorphic functions on $\mathbb{D}^n$ such that

$$\|f\|:=\left(\sup_{0\leqslant r<1}\int_{\mathbb{T}^n}|f(r\mathrm{e}^{\mathrm{i}\theta_1},r\mathrm{e}^{\mathrm{i}\theta_2},\dots,r\mathrm{e}^{\mathrm{i}\theta_n})|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2\dots\mathrm{d}\theta_n\right)^{1/2}<\infty,$$

where $\mathrm{d}\theta_1\,\mathrm{d}\theta_2\dots\mathrm{d}\theta_n$ indicates the normalized Lebesgue measure on the torus $\mathbb{T}^n$.

One can see the identification between the Hardy space $H^2(\mathbb{D}^n)$ and $H^2(\mathbb{T}^n)$ via the radial limits of functions in $H^2(\mathbb{D}^n)$ (see [9] and the references therein). From now onwards, by the analytic function in $L^2(\mathbb{T}^n)$ we mean that a function with Fourier coefficients $f_{m_1,m_2,\dots,m_n}=0$, whenever $m_j<0$ for at least one $j$, $1\leqslant j\leqslant n$. A function $g\in L^2(\mathbb{T}^n)$ is co-analytic if $\overline{g}$ is analytic in the above sense. Also, we denote the standard basis of $\mathbb{R}^n$ by $B_n$, i.e.

$$B_n=\{(1,0,0,\dots,0),(0,1,0,\dots,0),\dots,(0,0,0,\dots,1)\}.$$

Throughout the paper, $k$ and $n$ are chosen as integers such that $k\geqslant 2$ and $n\geqslant 1$.

## 2. Characterization of the compression of $k$th-order slant Toeplitz operator

We begin the section by recalling a few definitions and basic information related with $k$th-order slant Toeplitz operator.

**Definition 2.1** ([9]). Let $\varphi\in L^\infty(\mathbb{T}^n)$; then the Toeplitz operator $T_{\varphi,n}$, induced by symbol $\varphi$, on $H^2(\mathbb{T}^n)$ is defined as

$$T_{\varphi,n}(f)=PM_\varphi(f)\quad\text{for all } f\in H^2(\mathbb{T}^n),$$

where $M_\varphi$ is the multiplication operator, induced by $\varphi$, and $P$ is the orthogonal projection from the space $L^2(\mathbb{T}^n)$ onto the space $H^2(\mathbb{T}^n)$.

**Definition 2.2** ([3])**.** For $\varphi \in L^\infty(\mathbb{T}^n)$, the $k$th-order slant Toeplitz operator $A_{\varphi,k,n}$ on $L^2(\mathbb{T}^n)$ is given by

$$A_{\varphi,k,n}(f) = E_{k,n}M_\varphi(f) \quad \text{for all } f \in L^2(\mathbb{T}^n),$$

where $E_{k,n}$ is a bounded operator on $L^2(\mathbb{T}^n)$ for a fixed integer $k \geqslant 2$, given by

$$E_{k,n}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) = \begin{cases} z_1^{i_1/k} z_2^{i_2/k} \dots z_n^{i_n/k} & \text{if each } i_j \in \mathbb{Z} \text{ is a multiple of } k, \\ & 1 \leqslant j \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Now we are in a position to define the compression of $k$th-order slant Toeplitz operator on the Hardy space $H^2(\mathbb{T}^n)$.

**Definition 2.3.** Let $\varphi$ be an element of the space $L^\infty(\mathbb{T}^n)$. Then the *compression $V_{\varphi,k,n}$ of $k$th-order slant Toeplitz operator $A_{\varphi,k,n}$ to the Hardy space $H^2(\mathbb{T}^n)$* is defined as

$$V_{\varphi,k,n}(f) = PA_{\varphi,k,n}(f) \quad \text{for all } f \in H^2(\mathbb{T}^n),$$

where $P$ is the orthogonal projection from the space $L^2(\mathbb{T}^n)$ onto the space $H^2(\mathbb{T}^n)$. Equivalently, $V_{\varphi,k,n} = PA_{\varphi,k,n}|_{H^2(\mathbb{T}^n)}$.

Let

$$\varphi(z_1, z_2, \dots, z_n) = \sum_{(m_1,m_2,\dots,m_n)\in\mathbb{Z}^n} \varphi_{m_1,m_2,\dots,m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} \in L^\infty(\mathbb{T}^n).$$

In order to know the Toeplitz operator $T_{\varphi,n}$, we see that for $(i_1, i_2, \dots, i_n) \in \mathbb{Z}_+^n$,

$$T_{\varphi,n}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) = \sum_{(m_1,m_2,\dots,m_n)\in\mathbb{Z}_+^n} \varphi_{m_1-i_1,m_2-i_2,\dots,m_n-i_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}.$$

A simple calculation yields that $T^*_{\varphi,n} = PM^*_\varphi\big|_{H^2(\mathbb{T}^n)}$. Similarly, the action of the compression $V_{\varphi,k,n}$ of the $k$th-order slant Toeplitz operator on basis elements can be seen as

$$V_{\varphi,k,n}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) = \sum_{(m_1,m_2,\dots,m_n)\in\mathbb{Z}_+^n} \varphi_{km_1-i_1,km_2-i_2,\dots,km_n-i_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}.$$

On taking the adjoint in the definition of $V_{\varphi,k,n}$, we get that $V^*_{\varphi,k,n} = PA^*_{\varphi,k,n}\big|_{H^2(\mathbb{T}^n)}$. Again, simple computation yields that

$$V^*_{\varphi,k,n}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) = \sum_{(m_1,m_2,\dots,m_n)\in\mathbb{Z}_+^n} \bar{\varphi}_{ki_1-m_1,ki_2-m_2,\dots,ki_n-m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$$

for each $(i_1, i_2, \ldots, i_n) \in \mathbb{Z}_+^n$. With the help of Proposition 2.9 of the paper [3], the compression of $A_{\varphi,k,n}$ to the space $H^2(\mathbb{T}^n)$ can be expressed as

$$V_{\varphi,k,n} = PA_{\varphi,k,n}\big|_{H^2(\mathbb{T}^n)} = PE_{k,n}M_\varphi\big|_{H^2(\mathbb{T}^n)} = E_{k,n}PM_\varphi\big|_{H^2(\mathbb{T}^n)} = E_{k,n}T_{\varphi,n},$$

where $T_{\varphi,n}$ is the Toeplitz operator on $H^2(\mathbb{T}^n)$. Also, we observe that

$$T_{\varphi,n}E_{k,n}\big|_{H^2(\mathbb{T}^n)} = PE_{k,n}M_{\varphi(z_1^k, z_2^k, \ldots, z_n^k)}\big|_{H^2(\mathbb{T}^n)} = V_{\varphi(z_1^k, z_2^k, \ldots, z_n^k),k,n}.$$

The linearity of the mapping $\varphi \mapsto V_{\varphi,k,n}$ follows from the linearity of $A_{\varphi,k,n}$ and $P$. Further, we prove the following.

**Theorem 2.4.** *The linear correspondence $\varphi \mapsto V_{\varphi,k,n}$ is an injective mapping from the space $L^\infty(\mathbb{T}^n)$ to $B(H^2(\mathbb{T}^n))$, the space of all bounded operators on $H^2(\mathbb{T}^n)$.*

Proof. In order to prove the injectivity, assume that $V_{\varphi,k,n} = 0$. Then, for $(i_1, i_2, \ldots, i_n)$, $(j_1, j_2, \ldots, j_n) \in \mathbb{Z}_+^n$, we have

$$\begin{aligned} (2.1)\quad 0 &= \langle V_{\varphi,k,n}(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}),\ z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n}\rangle \\ &= \Big\langle \sum_{(m_1,\ldots,m_n)\in\mathbb{Z}_+^n} \varphi_{km_1-i_1,km_2-i_2,\ldots,km_n-i_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n},\ z_1^{j_1} z_2^{j_2} \ldots z_n^{j_n} \Big\rangle \\ &= \varphi_{kj_1-i_1,kj_2-i_2,\ldots,kj_n-i_n}. \end{aligned}$$

Now, for an arbitrary $n$-tuple $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n$, the substitution $j_t = |p_t|$ and the replacement $[k - \mathrm{sgn}(p_t)]|p_t|$ in place of $i_t$ for each integer $t$ such that $1 \leqslant t \leqslant n$ in the above expression give that

$$\varphi_{k|p_1|-[k-\mathrm{sgn}(p_1)]|p_1|,\ldots,k|p_n|-[k-\mathrm{sgn}(p_n)]|p_n|} = 0.$$

The function $\mathrm{sgn}(p)$, appearing in the above expression, is the sign or signum function. This reduces to $\varphi_{\mathrm{sgn}(p_1)|p_1|,\mathrm{sgn}(p_2)|p_2|,\ldots,\mathrm{sgn}(p_n)|p_n|} = 0$ for all $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n$. It yields that $\varphi_{p_1,p_2,\ldots,p_n} = 0$ for all $(p_1, p_2, \ldots, p_n) \in \mathbb{Z}^n$ and hence $\varphi = 0$. In the view of the above observation, the injectivity of the correspondence follows. □

An immediate corollary that follows from the above theorem is the following.

