Commentationes Mathematicae Universitatis Carolinae

Chèfiath A. Adegbindin; Alain Togbé
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Commentationes Mathematicae Universitatis Carolinae, Vol. 61 (2020), No. 3, 383–396

Persistent URL: http://dml.cz/dmlcz/148473

Comment.Math.Univ.Carolin. 61,3 (2020) 383–396

Can a Lucas number be a sum of three repdigits?

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Abstract. We give the answer to the question in the title by proving that

$$L_{18} = 5778 = 5555 + 222 + 1$$

is the largest Lucas number expressible as a sum of exactly three repdigits. Therefore, there are many Lucas numbers which are sums of three repdigits.

Keywords: Pell equation; repdigit; linear forms in complex logarithms

Classification: 11A25, 11B39, 11J86

1. Introduction

Let $\{L_m\}_{m\geq 0}$ be the sequence of Lucas numbers given by $L_{m+2} = L_{m+1} + L_m$ for $m \geq 0$, where $L_0 = 2$ and $L_1 = 1$. A few terms of this sequence are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521,$$
$$843, 1364, 2207, 3571, 5778, 9349, 15127, 24476, \ldots$$

The Binet formula for its general term is

$$L_m = \alpha^m + \beta^m \tag{1}$$

for all $m \geq 0$, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$ are the two roots of the characteristic equation $x^2 - x - 1 = 0$.

In this paper, we study the Diophantine equations

$$L_n = d_1\Big(\frac{10^{m_1}-1}{9}\Big) + d_2\Big(\frac{10^{m_2}-1}{9}\Big) + d_3\Big(\frac{10^{m_3}-1}{9}\Big) \tag{2}$$

for some integers $m_1 \leq m_2 \leq m_3$ and $d_1, d_2, d_3 \in \{1, 2, \ldots, 9\}$.

F. Luca and various co-authors have considered similar problems to the one addressed in this paper. The papers [9] and [7] give all Fibonacci, Lucas, Pell and Pell–Lucas numbers that are repdigits. The paper [5] gives all Fibonacci numbers that are sums of two repdigits, while the paper [1] provides all Pell and Pell–Lucas numbers that are sums of three repdigits. For other related problems, one can refer to [2], [3], [6], [8]–[12].

DOI 10.14712/1213-7243.2020.028

Our main result is the following.

Theorem 1.1. *The largest Lucas number which is a sum of exactly three repdigits is*

$$L_{18} = 5778 = 5555 + 222 + 1.$$

Remark. In fact, the only Lucas numbers that are sums of three repdigits are given in Table 1. The representations are not unique.

$5778 = 5555 + 222 + 1$
$843 = 666 + 111 + 66$
$521 = 333 + 111 + 77$
$322 = 222 + 99 + 1$
$199 = 111 + 77 + 11$
$123 = 99 + 22 + 2$
$76 = 66 + 9 + 1$
$47 = 44 + 2 + 1$
$29 = 22 + 5 + 2$
$18 = 11 + 5 + 2$
$11 = 7 + 3 + 1$
$7 = 4 + 2 + 1$
$4 = 2 + 1 + 1$
$3 = 1 + 1 + 1$

TABLE 1. All solutions of equation (2).

In the next section, we prove the above theorem in three parts. In the first part, we use a computational method to prove that there is no solution to the problem for $n \in [19, 1000]$. Moreover, we get an estimate of n in terms of m_3. The second part consists in the use of Baker's method to bound n, m_1, m_2, m_3. For that, we apply a result due to E. M. Matveev concerning a lower bound of linear forms of logarithms of algebraic numbers. In the last part, we complete the proof of the theorem by reducing the bounds obtained for n, m_1, m_2, m_3. To do this, we use a version of the Baker–Davenport reduction given by B. M. M. de Weger in [14].

2. Proof of Theorem 1.1

2.1 An elementary estimate. We assume that

$$L_n = d_1\Big(\frac{10^{m_1}-1}{9}\Big) + d_2\Big(\frac{10^{m_2}-1}{9}\Big) + d_3\Big(\frac{10^{m_3}-1}{9}\Big) \tag{3}$$

for some integers $m_1 \le m_2 \le m_3$ and $d_1, d_2, d_3 \in \{1, 2, \ldots, 9\}$. A quick computation with Maple reveals no solutions in the interval $n \in [19, 1000]$. For this computation, we first noted that L_{1000} has 209 digits. Thus, we generated the list of all numbers which are sums of at most 2 repdigits with at most 209 digits each, let us call it $\mathcal{A}$. Then, for every $n \in [19, 1000]$, we computed $M := \lfloor \log L_n / \log 10 \rfloor + 1$ (the number of digits of L_n) and then checked whether $L_n - d(10^m - 1)/9$ is a member of $\mathcal{A}$ for some digit $d \in \{1, \ldots, 9\}$ and some $m \in \{M-1, M\}$. This computation took a few minutes.

