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DIRICHLET BOUNDARY VALUE PROBLEM FOR AN IMPULSIVE FORCED PENDULUM EQUATION WITH VISCOUS AND DRY FRICTIONS

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Abstract. Sufficient conditions are given for the solvability of an impulsive Dirichlet boundary value problem to forced nonlinear differential equations involving the combination of viscous and dry frictions. Apart from the solvability, also the explicit estimates of solutions and their derivatives are obtained. As an application, an illustrative example is given, and the corresponding numerical solution is obtained by applying Matlab software.

Keywords: impulsive Dirichlet problem; Kakutani-Ky Fan fixed-point theorem; pendulum equation; dry friction

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1. Introduction

Let us consider the forced pendulum equation with dry friction and viscous damping term together with Dirichlet boundary conditions

$$\ddot{x}(t) + a\dot{x}(t) + b\sin x(t) + c\operatorname{sgn}\dot{x}(t) = h(t) \quad \text{for a.a. } t \in [0,T], \tag{1.1}$$

$$x(0) = x(T) = 0, \tag{1.2}$$

where a, b and c are real constants and the function $h\colon [0,T] \to \mathbb{R}$ plays the role of the forcing term.

Moreover, let a finite number of points $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$, $p \in \mathbb{N}$, be given.

In the paper, the solvability of the dry friction problem with the Dirichlet boundary conditions (1.1), (1.2) will be investigated in the presence of the following impulse

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conditions:

$$\begin{cases} x(t_i^+) - x(t_i) = J_i(x(t_i)), & i = 1, \dots, p, \\ \dot{x}(t_i^+) - \dot{x}(t_i^-) = M_i(x(t_i)), & i = 1, \dots, p, \end{cases} \tag{1.3}$$

where the notation $\lim\limits_{t\to a^+} x(t) = x(a^+)$, $\lim\limits_{t\to a^-} x(t) = x(a^-)$ is used and J_i, $M_i \in C(\mathbb{R}, \mathbb{R})$ for all $i = 1, \dots, p$.

The study of the forced mathematical pendulum equation

$$\ddot{x}(t) + b \sin x(t) = h(t) \quad \text{for a.a. } t \in [0, T] \tag{1.4}$$

(i.e. the case $b \neq 0$, $a = c = 0$ in (1.1)) was studied as far back as a century ago (see [8] for the case $b > 0$ and [6] for $b < 0$). It was shown that it is worth to consider Dirichlet boundary conditions, since the symmetry of the equation imply that such solutions are related to periodic solutions. It is known that the Dirichlet problem corresponding to the forced mathematical pendulum equation (1.4) is solvable under very mild assumptions, e.g. if p is measurable and satisfies $\int_0^T |p(t)|\,\mathrm{d}t < \infty$.

More recently, the pendulum equation was generalized introducing a nonzero viscous damping coefficient a or a nonzero friction coefficient c (see [1], [10], [12] for more details about this topic). As is shown in the mentioned papers, the problems in the generalized versions are formed mainly by the nonzero friction coefficient c; in such a case the criteria guaranteeing the existence of a solution are much more complicated (because of the discontinuous r.h.s.) than in the simplest case of the forced mathematical pendulum equation.

All of the mentioned papers and monographs studied the pendulum equation with prescribed boundary conditions, but without impulses. A few years ago, the attention started to be paid also to the impulse forced mathematical pendulum equation problem, since the impulses can model a rapid changes in evolution processes. Recently, in papers [4], [9], [14], an impulse problem was considered in the case $a = c = 0$, and in the presence of periodic boundary conditions.

As far as we know, the impulsive problem to the pendulum equation with a nonzero viscous damping coefficient a as well as with a nonzero friction coefficient c has not been studied yet. Therefore, the aim of the present paper is to study the forced pendulum equation with dry friction and viscous damping term (1.1) together with Dirichlet boundary conditions (1.2) in the case when the impulses described by (1.3) are involved.