**Corollary 2.5.** *The operator $V_{\varphi,k,n}$ is the zero operator if and only if $\varphi = 0$.*

Primarily, we intend to have a necessary condition for a bounded operator on $H^2(\mathbb{T}^n)$ to be the compression of $k$th-order slant Toeplitz operator. Secondly, we provide a characterization for the compression of $k$th-order slant Toeplitz operator on $H^2(\mathbb{T}^n)$ for a special kind of inducing function.

**Theorem 2.6.** *Let $V \in B(H^2(\mathbb{T}^n))$ be a compression of $k$th-order slant Toeplitz operator on the space $H^2(\mathbb{T}^n)$. Then, it satisfies*

$$V = T^*_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n} V T_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n}, n} \quad \textit{for each } (p_1, p_2, \dots, p_n) \in \mathbb{Z}_+^n. \tag{2.2}$$

Proof. Let $V$ be a compression of $k$th-order slant Toeplitz operator, that is, $V = V_{\varphi,k,n}$ for some $\varphi \in L^\infty(\mathbb{T}^n)$ given by

$$\varphi(z_1, z_2, \dots, z_n) = \sum_{(m_1, m_2, \dots, m_n) \in \mathbb{Z}^n} \varphi_{m_1, m_2, \dots, m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}.$$

Now, for $(i_1, i_2, \dots, i_n), (p_1, p_2, \dots, p_n) \in \mathbb{Z}_+^n$, we get

$$\begin{aligned}
&T^*_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n} V T_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n}, n}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) \\
&\quad = T^*_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n}\Bigg[\sum_{(m_1, m_2, \dots, m_n) \in \mathbb{Z}_+^n} \varphi_{km_1 - i_1 - kp_1, \dots, km_n - i_n - kp_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}\Bigg] \\
&\quad = P\Bigg[\sum_{(m_1, m_2, \dots, m_n) \in \mathbb{Z}_+^n} \varphi_{km_1 - i_1 - kp_1, \dots, km_n - i_n - kp_n} z_1^{m_1 - p_1} z_2^{m_2 - p_2} \dots z_n^{m_n - p_n}\Bigg].
\end{aligned}$$

On replacing $m_j$ by $m_j + p_j$ for each integer $j$, $1 \leqslant j \leqslant n$, we obtain that

$$\begin{aligned}
&T^*_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n} V T_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n}, n}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) \\
&\qquad = P\Bigg[\sum_{m_j = -p_j, 1 \leqslant j \leqslant n}^{\infty} \varphi_{km_1 - i_1, km_2 - i_2, \dots, km_n - i_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}\Bigg] \\
&\qquad = \sum_{(m_1, m_2, \dots, m_n) \in \mathbb{Z}_+^n} \varphi_{km_1 - i_1, km_2 - i_2, \dots, km_n - i_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} \\
&\qquad = V_{\varphi,k,n}(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) = V(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}),
\end{aligned}$$

which furnishes the desired result. □

Now, we look for a condition which not only acts as a necessary condition but also as a sufficient condition for a bounded operator on $H^2(\mathbb{T}^n)$ to be the compression of $A_{\varphi,k,n}$ for some specific $\varphi \in L^\infty(\mathbb{T}^n)$. The following result uses the fact that $E_{k,n}[f(z_1^k, \dots, z_n^k)g] = f[E_{k,n}(g)]$ for $f, g \in L^2(\mathbb{T}^n)$ satisfying $fg \in L^2(\mathbb{T}^n)$, which is derived in Proposition 2.2 of [3].

**Theorem 2.7.** *A necessary and sufficient condition for a bounded operator $V$ on $H^2(\mathbb{T}^n)$ to be the compression of $k$th-order slant Toeplitz operator induced by* *the symbol*

$$\varphi(z_1,z_2,\ldots,z_n)=\sum_{m_j=-(k-1),1\leqslant j\leqslant n}^{\infty}\varphi_{m_1,m_2\ldots,m_n}z_1^{m_1}z_2^{m_2}\ldots z_n^{m_n}\in L^{\infty}(\mathbb{T}^n), \tag{2.3}$$

*is that* $T_{z_1^{p_1}z_2^{p_2}\ldots z_n^{p_n},n}V=VT_{z_1^{kp_1}z_2^{kp_2}\ldots z_n^{kp_n},n}$ *for each* $(p_1,p_2,\ldots,p_n)\in B_n$.

P r o o f. Let $V$ $(=V_{\varphi,k,n})$ be the compression of a $k$th-order slant Toeplitz operator induced by $\varphi\in L^{\infty}(\mathbb{T}^n)$, given by

$$\varphi(z_1,z_2,\ldots,z_n)=\sum_{m_j=-(k-1),1\leqslant j\leqslant n}^{\infty}\varphi_{m_1,m_2\ldots,m_n}z_1^{m_1}z_2^{m_2}\ldots z_n^{m_n}.$$

The above expression of $\varphi$ can be rewritten as

$$\varphi(z_1,z_2,\ldots,z_n)=\sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n}\varphi_{m_1,m_2\ldots,m_n}z_1^{m_1}z_2^{m_2}\ldots z_n^{m_n}$$

with the condition that $\varphi_{m_1,m_2,\ldots,m_n}=0$ if $m_j\leqslant -k$ for some integer $j$, $1\leqslant j\leqslant n$. Now, for each $(i_1,i_2,\ldots,i_n)\in\mathbb{Z}_+^n$ and $(0,0,\ldots,0)\neq(p_1,p_2,\ldots,p_n)\in\mathbb{Z}_+^n$, the above form of $\varphi$ yields that

$$\begin{aligned}VT_{z_1^{kp_1}z_2^{kp_2}\ldots z_n^{kp_n},n}(z_1^{i_1}z_2^{i_2}\ldots z_n^{i_n})&=V_{\varphi,k,n}(z_1^{i_1+kp_1}z_2^{i_2+kp_2}\ldots z_n^{i_n+kp_n})\\&=\sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}_+^n}\varphi_{km_1-i_1-kp_1,\ldots,km_n-i_n-kp_n}z_1^{m_1}z_2^{m_2}\ldots z_n^{m_n}\\&=\sum_{m_j=p_j,1\leqslant j\leqslant n}^{\infty}\varphi_{km_1-i_1-kp_1,\ldots,km_n-i_n-kp_n}z_1^{m_1}z_2^{m_2}\ldots z_n^{m_n}\\&\quad+\sum_{\substack{(m_1,\ldots,m_n)\in\mathbb{Z}_+^n,\\ \text{at least one } m_{j_0}\leqslant p_{j_0}-1,\\ 1\leqslant j_0\leqslant n \text{ for which } p_{j_0}\neq 0}}\varphi_{km_1-i_1-kp_1,\ldots,km_n-i_n-kp_n}z_1^{m_1}\ldots z_n^{m_n}\\&=\sum_{m_j=p_j,1\leqslant j\leqslant n}^{\infty}\varphi_{km_1-i_1-kp_1,\ldots,km_n-i_n-kp_n}z_1^{m_1}z_2^{m_2}\ldots z_n^{m_n}.\end{aligned} \tag{2.4}$$

Again, for $(i_1,i_2,\ldots,i_n)\in\mathbb{Z}_+^n$ and $(p_1,p_2,\ldots,p_n)\in\mathbb{Z}_+^n$, we get

$$\begin{aligned}T_{z_1^{p_1}\ldots z_n^{p_n},n}V(z_1^{i_1}z_2^{i_2}\ldots z_n^{i_n})&=T_{z_1^{p_1}\ldots z_n^{p_n},n}V_{\varphi,k,n}(z_1^{i_1}z_2^{i_2}\ldots z_n^{i_n})\\&=T_{z_1^{p_1}\ldots z_n^{p_n},n}\Biggl[\sum_{(m_1,\ldots,m_n)\in\mathbb{Z}_+^n}\varphi_{km_1-i_1,\ldots,km_n-i_n}z_1^{m_1}\ldots z_n^{m_n}\Biggr]\\&=\sum_{(m_1,\ldots,m_n)\in\mathbb{Z}_+^n}\varphi_{km_1-i_1,\ldots,km_n-i_n}z_1^{m_1+p_1}z_2^{m_2+p_2}\ldots z_n^{m_n+p_n}.\end{aligned}$$

Replacing $m_j$ by $m_j - p_j$ for $1 \leqslant j \leqslant n$ in the above expression, we get

$$(2.5)\quad T_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n} V(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) = \sum_{m_j = p_j, 1 \leqslant j \leqslant n}^{\infty} \varphi_{km_1 - i_1 - kp_1, \dots, km_n - i_n - kp_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}.$$

The equations (2.5) and (2.4) apparently provide that

$$T_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n} V = V T_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n}, n}$$

for each $(0, \dots, 0) \neq (p_1, p_2, \dots, p_n) \in \mathbb{Z}_+^n$. Also, for $(p_1, \dots, p_n) = (0, \dots, 0)$, the preceding relation is vacuously satisfied. Hence, $T_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n} V = V T_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n}, n}$ for each $(p_1, p_2, \dots, p_n) \in \mathbb{Z}_+^n$ and in particular for $(p_1, p_2, \dots, p_n) \in B_n$.