So, from now on, we may assume that $n > 1000$.

We next investigate the size of m_1, m_2, m_3 versus n.

Lemma 2.1. *All solutions of equation* (2) *satisfy*

$$m_3 \log 10 - 4 < n \log\alpha < m_3 \log 10 + 3.$$

Proof: The proof follows easily from the fact that $\alpha^{n-1} < L_n < \alpha^{n+1}$. One can see that

$$\alpha^{n-1} < L_n < 3 \cdot 10^{m_3}.$$

Taking the logarithm on both sides, we get $(n-1)\log\alpha < \log 3 + m_3 \log 10$, which leads to

$$n \log\alpha < \log\alpha + \log 3 + m_3 \log 10 < m_3 \log 10 + 3.$$

Similarly, the lower bound follows. □

2.2 Bounds of n, m_1, m_2, m_3. To find bounds for n, m_1, m_2, m_3, we will use Baker's method. So we need a result from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. Thus, we recall here Theorem 9.4 of [4], which is a modified version of a result of E. M. Matveev [13]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1, \eta_2, \ldots, \eta_l \in \mathbb{L}$ not 0 or 1 and $b_1, \ldots, b_l$ be nonzero integers. We put

$$D = \max\{|b_1|, \ldots, |b_l|\},$$

and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{b_i} - 1.$$

Let $A_1, \ldots, A_l$ be positive integers such that

$$A_j \geq h'(\eta_j) := \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\} \qquad \text{for } \ j = 1, \ldots, l,$$

where for an algebraic number η of minimal polynomial

$$f(X) = a_0(X - \eta^{(1)}) \cdots (X - \eta^{(k)}) \in \mathbb{Z}[X]$$

over the integers with positive a_0, we write $h(\eta)$ for its Weil height given by

$$h(\eta) = \frac{1}{k}\left(\log a_0 + \sum_{j=1}^{k} \max\{0, \log |\eta^{(j)}|\}\right).$$

The following consequence of Matveev's theorem is Theorem 9.4 in [4].

Theorem 2.1. *If* $\Gamma \neq 0$ *and* $\mathbb{L} \subseteq \mathbb{R}$, *then*

$$\log |\Gamma| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 \cdots A_l.$$

To apply this result, we return to equation (2) and use the Binet formula (1) to get

$$\alpha^n + \beta^n = d_1\Big(\frac{10^{m_1} - 1}{9}\Big) + d_2\Big(\frac{10^{m_2} - 1}{9}\Big) + d_3\Big(\frac{10^{m_3} - 1}{9}\Big).$$

The equation (2) can be expressed

$$9(\alpha^n + \beta^n) - d_1 10^{m_1} - d_2 10^{m_2} - d_3 10^{m_3} = -(d_1 + d_2 + d_3). \tag{4}$$

We examine (4) in three different steps as follows.

Step 1: Equation (4) gives

$$9\alpha^n - d_3 10^{m_3} = d_1 10^{m_1} + d_2 10^{m_2} - 9\beta^n - (d_1 + d_2 + d_3), \tag{5}$$

which we rewrite as

$$|9\alpha^n - d_3 10^{m_3}| = |d_1 10^{m_1} + d_2 10^{m_2} - 9\beta^n - (d_1 + d_2 + d_3)| < 54 \cdot 10^{m_2}.$$

Thus, dividing both sides by $d_3 10^{m_3}$, we get

$$\Big|\Big(\frac{9}{d_3}\Big)\alpha^n 10^{-m_3} - 1\Big| < \frac{54}{10^{m_3 - m_2}}. \tag{6}$$