Conversely, suppose that $V$ is an operator on $H^2(\mathbb{T}^n)$ which satisfies

$$T_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n} V = V T_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n}, n}$$

for all $(p_1, p_2, \dots, p_n) \in B_n$. It is easy to verify that the preceding condition also holds for all $(p_1, p_2, \dots, p_n) \in \mathbb{Z}_+^n$. Let $f \in H^2(\mathbb{T}^n)$ be of the form

$$\begin{aligned} f(z_1, z_2, \dots, z_n) &= \sum_{(m_1, m_2, \dots, m_n) \in \mathbb{Z}_+^n} f_{m_1, m_2, \dots, m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} \\ &= \sum_{(m_1, m_2, \dots, m_n) \in \mathbb{Z}_+^n} f_{m_1, m_2, \dots, m_n} T_{z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}}(1)(z_1, z_2, \dots, z_n). \end{aligned}$$

For each $i_j \in \{0, 1, 2, \dots, k-1\}$, $1 \leqslant j \leqslant n$ and $f \in H^2(\mathbb{T}^n)$, the condition $T_{z_1^{p_1} z_2^{p_2} \dots z_n^{p_n}, n} V = V T_{z_1^{kp_1} z_2^{kp_2} \dots z_n^{kp_n}, n}$ helps to conclude that

$$\begin{aligned} (2.6)\quad & V[z_1^{i_1} z_2^{i_2} \dots z_n^{i_n} f(z_1^k, z_2^k, \dots, z_n^k)](z_1, z_2, \dots, z_n) \\ &= V\Bigg[\sum_{(m_1, \dots, m_n) \in \mathbb{Z}_+^n} f_{m_1, \dots, m_n} T_{z_1^{km_1 + i_1} z_2^{km_2 + i_2} \dots z_n^{km_n + i_n}}(1)\Bigg](z_1, \dots, z_n) \\ &= \sum_{(m_1, \dots, m_n) \in \mathbb{Z}_+^n} f_{m_1, \dots, m_n} [T_{z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}} V(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n})](z_1, \dots, z_n) \\ &= \sum_{(m_1, \dots, m_n) \in \mathbb{Z}_+^n} f_{m_1, \dots, m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} [V(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n})](z_1, \dots, z_n) \\ &= f(z_1, z_2, \dots, z_n) V[z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}](z_1, z_2, \dots, z_n). \end{aligned}$$

Let $\varphi_{i_1, i_2, \dots, i_n}$ represent the function $V(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n})$ for each $i_j \in \{0, 1, 2, \dots, k-1\}$, $1 \leqslant j \leqslant n$. Ultimately, we intend to prove that each function $\varphi_{i_1, i_2, \dots, i_n}$ belongs to $L^\infty(\mathbb{T}^n)$. Further, equation (2.6) gives that

$$V(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n} \cdot h) = f \cdot V(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}) = f \cdot \varphi_{i_1, i_2, \dots, i_n},$$

where $h(z_1, z_2, \dots, z_n) = f(z_1^k, z_2^k, \dots, z_n^k)$. The above expression provides that

$$\|f \cdot \varphi_{i_1,i_2,\dots,i_n}\|_2^2 = \|V(z_1^{i_1} z_2^{i_2} \dots z_n^{i_n} \cdot h)\|_2^2 \leqslant \|V\|^2 \|f\|_2^2 < \infty,$$

which implies that $f \cdot \varphi_{i_1,i_2,\dots,i_n} \in H^2(\mathbb{T}^n)$. Therefore, by the above observation and the solution of Problems 50 and 53 of [6], each function $\varphi_{i_1,i_2,\dots,i_n}$ belongs to the space $L^\infty(\mathbb{T}^n)$ for all $i_j \in \{0, 1, 2, \dots, k-1\}$ and $1 \leqslant j \leqslant n$.

Now we aim to construct a function $\varphi$ using these functions $\varphi_{i_1,i_2,\dots,i_n}$ so that $\varphi \in L^\infty(\mathbb{T}^n)$ and $V = V_{\varphi,k,n}$. For this, consider the function

$$\varphi = \sum_{i_1,i_2,\dots,i_n=0}^{k-1} \overline{e_{i_1,i_2,\dots,i_n}} g_{i_1,i_2,\dots,i_n},$$

where

$$\overline{e_{i_1,i_2,\dots,i_n}}(z_1, \dots, z_n) = \overline{z}_1^{i_1} \overline{z}_2^{i_2} \dots \overline{z}_n^{i_n}$$

and

$$g_{i_1,\dots,i_n}(z_1, \dots, z_n) = \varphi_{i_1,i_2,\dots,i_n}(z_1^k, \dots, z_n^k).$$

Thus, it yields the desired form of $\varphi$ as

$$\varphi(z_1, z_2, \dots, z_n) = \sum_{i_1,i_2,\dots,i_n=0}^{k-1} \overline{z}_1^{i_1} \overline{z}_2^{i_2} \dots \overline{z}_n^{i_n} \varphi_{i_1,i_2,\dots,i_n}(z_1^k, z_2^k, \dots, z_n^k),$$

which is an element of the space $L^\infty(\mathbb{T}^n)$.

Now we are left to prove that $V = V_{\varphi,k,n}$. For, let $f$ be an arbitrary element of $H^2(\mathbb{T}^n)$ given by

$$f(z_1, z_2, \dots, z_n) = \sum_{(m_1,m_2,\dots,m_n)\in\mathbb{Z}_+^n} f_{m_1,m_2,\dots,m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}.$$

We express $f$ as

$$f(z_1, z_2, \dots, z_n) = \sum_{i_1,i_2,\dots,i_n=0}^{k-1} z_1^{i_1} z_2^{i_2} \dots z_n^{i_n} \tilde{f}_{i_1,i_2,\dots,i_n}(z_1^k, z_2^k, \dots, z_n^k),$$

where

$$\begin{aligned} h_{i_1,\dots,i_n}(z_1, \dots, z_n) &= \tilde{f}_{i_1,\dots,i_n}(z_1^k, \dots, z_n^k) \\ &= \sum_{(m_1,\dots,m_n)\in\mathbb{Z}_+^n} f_{km_1+i_1,\dots,km_n+i_n} z_1^{km_1} \dots z_n^{km_n}. \end{aligned}$$

These expressions of $f$ and $\varphi$ along with Proposition 2.2 of [3] and relation (2.6) provide that

$$\begin{aligned}
V_{\varphi,k,n}f(z_1,z_2,\dots,z_n) &= PE_{k,n}M_\varphi f(z_1,z_2,\dots,z_n)\\
&= PE_{k,n}\{\varphi\cdot f\}(z_1,z_2,\dots,z_n)\\
&= PE_{k,n}\Bigg[\sum_{i_1,i_2,\dots,i_n=0}^{k-1} g_{i_1,i_2,\dots,i_n}\cdot h_{i_1,i_2\dots,i_n}\\
&\qquad + \begin{cases}\text{other terms which cannot be generated by the set}\\ \{z_1^{km_1}\dots z_n^{km_n} \text{ or } e_{km_1,\dots,km_n}\colon\ (m_1,m_2,\dots,m_n)\in\mathbb{Z}_+^n\}\end{cases}\Bigg](z_1,z_2\dots,z_n)\\
&= P\Bigg[\sum_{i_1,i_2,\dots,i_n=0}^{k-1}\tilde{f}_{i_1,i_2,\dots,i_n}\cdot\varphi_{i_1,i_2,\dots,i_n}\Bigg](z_1,z_2,\dots,z_n)\\
&= \sum_{i_1,i_2,\dots,i_n=0}^{k-1}\tilde{f}_{i_1,i_2,\dots,i_n}(z_1,z_2,\dots,z_n)V[z_1^{i_1}z_2^{i_2}\dots z_n^{i_n}](z_1,z_2,\dots,z_n)\\
&= \sum_{i_1,i_2,\dots,i_n=0}^{k-1}(V\{\tilde{f}_{i_1,\dots,i_n}(z_1^k,z_2^k,\dots,z_n^k)z_1^{i_1}z_2^{i_2}\dots z_n^{i_n}\})(z_1,z_2\dots,z_n)\\
&= V\Bigg[\sum_{i_1,i_2,\dots,i_n=0}^{k-1} z_1^{i_1}z_2^{i_2}\dots z_n^{i_n}\tilde{f}_{i_1,\dots,i_n}(z_1^k,z_2^k,\dots,z_n^k)\Bigg](z_1,z_2\dots,z_n)\\
&= Vf(z_1,z_2\dots,z_n).
\end{aligned}$$

Thus, we have $V=V_{\varphi,k,n}$ for $\varphi\in L^\infty(\mathbb{T}^n)$. This completes the proof. □

The proof of the above theorem suggests the following without any extra effort.

**Theorem 2.8.** *A bounded operator $V$ on $H^2(\mathbb{T}^n)$ is the compression of $k$th-order slant Toeplitz operator with symbol $\varphi$ given in* (2.3) *if and only if it satisfies*

$$T_{z_1^{p_1}z_2^{p_2}\dots z_n^{p_n},n}V = VT_{z_1^{kp_1}z_2^{kp_2}\dots z_n^{kp_n},n} \quad \textit{for each } (p_1,p_2,\dots,p_n)\in\mathbb{Z}_+^n. \tag{2.7}$$

It is important to note that the characterizations provided in Theorems 2.7 and 2.8 are valid only for the compression of $k$th-order slant Toeplitz operators that are induced by symbols given in (2.3). We can see that the compressions may fail to satisfy the characterizations given in above theorems. For choose $\varphi=z_1^{-k}$ and $V=V_{\varphi,k,n}$. Then $\varphi\in L^\infty(\mathbb{T}^n)$ but is not of the form given in (2.3). Clearly $V$ is a bounded operator on $H^2(\mathbb{T}^n)$ and is the compression of $k$th-order slant Toeplitz operator with symbol $\varphi$. For $(p_1,p_2,\dots,p_n)=(1,0,0,\dots,0)$, the expressions $VT_{z_1^{kp_1}\dots z_n^{kp_n},n}$ and

$T_{z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}, n} V$ are given by

$$V_{z_1^{-k},k,n} T_{z_1^k,n}(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}) = \begin{cases} z_1^{i_1/k} z_2^{i_2/k} \ldots z_n^{i_n/k} & \text{if each } i_j \text{ is a multiple of } k, \\ & 1 \leqslant j \leqslant n, \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_{z_1,n} V_{z_1^{-k},k,n}(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}) = T_{z_1,n} P \begin{cases} z_1^{i_1/k-1} z_2^{i_2/k} \ldots z_n^{i_n/k} & \text{if each } i_j \text{ is a multiple} \\ & \text{of } k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for $i_1 = 0$, $i_2 = k$, $i_3 = i_4 = \ldots = i_n = 0$, the above expressions show that

$$T_{z_1,n} V_{z_1^{-k},k,n}(z_1^{i_1} \ldots z_n^{i_n}) = 0 \neq z_2 = V_{z_1^{-k},k,n} T_{z_1^k,n}(z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}).$$

This justifies that the operator $V$ fails to satisfy the characterizations obtained in Theorems 2.7 and 2.8.

**Remark 2.9.** It is evident to see that any bounded operator $V$ on $H^2(\mathbb{T}^n)$ satisfying (2.7) satisfies (2.2). However, the above example proves that the converse is not true.

It can be shown that a Toeplitz operator $T_{\varphi,n}$ on $H^2(\mathbb{T}^n)$ is compact if and only if $\varphi = 0$. In order to prove this, consider $f \in H^2(\mathbb{T}^n)$, which is given by

$$f(z_1, z_2, \ldots, z_n) = \sum_{(m_1, m_2, \ldots, m_n) \in \mathbb{Z}_+^n} \langle f, e_{m_1,m_2,\ldots,m_n} \rangle e_{m_1,m_2,\ldots,m_n}(z_1, \ldots, z_n)$$

and satisfies $\sum_{(m_1,\ldots,m_n) \in \mathbb{Z}_+^n} |\langle f, e_{m_1,m_2,\ldots,m_n} \rangle|^2 < \infty$. As a consequence of the absolute convergence of the preceding series, one can conclude that for each $f \in H^2(\mathbb{T}^n)$, $\langle f, e_{m_1,m_2,\ldots,m_n} \rangle$ converges to 0 as each $m_i \to \infty$ for $1 \leqslant i \leqslant n$. This means that the sequence $\{e_{m_1,m_2,\ldots,m_n}\}$ converges to 0 weakly as each $m_i \to \infty$ for $1 \leqslant i \leqslant n$. Since $T_{\varphi,n}$ is compact, it follows that $T_{\varphi,n}(e_{m_1,m_2,\ldots,m_n}) \to 0$ strongly as all $m_i$'s approach to $\infty$. Now, for given $(i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n$, we construct two $n$-tuples $(p_1, p_2, \ldots, p_n)$, $(q_1, q_2, \ldots, q_n) \in \mathbb{Z}_+^n$ such that

$$p_j = \begin{cases} 0 & \text{if } i_j \geqslant 0, \\ -i_j & \text{if } i_j < 0 \end{cases} \quad \text{and} \quad q_j = \begin{cases} i_j & \text{if } i_j \geqslant 0, \\ 0 & \text{if } i_j < 0 \end{cases} \quad \text{for } 1 \leqslant j \leqslant n.$$

Clearly, we have $i_j = q_j - p_j$ for $1 \leqslant j \leqslant n$. Now,

$$\begin{aligned}|\varphi_{q_1-p_1,q_2-p_2,\ldots,q_n-p_n}| &= |\langle T_{\varphi,n}(z_1^{p_1+m} z_2^{p_2+m} \ldots z_n^{p_n+m}), z_1^{q_1+m} z_2^{q_2+m} z_n^{q_n+m}\rangle| \\ &\leqslant \|T_{\varphi,n}(e_{p_1+m,p_2+m,\ldots,p_n+m})\| \to 0 \quad \text{as } m \to \infty.\end{aligned}$$

It shows that $\varphi_{i_1,i_2,\ldots,i_n} = 0$ and hence $\varphi = 0$.

Now we investigate the connections between the compression of $k$th-order slant Toeplitz operators and Toeplitz operators. Further, we also extract the inducing function $\varphi \in L^\infty(\mathbb{T}^n)$ for $V_{\varphi,k,n}$ to be a compact operator. The following theorem uses a relation $E_{k,n} M_\varphi E_{k,n}^* = M_{E_{k,n}(\varphi)}$, which can be seen by applying operators on basis elements and is shown in [3].

**Theorem 2.10.** *For $\varphi \in L^\infty(\mathbb{T}^n)$, the following conclusion can be made:*
(1) $E_{k,n} V_{\varphi,k,n}^* = T_{E_{k,n}(\bar{\varphi}),n}$.
(2) *If $\varphi$ is co-analytic then $V_{\varphi,k,n} V_{\varphi,k,n}^* = T_{E_{k,n}(|\varphi|^2),n}$.*
(3) *$V_{\varphi,k,n}$ is compact if and only if $\varphi = 0$.*

Proof. (1) For $\varphi \in L^\infty(\mathbb{T}^n)$, in the view of proof of the Lemma 3.12 of [3], one can observe that

$$\begin{aligned}E_{k,n} V_{\varphi,k,n}^* &= E_{k,n} P A_{\varphi,k,n}^*|_{H^2(\mathbb{T}^n)} = P E_{k,n} M_{\bar{\varphi}} E_{k,n}^*|_{H^2(\mathbb{T}^n)} \\ &= P M_{E_{k,n}(\bar{\varphi})}|_{H^2(\mathbb{T}^n)} = T_{E_{k,n}(\bar{\varphi}),n}.\end{aligned}$$

(2) Suppose that $\varphi$ is co-analytic. Then, again by the Lemma 3.12 of [3], we obtain that

$$\begin{aligned}V_{\varphi,k,n} V_{\varphi,k,n}^* &= P A_{\varphi,k,n} P A_{\varphi,k,n}^*|_{H^2(\mathbb{T}^n)} = P E_{k,n} M_\varphi P M_{\bar{\varphi}} E_{k,n}^*|_{H^2(\mathbb{T}^n)} \\ &= P E_{k,n} M_{|\varphi|^2} E_{k,n}^*|_{H^2(\mathbb{T}^n)} = T_{E_{k,n}(|\varphi|^2),n}.\end{aligned}$$

(3) Assume that $V_{\varphi,k,n}$ is a compact operator for $\varphi \in L^\infty(\mathbb{T}^n)$, given by

$$\varphi(z_1, z_2, \ldots, z_n) = \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n} \varphi_{m_1,m_2,\ldots,m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}.$$

Then this implies that $E_{k,n}(V_{\varphi,k,n} T_{z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n},n})^*$ is also compact operator for $p_j \in \{0, 1, 2, \ldots, k-1\}$ with $1 \leqslant j \leqslant n$. Now, by using the part (1) of the theorem, we get that

$$E_{k,n}(V_{\varphi,k,n} T_{z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n},n})^* = E_{k,n} V_{(z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}\varphi),k,n}^* = T_{E_{k,n}\overline{(z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}\varphi)},n}.$$

The preceding expression provides that $T_{E_{k,n}\overline{(z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}\varphi)}}$ is compact operator and hence $E_{k,n}\overline{(z_1^{p_1} z_2^{p_2} \ldots z_n^{p_n}\varphi)} = 0$, for all $p_j \in \{0, 1, 2, \ldots, k-1\}$ with $1 \leqslant j \leqslant n$

because of the observation made on the above proposition. Consequently, we obtain that

$$\sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n} \bar{\varphi}_{km_1-p_1,km_2-p_2,\ldots,km_n-p_n} \overline{z}_1^{m_1}\overline{z}_2^{m_2}\ldots\overline{z}_n^{m_n} = 0,$$

which implies that $\bar{\varphi}_{km_1-p_1,km_2-p_2,\ldots,km_n-p_n} = 0$ for each integer $p_j$ such that $0 \leqslant p_j \leqslant k-1$, $1 \leqslant j \leqslant n$. Hence, we get $\varphi = 0$. □

The following theorem points out the condition on inducing function so that the product of the Toeplitz operator and compression of $k$th-order slant Toeplitz operator is again a compression of $k$th-order slant Toeplitz operator.