Let

$$\Gamma_1 := \Big(\frac{9}{d_3}\Big)\alpha^n 10^{-m_3} - 1. \tag{7}$$

Suppose that $\Gamma_1 = 0$. Then, we have

$$\alpha^n = \frac{d_3 10^{m_3}}{9}.$$

Conjugating in $\mathbb{Q}(\alpha)$, we get

$$\beta^n = \frac{d_3 10^{m_3}}{9}.$$

Consequently, we obtain

$$\frac{10^{m_3}}{9} \leq \frac{d_3 10^{m_3}}{9} = |\beta|^n < 1,$$

which leads to $10^{m_3}/9 < 1$ which is false. Thus, $\Gamma_1 \neq 0$. With the notations of Theorem 2.1, we take

$$\eta_1 = \frac{9}{d_3}, \qquad \eta_2 = \alpha, \qquad \eta_3 = 10, \qquad b_1 = 1, \qquad b_2 = n, \qquad b_3 = -m_3.$$

Since $10^{m_3-1} < L_n < \alpha^{n+1}$, we have that $m_3 \leq n$. Therefore, we can take $D = n$. Observe that $\mathbb{L} := \mathbb{Q}(\eta_1, \eta_2, \eta_3) = \mathbb{Q}(\alpha)$, so $d_{\mathbb{L}} = 2$. We now need to take A_j for $j = 1, 2, 3$ such that

$$A_j \geq \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}.$$

Note that

$$h(\eta_1) \leq h(9) + h(d_3) \leq h(9) + h(9) \leq 2h(9).$$

This implies that

$$2h(\eta_1) < 8.8.$$

Thus, we can take

$$A_1 = 8.8.$$

Clearly,

$$h(\eta_2) = \frac{1}{2}\log\alpha, \qquad h(\eta_3) = \log(10).$$

We have

$$\max\{2h(\eta_2), |\log \eta_2|, 0.16\} = \log(\alpha) < 0.49 := A_2, \tag{8}$$

$$\max\{2h(\eta_3), |\log \eta_3|, 0.16\} = 2\log(10) < 4.7 := A_3. \tag{9}$$

We apply Theorem 2.1 to obtain

$$\log|\Gamma_1| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (6) leads to

$$(m_3 - m_2)\log(10) < \log(54) + 1.97 \cdot 10^{13}(1 + \log n),$$

giving

$$m_3 - m_2 < 8.6 \cdot 10^{12}(1 + \log n). \tag{10}$$

Step 2: Equation (4) becomes

$$9\alpha^n - d_3 10^{m_3} - d_2 10^{m_2} = d_1 10^{m_1} - 9\beta^n - (d_1 + d_2 + d_3), \tag{11}$$

which we rewrite as

$$\begin{aligned}|9\alpha^n - 10^{m_2}(d_3 10^{m_3-m_2} + d_2)| &= |d_1 10^{m_1} - 9\beta^n - (d_1 + d_2 + d_3)| \\ &< 45 \cdot 10^{m_1}.\end{aligned}$$

Thus, dividing both sides by $10^{m_2}(d_3 10^{m_3-m_2} + d_2)$, we get

$$\Big|\Big(\frac{9}{d_3 10^{m_3-m_2} + d_2}\Big)\alpha^n 10^{-m_2} - 1\Big| < \frac{45}{10^{m_2-m_1}}. \tag{12}$$

Let

$$\Gamma_2 := \Big(\frac{9}{d_3 10^{m_3-m_2} + d_2}\Big)\alpha^n 10^{-m_2} - 1. \tag{13}$$

Suppose that $\Gamma_2 = 0$. Then, we have

$$\alpha^n = \frac{d_2 10^{m_2}}{9} + \frac{d_3 10^{m_3}}{9}.$$

Conjugating in $\mathbb{Q}(\alpha)$, we get

$$\beta^n = \frac{d_2 10^{m_2}}{9} + \frac{d_3 10^{m_3}}{9}.$$

Consequently, we obtain

$$\frac{10^{m_3}}{9} \leq \frac{d_2 10^{m_2}}{9} + \frac{d_3 10^{m_3}}{9} = |\beta|^n < 1,$$

the same contradiction as when we assumed that $\Gamma_1 = 0$. Thus, $\Gamma_2 \neq 0$. To apply Theorem 2.1, we take

$$\eta_1 = \frac{9}{d_3 10^{m_3-m_2} + d_2}, \quad \eta_2 = \alpha, \quad \eta_3 = 10, \quad b_1 = 1, \quad b_2 = n, \quad b_3 = -m_2.$$