**Theorem 2.11.** *Let $\varphi$ and $\psi$ be two elements of the space $L^\infty(\mathbb{T}^n)$. Then the following statements are true.*

(1) *If either $\bar{\varphi}$ or $\psi$ is analytic then $V_{\varphi,k,n}T_{\psi,n} = V_{\varphi\psi,k,n}$.*

(2) *If either $\overline{\psi}$ or $\varphi$ is analytic then $T_{\psi,n}V_{\varphi,k,n} = V_{\psi(z_1^k,z_2^k,\ldots,z_n^k)\varphi,k,n}$.*

Proof. In order to prove part (1), we initially claim that $T_{\varphi,n}T_{\psi,n} = T_{\varphi\psi,n}$ whenever either $\bar{\varphi}$ or $\psi$ is analytic. We also know that $T_{\varphi,n}T_{\psi,n} = PM_\varphi PM_\psi|_{H^2(\mathbb{T}^n)}$. If $\psi$ is analytic, then the preceding expression reduces to $T_{\varphi,n}T_{\psi,n} = PM_\varphi M_\psi|_{H^2(\mathbb{T}^n)} = T_{\varphi\psi,n}$. Again, if $\bar{\varphi}$ is analytic, then we can observe that

$$(T_{\varphi,n}T_{\psi,n})^* = PM_{\overline{\psi}}PM_{\bar{\varphi}}\big|_{H^2(\mathbb{T}^n)} = PM_{\overline{\psi}}M_{\bar{\varphi}}\big|_{H^2(\mathbb{T}^n)} = T^*_{\varphi\psi,n}.$$

From the above observation, we get the claim. Now, consider the expression

$$V_{\varphi,k,n}T_{\psi,n} = E_{k,n}T_{\varphi,n}T_{\psi,n} = V_{\varphi\psi,k,n},$$

which implies the desired result.

(2) Since either $\overline{\psi}$ or $\varphi$ is analytic. Therefore, from the observation made in part (1), we get that

$$T_{\psi,n}V_{\varphi,k,n} = T_{\psi,n}E_{k,n}T_{\varphi,n} = E_{k,n}T_{\psi(z_1^k,z_2^k,\ldots,z_n^k)\varphi,n} = V_{\psi(z_1^k,z_2^k,\ldots,z_n^k)\varphi,k,n}.$$

Hence, the result follows. □

The next theorem provides a necessary and sufficient condition for $V^*_{\varphi,k,n}$ to be an isometry.

**Theorem 2.12.** *The adjoint $V^*_{\varphi,k,n}$ of $V_{\varphi,k,n}$ is an isometry if and only if $\varphi$ is co-analytic and $E_{k,n}(|\varphi|^2) = 1$.*

P r o o f. Assume that $\varphi$ is co-analytic and $E_{k,n}(|\varphi|^2) = 1$. Then, by Theorem 2.10, we get that

$$V_{\varphi,k,n}V^*_{\varphi,k,n} = T_{E_{k,n}(|\varphi|^2),n} = I,$$

which implies that $V^*_{\varphi,k,n}$ is an isometry.

Conversely, suppose that $V^*_{\varphi,k,n}$ is an isometry for $\varphi \in L^\infty(\mathbb{T}^n)$, given by

$$\varphi(z_1, z_2, \ldots, z_n) = \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n} \varphi_{m_1,m_2,\ldots,m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n}.$$

Then we have $\|V^*_{\varphi,k,n}(f)\|_2 = \|f\|_2$ for all $f \in H^2(\mathbb{T}^n)$. In particular, if we choose $f(z_1, z_2, \ldots, z_n) = z_1^{i_1} z_2^{i_2} \ldots z_n^{i_n}$ for $(i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n_+$, we get

$$\begin{aligned} (2.8)\qquad 1 = \|f\|_2^2 &= \Bigl\| \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n_+} \bar{\varphi}_{ki_1-m_1,ki_2-m_2,\ldots,ki_n-m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} \Bigr\|_2^2 \\ &= \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n_+} |\bar{\varphi}_{ki_1-m_1,ki_2-m_2,\ldots,ki_n-m_n}|^2. \end{aligned}$$

On substituting $i_j = 0$ for each integer $1 \leqslant j \leqslant n$, relation (2.8) reduces to

$$(2.9)\qquad 1 = \sum_{(m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n_+} |\bar{\varphi}_{-m_1,-m_2,\ldots,-m_n}|^2.$$

Again, for $i_j \geqslant 1$, $1 \leqslant j \leqslant n$, relation (2.8) can be rewritten as

$$\begin{aligned} (2.10)\qquad 1 = &\sum_{\substack{\text{at least for one } j,\\ 0\leqslant m_j\leqslant ki_j-1,\\ (m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n_+}} |\bar{\varphi}_{ki_1-m_1,ki_2-m_2,\ldots,ki_n-m_n}|^2 \\ &+ \sum_{m_j=ki_j,1\leqslant j\leqslant n}^{\infty} |\bar{\varphi}_{ki_1-m_1,ki_2-m_2,\ldots,ki_n-m_n}|^2. \end{aligned}$$

On observing relations (2.9) and (2.10), one can conclude that

$$\sum_{\substack{\text{at least for one } j,\\ 0\leqslant m_j\leqslant ki_j-1,\\ (m_1,m_2,\ldots,m_n)\in\mathbb{Z}^n_+}} |\bar{\varphi}_{ki_1-m_1,ki_2-m_2,\ldots,ki_n-m_n}|^2 = 0$$

for all integers $i_j \geqslant 1$, $1 \leqslant j \leqslant n$. Consequently, this gives that

$$(2.11)\qquad \bar{\varphi}_{ki_1-m_1,ki_2-m_2,\ldots,ki_n-m_n} = 0$$

for all integers $i_j \geqslant 1$ and for each $(m_1, m_2, \dots, m_n) \in \mathbb{Z}_+^n$ provided that there is at least one $t$, $1 \leqslant t \leqslant n$ such that $0 \leqslant m_t \leqslant ki_t - 1$. Therefore, equation (2.11) provides that

$$\varphi_{m_1,m_2,\dots,m_n} = 0 \quad \text{for each } (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n \text{ such that at least one } m_j \geqslant 1.$$

Thus, we get that $\varphi$ is co-analytic. Again, using the Theorem 2.10, we get

$$V_{\varphi,k,n} V_{\varphi,k,n}^* = T_{E_{k,n}(|\varphi|^2),n} = I,$$

which implies that $E_{k,n}(|\varphi|^2) = 1$. This completes the proof. □

Now, we provide an illustration in support of the preceding theorem.

**Example 2.13.** Let $\varphi(z_1, z_2, \dots, z_n) = (\overline{z_1 z_2 \dots z_n} + 1)/\sqrt{2}$. Then, obviously, it is a co-analytic function in the space $L^\infty(\mathbb{T}^n)$ and

$$|\varphi(z_1, z_2, \dots, z_n)|^2 = \frac{z_1 z_2 \dots z_n + \overline{z_1 z_2 \dots z_n} + 2}{2},$$

which yields that $E_{k,n}(|\varphi(z_1, z_2, \dots, z_n)|^2) = 1$ and hence $V_{\varphi,k,n} V_{\varphi,k,n}^* = I$. This points out that $V_{\varphi,k,n}^*$ is an isometry. Thus, the Theorem 2.12 is satisfied for $\varphi(z_1, z_2, \dots, z_n) = (\overline{z_1 z_2 \dots z_n} + 1)/\sqrt{2}$.

## 3. Spectrum of $V_{\varphi,k,n}$

In this section, we focus on the investigation of the spectrum and spectral radius of the compression of $k$th-order slant Toeplitz operator. In order to attain our results in an $n$-dimensional structure, we adopt the methodology provided in [2], [5]. We shall show that the spectral radius of $V_{\varphi,k,n}$ is same as that of $A_{\varphi,k,n}$ for analytic or co-analytic $\varphi \in L^\infty(\mathbb{T}^n)$. Prior to the main theorem, initially we investigate a few prerequisites for the accomplishment of the main results and certain other consequences.

**Lemma 3.1.** *The operator $(I - P)M_{z_1 z_2 \dots z_n}^q$ converges to 0 strongly as $q \to \infty$, where $M_{z_1 z_2 \dots z_n}$ is the multiplication operator induced by $z_1 z_2 \dots z_n \in L^\infty(\mathbb{T}^n)$ and $P$ is the orthogonal projection from the space $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{T}^n)$.*

Proof. Let $f$ be a function of the space $L^2(\mathbb{T}^n)$, given by

$$f(z_1, z_2, \dots, z_n) = \sum_{(m_1, m_2, \dots, m_n) \in \mathbb{Z}^n} f_{m_1,m_2,\dots,m_n} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}.$$

Then, we observe that

$$\|(I-P)M^q_{z_1z_2\dots z_n}(f)\|^2 = \left\|(I-P)\Bigl(\sum_{(m_1,\dots,m_n)\in\mathbb{Z}^n} f_{m_1,m_2,\dots,m_n} z_1^{m_1+q} z_2^{m_2+q}\dots z_n^{m_n+q}\Bigr)\right\|^2$$
$$= \sum_{m_j=-\infty,1\leqslant j\leqslant n}^{-1} |f_{m_1-q,m_2-q,\dots,m_n-q}|^2 = \sum_{m_j=-\infty,1\leqslant j\leqslant n}^{-q-1} |f_{m_1,\dots,m_n}|^2.$$

Being $f$ in $L^2(\mathbb{T}^n)$,

$$\sum_{m_j=-\infty,1\leqslant j\leqslant n}^{0} |f_{m_1,m_2,\dots,m_n}|^2 \leqslant \sum_{m_j=-\infty,1\leqslant j\leqslant n}^{\infty} |f_{m_1,m_2,\dots,m_n}|^2 < \infty.$$

Therefore, by the definition of convergence of series, we can conclude that

$$\|(I-P)M^q_{z_1z_2\dots z_n}(f)\| \to 0 \quad \text{as } q\to\infty \text{ for all } f\in L^2(\mathbb{T}^n).$$

Hence, the result follows. □

The next outcome utilizes a theorem proved in [3], which states that a bounded operator $A$ is the $k$th-order slant Toeplitz operator if and only if $A = M^*_{z_1^{p_1} z_2^{p_2}\dots z_n^{p_n}} AM_{z_1^{kp_1} z_2^{kp_2}\dots z_n^{kp_n}}$ for all $(p_1,p_2,\dots,p_n)\in\mathbb{Z}^n$.

**Lemma 3.2.** *The operator $M^q_{\overline{z}_1\overline{z}_2\dots\overline{z}_n} V_{\varphi,k,n} P M^{kq}_{z_1z_2\dots z_n}$ converges to $A_{\varphi,k,n}$ as $q\to\infty$ in the strong operator topology.*

Proof. Initially, from the Lemma 3.1, we know that $(I-P)M^q_{z_1z_2\dots z_n}$ converges to 0 as $q\to\infty$ in the strong operator topology. Therefore, $M^q_{\overline{z}_1\overline{z}_2\dots\overline{z}_n}(I-P)M^q_{z_1z_2\dots z_n}$ also converges to 0 strongly as $q\to\infty$. Consequently, we obtained that $M^q_{\overline{z}_1\overline{z}_2\dots\overline{z}_n} PM^q_{z_1z_2\dots z_n} \to I$ strongly as $q\to\infty$. Now, we see that

$$M^q_{\overline{z}_1\overline{z}_2\dots\overline{z}_n} V_{\varphi,k,n} P M^{kq}_{z_1z_2\dots z_n} = M^q_{\overline{z}_1\overline{z}_2\dots\overline{z}_n} P A_{\varphi,k,n} P M^{kq}_{z_1z_2\dots z_n}$$
$$= (M^q_{\overline{z}_1\overline{z}_2\dots\overline{z}_n} P M^q_{z_1z_2\dots z_n})(M^q_{\overline{z}_1\overline{z}_2\dots\overline{z}_n} A_{\varphi,k,n} M^{kq}_{z_1z_2\dots z_n})(M^{kq}_{\overline{z}_1\overline{z}_2\dots\overline{z}_n} P M^{kq}_{z_1z_2\dots z_n}).$$

By the use of above observations and the characterization of the $k$th-order slant Toeplitz operator given in [3], the desired result follows. □

The following theorem derives the norm of $V_{\varphi,k,n}$ in term of inducing function. Moreover, it shows that the norms of $V_{\varphi,k,n}$ and $A_{\varphi,k,n}$ are equal. In order to prove this, we require Lemma 3.12 of [3], which proves that $\|A^m_{\varphi,k,n}\| = \|\psi_m\|^{1/2}_\infty$, where $\psi_m$ is given by

$$\psi_m = \underbrace{E_{k,n}(|\varphi|^2 E_{k,n}(|\varphi|^2 E_{k,n}(\dots E_{k,n}}_{m\text{-times}}(|\varphi|^2)\dots))). \tag{3.1}$$

**Theorem 3.3.** *Let $V_{\varphi,k,n}$ be a compression of the $k$th-order slant Toeplitz operator $A_{\varphi,k,n}$. Then, $\|V_{\varphi,k,n}\| = \|A_{\varphi,k,n}\| = \|E_{k,n}(|\varphi|^2)\|_\infty^{1/2}$.*

P r o o f. For each $q \in \mathbb{Z}_+ \setminus \{0\}$, we have $\|M^q_{\overline{z}_1\overline{z}_2\ldots\overline{z}_n} V_{\varphi,k,n} P M^{kq}_{z_1z_2\ldots z_n}\| \leqslant \|V_{\varphi,k,n}\|$. On the basis of Lemma 3.2 and the above expression, we conclude that $\|A_{\varphi,k,n}\| \leqslant \|V_{\varphi,k,n}\|$. Since $V_{\varphi,k,n}$ is a compression of $k$th-order slant Toeplitz operator $A_{\varphi,k,n}$, we get that $\|V_{\varphi,k,n}\| \leqslant \|A_{\varphi,k,n}\|$. Finally, in the view of Lemma 3.12 of the paper [3], this implies that $\|V_{\varphi,k,n}\| = \|A_{\varphi,k,n}\| = \|E_{k,n}(|\varphi|^2)\|_\infty^{1/2}$. □

The next theorem shows that the spectral radius $r(V_{\varphi,k,n})$ of $V_{\varphi,k,n}$ is same as that of $A_{\varphi,k,n}$ for co-analytic inducing function. But, subsequently, we shall prove that the following result is also true for analytic $\varphi \in L^\infty(\mathbb{T}^n)$.

**Theorem 3.4.** *If $\varphi \in L^\infty(\mathbb{T}^n)$ is co-analytic then $r(V_{\varphi,k,n}) = r(A_{\varphi,k,n})$.*

P r o o f. Let $\varphi \in L^\infty(\mathbb{T}^n)$ be a co-analytic function. Primarily, with the help of the principle of mathematical induction on "$m$", we prove that the relation $V^m_{\varphi,k,n} V^{*m}_{\varphi,k,n} = T_{\psi_m,n}$, where $\psi_m$ is same as defined in (3.1). For $m = 1$, we have already proved the desired relation in part (2) of the Theorem 2.10. Now, assume that the relation is true for all $j \leqslant m-1$. Again, in the view of assumption and Theorem 2.10, we have

$$\begin{aligned} V^m_{\varphi,k,n} V^{*m}_{\varphi,k,n} &= V_{\varphi,k,n} V^{(m-1)}_{\varphi,k,n} V^{*(m-1)}_{\varphi,k,n} V^*_{\varphi,k,n} = V_{\varphi,k,n} T_{\psi_{m-1},n} V^*_{\varphi,k,n} \\ &= E_{k,n} T_{|\varphi|^2\psi_{m-1},n} E^*_{k,n} = E_{k,n} V^*_{|\varphi|^2\psi_{m-1}} = T_{\psi_m,n}. \end{aligned}$$

The above expression gives that $\|V^m_{\varphi,k,n}\| = \|V^m_{\varphi,k,n} V^{*m}_{\varphi,k,n}\|^{1/2} = \|\psi_m\|_\infty^{1/2}$. With the help of Gelfand's formula and the Lemma 3.12 of the paper [3], the result follows. □

The following example illustrates the preceding theorem.

**Example 3.5.** For the function $\varphi(z_1, z_2, \ldots, z_n) = \overline{z}_1^k \overline{z}_2^k \ldots \overline{z}_n^k + 1$, the operator $V_{\varphi,k,n}$ satisfies the conclusion of Theorem 3.4. Also, for this function $\varphi$, $V_{\varphi,k,n}$ is not a normaloid.