Again we take $D = n$. Furthermore, we have

$$\begin{aligned}h(\eta_1) &= h\Big(\frac{9}{d_3 10^{m_3-m_2} + d_2}\Big) \\ &\leq h(9) + h(d_3 10^{m_3-m_2} + d_2) \\ &\leq h(9) + h(d_3) + h(d_2) + (m_3 - m_2)h(10) + \log 2 \\ &\leq 7.3 + 2.4(m_3 - m_2).\end{aligned}$$

That is,

$$2h(\eta_1) < 14.6 + 4.8(m_3 - m_2).$$

Thus, we take

$$A_1 = 14.6 + 4.8(m_3 - m_2).$$

Since η_2, η_3 are the same as in Γ_1, we use the same values for A_2, A_3. From Theorem 2.1, we obtain

$$\log|\Gamma_2| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (12) leads to

$$(m_2 - m_1)\log(10) < \log(45) + 2.3 \cdot 10^{12}(14.6 + 4.8(m_3 - m_2))(1 + \log n).$$

Hence, using inequality (10), we obtain

$$(m_2 - m_1)\log(10) - \log(45) < 2.3 \cdot 10^{12}\big(14.6 + 4.8(8.6 \cdot 10^{12}(1 + \log n))\big) \times (1 + \log n).$$

The above inequality gives us

$$m_2 - m_1 < 4.21 \cdot 10^{25}(1 + \log n)^2. \tag{14}$$

Step 3: Equation (4) becomes

$$9\alpha^n - d_3 10^{m_3} - d_2 10^{m_2} - d_1 10^{m_1} = -9\beta^n - (d_1 + d_2 + d_3), \tag{15}$$

which we rewrite as

$$\left|\alpha^n - 10^{m_3}\frac{d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_3} + d_3}{9}\right| = \left| -\beta^n - \frac{d_1 + d_2 + d_3}{9}\right| < 4.$$

Thus, dividing both sides by α^n, we get

$$\left|1 - \alpha^{-n} 10^{m_3}\frac{d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_3} + d_3}{9}\right| < \frac{1}{\alpha^{n-2.9}}. \tag{16}$$

Put

$$\Gamma_3 := 1 - \alpha^{-n} 10^{m_3}\frac{d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_3} + d_3}{9}. \tag{17}$$

The fact that $\Gamma_3 \neq 0$ can be justified by a similar argument as the fact that $\Gamma_1 \neq 0$. In order to apply Theorem 2.1, we take

$$\eta_1 = \frac{d_2 10^{m_2 - m_3} + d_1 10^{m_1 - m_3} + d_3}{9}, \qquad \eta_2 = \alpha, \qquad \eta_3 = 10,$$
$$b_1 = 1, \qquad b_2 = -n, \qquad b_3 = m_3.$$

We have $D = n$, and A_2 and A_3 are as in (8) and (9). As for A_1, we have

$$\begin{aligned} h(\eta_1) &= h\Big(\frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3}{9}\Big) \\ &\le h\Big(\frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_2} + d_3}{9}\Big) \\ &\le h(9) + h(d_2 10^{m_2-m_3} + d_1 10^{m_1-m_2} + d_3) \\ &\le h(9) + h(d_1) + h(d_2) + h(d_3) + (m_3 - m_2)h(10) \\ &\quad + (m_2 - m_1)h(10) + 2\log 2 \\ &\le 10.2 + 2.4(m_3 - m_2) + 2.4(m_2 - m_1). \end{aligned}$$

That is,

$$2h(\eta_1) < 20.4 + 4.8(m_3 - m_2) + 4.8(m_2 - m_1).$$

Thus, we can take

$$A_1 = 20.4 + 4.8(m_3 - m_2) + 4.8(m_2 - m_1).$$

Theorem 2.1 tells us that

$$\log|\Gamma_4| > -1.4 \cdot 30^{l+3} l^{4.5} d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 A_2 A_3.$$

Comparing this last inequality with (16) leads to

$$n\log(\alpha) - \log(4) < 2.3 \cdot 10^{12}(20.4 + 4.8(m_3 - m_2) + 4.8(m_2 - m_1))(1 + \log n).$$

Hence, using inequality (10) and (14), we obtain

$$\begin{aligned} n\log(\alpha) - \log(\alpha^{2.9}) < 2.3 \cdot 10^{12}(&20.4 + 4.8(8.6 \cdot 10^{12}(1 + \log n)) \\ &+ 4.8(4.21 \cdot 10^{25}(1 + \log n)^2))(1 + \log n). \end{aligned}$$

The above inequality gives us

$$n < 4.8233 \cdot 10^{41}.$$

Lemma 2.1 implies

$$m_1 \le m_2 \le m_3 < 1.0080 \cdot 10^{41}.$$

We summarize what we have proved so far in the following lemma.