P r o o f. The given function $\varphi$ is of the form $\varphi(z_1, z_2, \ldots, z_n) = \overline{z}_1^k \overline{z}_2^k \ldots \overline{z}_n^k + 1$, for a fixed integer $k \geqslant 2$. Clearly, $\varphi \in L^\infty(\mathbb{T}^n)$ and $\varphi$ is co-analytic. Now, consider $|\varphi(z_1, z_2, \ldots, z_n)|^2 = \overline{z}_1^k \overline{z}_2^k \ldots \overline{z}_n^k + z_1^k z_2^k \ldots z_n^k + 2$, which gives that

$$E_{k,n}(|\varphi|^2) = \overline{z}_1 \overline{z}_2 \ldots \overline{z}_n + z_1 z_2 \ldots z_n + 2.$$

Again, consider the following expression:

$$|\varphi|^2 E_{k,n}(|\varphi|^2) = 2(\overline{z}_1^k \overline{z}_2^k \ldots \overline{z}_n^k + z_1^k z_2^k \ldots z_n^k + 2) + \begin{cases} \text{other terms which cannot be} \\ \text{generated by terms having} \\ \text{exponent in the multiple of } k. \end{cases}$$

Subsequently, the above expression provides that

$$E_{k,n}(|\varphi|^2 E_{k,n}(|\varphi|^2)) = 2(\overline{z}_1\overline{z}_2\ldots\overline{z}_n + z_1z_2\ldots z_n + 2).$$

Similarly, one can obtain that

$$\begin{aligned}\psi_m &= \underbrace{E_{k,n}(|\varphi|^2 E_{k,n}(|\varphi|^2 E_{k,n}(\ldots E_{k,n}}_{m\text{-times}}(|\varphi|^2)\ldots)))\\ &= 2^{m-1}(\overline{z}_1\overline{z}_2\ldots\overline{z}_n + z_1z_2\ldots z_n + 2),\end{aligned}$$

which yields that $\|\psi_m\|_\infty = 2^{m+1}$.

By the use of Gelfand's formula for spectral radius, we get that

$$r(V_{\varphi,k,n}) = r(A_{\varphi,k,n}) = \lim_{m\to\infty}\|\psi_m\|_\infty^{1/2m} = \lim_{m\to\infty} 2^{(m+1)/2m} = \sqrt{2}. \tag{3.2}$$

Now, the norm of $V_{\varphi,k,n}$ is given by

$$\|V_{\varphi,k,n}\| = \|\psi_1\|_\infty^{1/2} = \|\overline{z}_1\overline{z}_2\ldots\overline{z}_n + z_1z_2\ldots z_n + 2\|_\infty^{1/2} = 2,$$

which implies that $r(V_{\varphi,k,n}) \neq \|V_{\varphi,k,n}\|$. Thus, we can conclude that the operator $V_{\varphi,k,n}$ may not be a normaloid in general. $\square$

**Theorem 3.6.** *If $\varphi \in L^\infty(\mathbb{T}^n)$ is analytic then $r(V_{\varphi,k,n}) = r(A_{\varphi,k,n})$.*

Proof. We know that $\|A^m_{\varphi,k,n}\| = \sup\limits_{\|f\|=1}\|A^m_{\varphi,k,n}(f)\|$, so for every $\varepsilon > 0$, we have

$$\|A^m_{\varphi,k,n}\| - \frac{\varepsilon}{2} \leqslant \|A^m_{\varphi,k,n}(f)\| \quad \text{for some } f \in L^2(\mathbb{T}^n) \text{ and } \|f\| = 1. \tag{3.3}$$

Also, the operator $A_{\varphi,k,n}$ satisfies the operator equation

$$M^q_{z_1z_2\ldots z_n}A_{\varphi,k,n} = A_{\varphi,k,n}M^{kq}_{z_1z_2\ldots z_n}.$$

Using it repeatedly, we get the following:

$$M^q_{z_1z_2\ldots z_n}A^m_{\varphi,k,n} = A^m_{\varphi,k,n}M^{k^mq}_{z_1z_2\ldots z_n}.$$

From the Lemma 3.1, we know that $(I-P)M^{k^mq}_{z_1z_2\ldots z_n}$ converges to 0 strongly as $q\to\infty$. Consequently, $A^m_{\varphi,k,n}(I-P)M^{k^mq}_{z_1z_2\ldots z_n}$ converges to 0 as $q\to\infty$ in the strong operator topology. Using the invertibility of $M^q_{z_1z_2\ldots z_n}$, the above observation brings out that

$$\begin{aligned}\big|\|A^m_{\varphi,k,n}(f)\| - \|A^m_{\varphi,k,n}PM^{k^mq}_{z_1z_2\ldots z_n}(f)\|\big| \\ &= \big|\|M^q_{z_1z_2\ldots z_n}A^m_{\varphi,k,n}(f)\| - \|A^m_{\varphi,k,n}PM^{k^mq}_{z_1z_2\ldots z_n}(f)\|\big| \\ &= \big|\|A^m_{\varphi,k,n}M^{k^mq}_{z_1z_2\ldots z_n}(f)\| - \|A^m_{\varphi,k,n}PM^{k^mq}_{z_1z_2\ldots z_n}(f)\|\big| \\ &\leqslant \|A^m_{\varphi,k,n}M^{k^mq}_{z_1z_2\ldots z_n}(f) - A^m_{\varphi,k,n}PM^{k^mq}_{z_1z_2\ldots z_n}(f)\| \to 0,\end{aligned}$$

as $q\to\infty$. With the help of the $\varepsilon-\delta$ definition of the limit, the above expression yields that

$$\|A^m_{\varphi,k,n}(f)\| - \|A^m_{\varphi,k,n}PM^{k^mq}_{z_1z_2\ldots z_n}(f)\| \leqslant \big|\|A^m_{\varphi,k,n}(f)\| - \|A^m_{\varphi,k,n}PM^{k^mq}_{z_1z_2\ldots z_n}(f)\|\big| < \frac{\varepsilon}{2}$$

for sufficiently large value of $q$. Equivalently, for sufficiently larger value of $q$, we have

$$\|A^m_{\varphi,k,n}(f)\| < \|A^m_{\varphi,k,n}PM^{k^mq}_{z_1z_2\ldots z_n}(f)\| + \frac{\varepsilon}{2}. \tag{3.4}$$

Let $g = PM^{k^mq}_{z_1z_2\ldots z_n}(f)$, clearly $g\in H^2(\mathbb{T}^n)$ and $\|g\|\leqslant 1$. In the view of (3.4) and $\varphi$ being analytic, relation (3.3) reduces to

$$\|A^m_{\varphi,k,n}\| < \|A^m_{\varphi,k,n}PM^{k^mq}_{z_1z_2\ldots z_n}(f)\| + \varepsilon = \|V^m_{\varphi,k,n}(g)\| + \varepsilon \leqslant \|V^m_{\varphi,k,n}\| + \varepsilon.$$

Since $\varepsilon>0$ is arbitrary, so $\|A^m_{\varphi,k,n}\|\leqslant\|V^m_{\varphi,k,n}\|$ for each integer $m\geqslant 0$. Also, the reverse inequality is trivial. Therefore, we get that $\|A^m_{\varphi,k,n}\| = \|V^m_{\varphi,k,n}\|$ and hence $r(V_{\varphi,k,n}) = r(A_{\varphi,k,n})$. This completes the proof. □

The subsequent example is to illustrate the preceding theorem.

**Example 3.7.** For the function $\varphi(z_1,z_2,\ldots,z_n) = 1 + z_1^{k-1}z_2^{k-1}\ldots z_n^{k-1}$ for a fixed integer $k\geqslant 2$, the corresponding $V_{\varphi,k,n}$ verifies the conclusion of the Theorem 3.6. Moreover, for the given $\varphi$, $V_{\varphi,k,n}$ is a normaloid.

P r o o f. Given that $\varphi(z_1,z_2,\ldots,z_n) = 1 + z_1^{k-1}z_2^{k-1}\ldots z_n^{k-1}$ for a fixed integer $k\geqslant 2$. Now, we get $|\varphi|^2 = 2 + z_1^{k-1}z_2^{k-1}\ldots z_n^{k-1} + \overline{z}_1^{(k-1)}\overline{z}_2^{(k-1)}\ldots\overline{z}_n^{(k-1)}$, which gives that $E_{k,n}(|\varphi|^2) = 2$. Similarly, one can have

$$\psi_m = \underbrace{E_{k,n}(|\varphi|^2E_{k,n}(|\varphi|^2E_{k,n}(\ldots E_{k,n}}_{m\text{-times}}(|\varphi|^2)\ldots))) = 2^m.$$

Then, the spectral radius $r(V_{\varphi,k,n})$ of $V_{\varphi,k,n}$ is given by

$$r(V_{\varphi,k,n}) = \lim_{m\to\infty}\|\psi_m\|_\infty^{1/2m} = \sqrt{2} = \|V_{\varphi,k,n}\|.$$

This shows that $V_{\varphi,k,n}$ is normaloid for the function $\varphi$ defined above. □

The next result establishes the relationship between the point spectrums of compression of $k$th-order slant Toeplitz operators.