Lemma 2.2. *All solutions of equation* (2) *satisfy*

$$m_1 \le m_2 \le m_3 < 1.0080 \cdot 10^{41}, \qquad n < 4.8233 \cdot 10^{41}.$$

2.3 Reducing the bound. As the above bounds are high, we need to reduce them by using a reduction method. Here, we present a variant of the reduction method of Baker and Davenport due to B. M. M. de Weger [14].

Let $\vartheta_1, \vartheta_2, \beta \in \mathbb{R}$ be given, and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \beta + x_1\vartheta_1 + x_2\vartheta_2. \tag{18}$$

Let c, δ be positive constants. Set $X = \max\{|x_1|, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y), \tag{19}$$

$$X \le X_0. \tag{20}$$

We put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$ We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 2.3 (see Lemma 3.2 in [14]). *Let*

$$A = \max_{0 \le k \le Y_0} a_{k+1}.$$

If (19) *and* (20) *hold for* x_1, x_2 *and* $\beta = 0$, *then*

$$Y < \frac{1}{\delta} \log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right). \tag{21}$$

When $\beta \neq 0$ in (18), we put $\psi = \beta/\vartheta_2$. Then we have

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x we let $\|x\| = \min\{|x-n| \colon n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 2.4 (see Lemma 3.3 in [14]). *Suppose that*

$$\|q\psi\| > \frac{2X_0}{q}.$$

Then, the solutions of (19) *and* (20) *satisfy*

$$Y < \frac{1}{\delta} \log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

Now, we are ready to lower the above bounds. Thus, we return to equation (2) We rewrite it into the form

$$L_n = \frac{d_3 10^{m_3}}{9} + \left(d_1 \frac{10^{m_1}-1}{9} + d_2 \frac{10^{m_2}-1}{9} - \frac{d_3}{9}\right).$$

Observe that the term in parentheses is always positive as

$$\left(d_1 \frac{10^{m_1}-1}{9} + d_2 \frac{10^{m_2}-1}{9} - \frac{d_3}{9}\right) \geq 2\frac{10^{m_1}-1}{9} - \frac{1}{9} \geq 2 - \frac{1}{9} \geq \frac{7}{4} > 0.$$

Hence, we have

$$\alpha^n - \frac{d_3 10^{m_3}}{9} = \left(d_1 \frac{10^{m_1}-1}{9} + d_2 \frac{10^{m_2}-1}{9} - \frac{d_3}{9}\right) - \beta^n \geq \frac{7}{4} - \frac{1}{\alpha^{1000}} > 0.$$

Thus, the number Γ_1 from (7) appearing inside the absolute value in inequality (6) is positive. Hence, with the above notations, we have

$$\alpha^n - \frac{d_3 10^{m_3}}{9} = \frac{d_3 10^{m_3}}{9}(\mathrm{e}^{\Lambda_1} - 1) > 0.$$

Let

$$\Lambda_1 = n \log \eta_2 - m_3 \log \eta_3 + \log \eta_1.$$

Therefore, we obtain

$$0 < \Lambda_1 < \exp(\Lambda_1) - 1 = \Gamma_1 < \frac{54}{10^{m_3 - m_2}},$$

which implies that

$$\begin{aligned} 0 < \log\left(\frac{9}{d_3}\right) + m_3(-\log 10) + n \log \alpha &< \frac{54}{10^{m_3-m_2}} \\ &< 10^{1.74} \exp(-2.30 \cdot (m_3 - m_2)). \end{aligned}$$

Thus

$$\Lambda_1 < 10^{1.74} \exp(-2.30 \cdot (m_3 - m_2)),$$

with $Y := m_3 - m_2 < 1.0080 \cdot 10^{41}$.

Therefore, to apply Lemma 2.4 we take

$$c = 10^{1.74}, \qquad \delta = 2.3, \qquad X_0 = 1.0080 \cdot 10^{41}, \qquad \psi = \frac{\log(9/d_3)}{\log 10},$$

$$\vartheta = -\frac{\log \alpha}{\log 10}, \qquad \vartheta_1 = -\log \alpha, \qquad \vartheta_2 = \log 10, \qquad \beta = \log(9/d_3).$$

The smallest value of $q > X_0$ is $q = q_{86}$. We find that q_{90} satisfies the hypothesis of Lemma 2.4. Applying Lemma 2.4, we get $m_3 - m_2 \leq 46$ (over all the values of $d_3 \neq 9$).

When $d_3 = 9$, we get that $\beta = 0$. The largest partial quotient a_k for $0 \leq k \leq 197$ is $a_{139} = 770$. Applying Lemma 2.3, $m_3 - m_2 = Y < m_3 \leq X_0 := 1.0080 \cdot 10^{41}$ implies that

$$m_3 - m_2 < \frac{1}{2.3} \log\left(\frac{10^{1.74}(770+2) \cdot 1.0080 \cdot 10^{41}}{|\log 10|}\right),$$

We obtain $m_3 - m_2 \leq 45$, so we get the same conclusion as before, namely that $m_3 - m_2 \leq 46$.

We now take $0 \leq m_3 - m_2 \leq 46$. Let

$$\Lambda_2 = n \log \eta_2 - m_2 \log \eta_3 + \log \eta_1.$$

From equation (4), we have that

$$\begin{aligned} \frac{d_3 10^{m_3} + d_2 10^{m_2}}{9}(e^{\Lambda_2} - 1) &= -\beta^n + d_1 \frac{10^{m_1} - 1}{9} - \left(\frac{d_3 + d_2}{9}\right) \\ &> -\frac{(-1)^n}{\alpha^n} + \frac{10^{m_1}}{9} - \frac{1}{3}. \end{aligned}$$

Furthermore, we get

$$-\frac{(-1)^n}{\alpha^n} + \frac{10^{m_1}}{9} - \frac{1}{3} > -\frac{1}{\alpha^n} + \frac{7}{9} > -\frac{1}{\alpha^{1000}} + \frac{7}{9} > 0.$$

Thus, we have

$$e^{\Lambda_2} - 1 > 0.$$

So, from (11) we see that

$$\alpha^n - \frac{d_3 10^{m_3}}{9} - \frac{d_2 10^{m_2}}{9} = \left(\frac{d_3 10^{m_3}}{9} + \frac{d_2 10^{m_2}}{9}\right)(e^{\Lambda_2} - 1) > 0,$$

then

$$0 < \Lambda_2 < e^{\Lambda_2} - 1 = \Gamma_2 < \frac{45}{10^{m_2 - m_1}},$$

which implies that

$$\begin{aligned} 0 &< \log\left(\frac{9}{d_3 10^{m_3 - m_2} + d_2}\right) + m_2(-\log 10) + n \log \alpha \\ &< \frac{45}{10^{m_2 - m_1}} < 10^{1.66} \exp(-2.30 \cdot (m_2 - m_1)). \end{aligned}$$

Thus, we get

$$\Lambda_2 < 10^{1.66} \exp(-2.30 \cdot (m_2 - m_1)),$$

with $Y := m_2 - m_1 < 1.0080 \cdot 10^{41}$.

Therefore, in order to apply Lemma 2.4 we take

$$c = 10^{1.66}, \qquad \delta = 2.3, \qquad X_0 = 1.0080 \cdot 10^{41}, \qquad \psi = \frac{\log(9/(d_3 10^{m_3-m_2} + d_2))}{\log 10},$$
$$\vartheta = -\frac{\log\alpha}{\log 10}, \qquad \vartheta_1 = -\log\alpha, \qquad \vartheta_2 = \log 10, \qquad \beta = \log\Big(\frac{9}{d_3 10^{m_3-m_2} + d_2}\Big).$$

We get $q = q_{96} > X_0$. By applying Lemma 2.4, over all the possibilities for the digits $d_2, d_3 \in \{1, \dots, 9\}$ and $m_3 - m_2 \in \{0, \dots, 46\}$ except for $m_3 = m_2$ and $d_2 + d_3 = 9$, we get $m_2 - m_1 \leq 51$.