**Lemma 3.8.** *Let $\varphi$ be a function in the space $L^\infty(\mathbb{T}^n)$. If $T_{\varphi,n}$ is invertible, then $\sigma_p(V_{\varphi,k,n}) = \sigma_p(V_{\varphi(z_1^k,z_2^k,\ldots,z_n^k),k,n})$. In fact, in this case $0 \in \sigma_p(V_{\varphi,k,n})$.*

P r o o f. Suppose that $\lambda$ is a nonzero element in $\sigma_p(V_{\varphi,k,n})$, the point spectrum of $V_{\varphi,k,n}$. Then, there exists a nonzero vector $f$ in $H^2(\mathbb{T}^n)$ such that $V_{\varphi,k,n}f = \lambda f$, i.e., $E_{k,n}T_{\varphi,n}(f) = \lambda f$. Since $T_{\varphi,n}$ is invertible and $f \neq 0$, therefore we have $T_{\varphi,n}f \neq 0$. Again, consider $T_{\varphi,n}E_{k,n}T_{\varphi,n}(f) = \lambda T_{\varphi,n}(f)$, which can be rewritten as

$$V_{\varphi(z_1^k,z_2^k,\ldots,z_n^k),k,n}(T_{\varphi,n}(f)) = \lambda T_{\varphi,n}(f).$$

Thus, the above expression provides that $\lambda \in \sigma_p(V_{\varphi(z_1^k,z_2^k,\ldots,z_n^k),k,n})$.

Conversely, assume that $0 \neq \lambda \in \sigma_p(V_{\varphi(z_1^k,z_2^k,\ldots,z_n^k),k,n})$. Then, there exists a nonzero element $g \in H^2(\mathbb{T}^n)$ such that $V_{\varphi(z_1^k,z_2^k,\ldots,z_n^k),k,n}(g) = \lambda g$. Equivalently, $T_{\varphi,n}E_{k,n}(g) = \lambda g$. Since $T_{\varphi,n}$ is invertible, this implies that $E_{k,n}g \neq 0$. Further, we get that

$$E_{k,n}T_{\varphi,n}(E_{k,n}(g)) = \lambda E_{k,n}g,$$

which yields that $\lambda \in \sigma_p(V_\varphi, k, n)$. Ultimately, we observe that

$$V_{\varphi(z_1^k,z_2^k,\ldots,z_n^k),k,n}(z_1z_2\ldots z_n) = PE_{k,n}(z_1z_2\ldots z_n\varphi(z_1^k, z_2^k, \ldots, z_n^k)) = 0$$

and

$$V_{\varphi,k,n}[T_{\varphi,n}^{-1}(z_1z_2\ldots z_n)] = E_{k,n}[T_{\varphi,n}T_{\varphi,n}^{-1}(z_1z_2\ldots z_n)] = 0.$$

From the preceding expressions, we can deduce that $0 \in \sigma_p(V_{\varphi(z_1^k,z_2^k,\ldots,z_n^k),k,n})$ and $0 \in \sigma_p(V_{\varphi,k,n})$. This completes the proof. $\square$

Now we investigate the spectrum of the compression of $k$th-order slant Toeplitz operator. More precisely, we show that a closed disc lies inside the spectrum of $V_{\varphi,k,n}$, whenever $T_{\varphi,n}$ is invertible.

**Theorem 3.9.** *Let the Toeplitz operator $T_{\varphi,n}$, $\varphi \in L^\infty(\mathbb{T}^n)$, be invertible. Then a closed disc is contained in the spectrum of $V_{\varphi,k,n}$, the compression of $k$th-order slant Toeplitz operator and the interior of the disc consists of eigenvalues with infinite multiplicity.*

P r o o f. Let $\lambda$ be a nonzero complex number and the operator $(E_{k,n}^*T_{\varphi,n}^{-1} - \lambda I)$ is onto. Then, for any $h \in H^2(\mathbb{T}^n)$, we get

$$(E_{k,n}^*T_{\varphi,n}^{-1} - \lambda I)h = (E_{k,n}^*T_{\varphi,n}^{-1} - \lambda P_k)(h) - \lambda(I - P_k)(h),$$

where $P_k$ is the projection of the space $H^2(\mathbb{T}^n)$ onto the closed subspace generated by the set $\{z_1^{km_1} z_2^{km_2} \dots z_n^{km_n} \colon m_i \in \mathbb{Z}_+, 1 \leqslant i \leqslant n\}$. Let $\widetilde{P}_k$ express $I - P_k$. By the assumption, for $0 \neq g \in \widetilde{P}_k(H^2(\mathbb{T}^n))$, there exists a nonzero function $f \in H^2(\mathbb{T}^n)$ such that $(E^*_{k,n} T^{-1}_{\varphi,n} - \lambda I)(f) = g$. Again, employing the fact that $0 \neq g \in \widetilde{P}_k(H^2(\mathbb{T}^n))$, one can see that $(E^*_{k,n} T^{-1}_{\varphi,n} - \lambda P_k)(f) = 0$. Equivalently, we obtain that

$$\lambda E^*_{k,n} T^{-1}_{\varphi,n} (\lambda^{-1} - T_{\varphi,n} E_{k,n})(f) = 0. \tag{3.5}$$

Given that $T_{\varphi,n}$ is invertible and $\lambda \neq 0$. Also, we know that $E^*_{k,n}$ is an isometry and $T_{\varphi,n} E_{k,n} = V_{\varphi(z_1^k, z_2^k, \dots, z_n^k),k,n}$. From (3.5), we conclude that

$$(\lambda^{-1} - V_{\varphi(z_1^k, z_2^k, \dots, z_n^k),k,n})(f) = 0,$$

which gives that $\lambda^{-1} \in \sigma_p(V_{\varphi(z_1^k, z_2^k, \dots, z_n^k),k,n})$.

Now, let $\lambda \in \varrho(E^*_{k,n} T^{-1}_{\varphi,n})$, the resolvent of the operator $(E^*_{k,n} T^{-1}_{\varphi,n})$. Then, the operator $(E^*_{k,n} T^{-1}_{\varphi,n} - \lambda I)$ is invertible and hence onto. Therefore, in the view of the above discussion, we get that

$$D = \{\lambda^{-1} \colon \lambda \in \varrho(E^*_{k,n} T^{-1}_{\varphi,n})\} \subset \sigma_p(V_{\varphi(z_1^k, z_2^k, \dots, z_n^k),k,n}).$$

With the help of preceding Lemma 3.8, we obtain that $D \subset \sigma_p(V_{\varphi,k,n})$. The resolvent and the spectrum of a bounded operator are respectively open and compact subsets of the complex plane. Therefore, $D$ is open and contains a open disc. By the compactness of spectrum, one can conclude that the spectrum of $V_{\varphi,k,n}$ contains a closed disc. From the above discussion, it follows that for a fixed $\lambda \in D$, i.e., $(\lambda^{-1} \in \varrho(E^*_{k,n} T^{-1}_{\varphi,n}))$ and for nonzero $g \in \widetilde{P}_k H^2(\mathbb{T}^n)$, there exists nonzero $f \in H^2(\mathbb{T}^n)$ such that $(\lambda - V_{\varphi(z_1^k, z_2^k, \dots, z_n^k),k,n})(f) = 0$. It means that $f$ is an eigenvector of $V_{\varphi(z_1^k, z_2^k, \dots, z_n^k),k,n}$ corresponding to eigenvalue $\lambda$. Hence, taking the invertibility of $T_{\varphi,n}$ into consideration, the observation made in Lemma 3.8 yields that $E_{k,n}(f)$ is an eigenvector of $V_{\varphi,k,n}$ corresponding to eigenvalue $\lambda$. Since $\dim[\widetilde{P}_k(H^2(\mathbb{T}^n))] = \infty$ and $\sigma_p(V_{\varphi,k,n}) = \sigma_p(V_{\varphi(z_1^k, z_2^k, \dots, z_n^k),k,n})$, we can conclude that each $\lambda \in D$ is an eigenvalue of $V_{\varphi,k,n}$ with infinite multiplicity. □

**Remark 3.10.** The radius of the closed disc contained in the spectrum $\sigma(V_{\varphi,k,n})$ is equal to $[r(E^*_{k,n} T^{-1}_{\varphi,n})]^{-1}$ if $T_{\varphi,n}$ is invertible.

P r o o f. Let

$$D_0 = \{0\} \cup \{\lambda^{-1} \colon \lambda \in \varrho(E^*_{k,n} T^{-1}_{\varphi,n})\} \supseteq \{0\} \cup \{\lambda^{-1} \colon |\lambda| > r(E^*_{k,n} T^{-1}_{\varphi,n})\}.$$

Let $r_0 = [r(E^*_{k,n} T^{-1}_{\varphi,n})]^{-1}$. Then, clearly $D_0 \supseteq B(0, r_0)$, where $B(0, r_0)$ is the open ball in $\mathbb{C}$. Also, we know that $D_0 \subset \sigma_p(V_{\varphi,k,n}) \subset \sigma(V_{\varphi,k,n})$. Therefore, the radius of the closed disc which is contained in the spectrum $\sigma(V_{\varphi,k,n})$, is equal to $[r(E^*_{k,n} T^{-1}_{\varphi,n})]^{-1}$. Moreover, $r(V_{\varphi,k,n}) \geqslant [r(E^*_{k,n} T^{-1}_{\varphi,n})]^{-1}$. □

**Corollary 3.11.** *If $\varphi$ is unimodular, then $r(V_{\varphi,k,n}) = r(V_{\varphi^{-1},k,n}) = 1$. In particular, if $\varphi$ is an inner function, then $r(V_{\varphi,k,n}) = r(V_{\varphi^{-1},k,n}) = 1$.*

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