In the exceptional cases $m_3 = m_2$ and $d_3 + d_2 = 9$, one actually gets that $\beta = 0$, and the largest partial quotient a_k for $0 \leq k \leq 197$ is $a_{139} = 770$. Applying Lemma 2.3 with $m_2 - m_1 = Y < m_2 \leq X_0 := 1.0080 \cdot 10^{41}$,

$$m_2 - m_1 < \frac{1}{2.3}\log\Big(\frac{10^{1.66}(770+2)\cdot 1.0080 \cdot 10^{41}}{|\log 10|}\Big),$$

we obtain $m_2 - m_1 \leq 45$. So we get the same conclusion as before, namely that $m_2 - m_1 \leq 51$.

We now take $0 \leq m_3 - m_1 \leq 97$ and $0 \leq m_3 - m_2 \leq 46$. Let

$$\Lambda_3 = m_3 \log\eta_3 - n\log\eta_2 + \log\eta_1.$$

From equation (4), we have that

$$\alpha^n(1 - e^{\Lambda_3}) = -\beta^n - \frac{d_1 + d_2 + d_3}{9} = -\Big(\beta^n + \frac{d_1 + d_2 + d_2}{9}\Big).$$

Furthermore,
$$\beta^n + \frac{d_1 + d_2 + d_3}{9} > -\frac{1}{\alpha^n} + \frac{1}{3} > -\frac{1}{\alpha^{1000}} + \frac{1}{3} > 0.$$
Thus,
$$\mathrm{e}^{\Lambda_3} - 1 > 0.$$
So,
$$0 < \Lambda_3 < \mathrm{e}^{\Lambda_3} - 1 = |\Gamma_3| < \frac{4}{\alpha^n} < \frac{1}{\alpha^{n-2.9}}.$$
which implies that

$$0 < \log\Big(\frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3}{9}\Big) + m_3\log 10 + n(-\log\alpha)$$
$$< \frac{4}{\alpha^n} < \alpha^{2.9}\exp(-0.48\cdot n).$$

We keep the value for $X_0 = 4.8233 \cdot 10^{41}$, and only change ψ to

$$\psi' = \frac{\log\big((d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3)/9\big)}{\log 10}, \quad c = \alpha^{2.9}, \quad \delta = 0.48, \quad v = \frac{\log\alpha}{\log 10},$$

$$v_1 = \log\alpha, \qquad v_2 = \log 10, \qquad \beta = \log\Big(\frac{d_2 10^{m_2-m_3} + d_1 10^{m_1-m_3} + d_3}{9}\Big).$$

We get $q = q_{99} > X_0$ and by Lemma 2.4, we get $n \le 263$. This holds for all choices of $d_1, d_2, d_3 \in \{1, \dots, 9\}$, $m_3 - m_2 \in [0, 46]$ and $m_3 - m_1 \in [0, 97]$ except when $m_1 = m_2 = m_3$, $m_1 = m_2 = m_3 + 1$, $d_1 + d_2 = 10$, $d_3 = 8$ and $d_1 + d_2 + d_3 = 9$.

For the exceptional cases $m_3 = m_2$, $m_3 = m_1$, $m_1 = m_2 = m_3 + 1$, $d_1 + d_2 = 10$, $d_3 = 8$ and $d_1 + d_2 + d_3 = 9$, one actually gets that $\beta = 0$, so the largest partial quotient a_k for $0 \le k \le 201$ is $a_{138} = 770$. Applying again Lemma 2.3 with $n = Y < m_1 \le X_0 := 4.8233 \cdot 10^{41}$,

$$n < \frac{1}{0.48}\log\Big(\frac{\alpha^{2.9}(770+2)\cdot 4.8233\cdot 10^{41}}{|\log 10|}\Big),$$

we obtain $n \le 214$, so we get the same conclusion as before, namely that $n \le 263$. But this contradicts the assumption that $n > 1000$. Hence, the theorem is proved.

Acknowledgement. The authors thank the referee for a careful reading of the manuscript and for comments which improved its quality. The second author is partially supported by Purdue University Northwest.

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Ch. A. Adegbindin:
Institut de Mathématiques et de Science Physiques, Quartier Avakpa,
BP 613, Porto-Novo, Dangbo, Bénin

E-mail: adegbindinchefiath@gmail.com

A. Togbé:
Department of Mathematics, Statistics and Computer Science,
Purdue University Northwest, 1401 S, U.S. 421, Westville, Indiana,
IN 46391, USA

E-mail: atogbe@pnw.edu

(*Received* March 8, 2019*, revised* August 6, 2019)