Let us point out that because of discontinuity at $y = 0$ in $\operatorname{sgn} y$, we should consider a Filippov solutions of (1.1) which can be identified as Carathéodory solutions of the following inclusion with a Filippov regularized right-hand side (see [5])

$$\ddot{x}(t) + a\dot{x}(t) + b \sin x(t) \in h(t) - c \operatorname{Sgn} \dot{x}(t), \tag{1.5}$$

where

$$\operatorname{Sgn} y := \begin{cases} -1 & \text{for } y < 0, \\ [-1,1] & \text{for } y = 0, \\ 1 & \text{for } y > 0. \end{cases}$$

By a *Filippov solution* of problem (1.1)–(1.3) we shall mean a function $x \in PAC^1([0,T],\mathbb{R}^n)$ (see Section 2 for the definition) satisfying (1.2)–(1.5) for a.a. $t \in [0,T]$.

In this paper, the solvability of the forced pendulum equation with dry friction and viscous damping term together with Dirichlet boundary conditions and impulses (1.1)–(1.3) will be investigated using the Kakutani-Ky Fan fixed-point theorem. Apart from the solvability, also the explicit estimates of solutions and their derivatives will be obtained.

The paper is organized as follows. In the second section, suitable definitions and statements which will be used in the sequel are recalled. In Section 3, Schauder linearization device will be combined with the Kakutani-Ky Fan fixed-point theorem and the existence and localization result will be obtained in this way for the impulsive Dirichlet problem (1.1)–(1.3). Finally, the obtained result will be illustrated by an example whose numerical solution will be obtained by applying Matlab software.

2. Preliminaries

Let us start with notations we use in the paper. For a given compact real interval J, we denote by $C(J,\mathbb{R}^n)$ (by $C^1(J,\mathbb{R}^n)$) the set of all functions $x\colon J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $AC^1(J,\mathbb{R}^n)$, we shall mean the set of all functions $x\colon J \to \mathbb{R}^n$ with absolutely continuous first derivatives on J.

Let $PAC^1([0,T],\mathbb{R}^n)$ be the space of all functions $x\colon [0,T] \to \mathbb{R}^n$ such that

$$x(t) = \begin{cases} x_{[0]}(t) & \text{for } t \in [0,t_1], \\ x_{[1]}(t) & \text{for } t \in (t_1,t_2], \\ \vdots & \\ x_{[p]}(t) & \text{for } t \in (t_p,T], \end{cases}$$

where $x_{[0]} \in AC^1([0,t_1],\mathbb{R}^n)$, $x_{[i]} \in AC^1((t_i,t_{i+1}],\mathbb{R}^n)$, $x(t_i^+) = \lim\limits_{t\to t_i^+} x(t) \in \mathbb{R}$ and $\dot{x}(t_i^+) = \lim\limits_{t\to t_i^+} \dot{x}(t) \in \mathbb{R}$, for every $i = 1,\dots,p$. The space $PAC^1([0,T],\mathbb{R}^n)$ is a normed space with the norm

$$\|x\|_E := \sup_{t\in[0,T]} |x(t)| + \sup_{t\in[0,T]} |\dot{x}(t)|. \tag{2.1}$$

In the sequel, it will be denoted by $(E, \|\cdot\|_E)$. In a similar way, we can define the spaces $PC([0,T],\mathbb{R}^n)$ and $PC^1([0,T],\mathbb{R}^n)$ as the spaces of functions $x\colon [0,T] \to \mathbb{R}^n$ satisfying the previous definition with $x_{[0]} \in C([0,t_1],\mathbb{R}^n), x_{[i]} \in C((t_i,t_{i+1}],\mathbb{R}^n)$ or with $x_{[0]} \in C^1([0,t_1],\mathbb{R}^n)$, $x_{[i]} \in C^1((t_i,t_{i+1}],\mathbb{R}^n)$, for every $i = 1,\dots,p$, respectively. The space $PC^1([0,T],\mathbb{R}^n)$ with the norm defined by (2.1) is a Banach space (see [11], p. 128). In the following, a compactness result for subsets of $PC^1([0,T],\mathbb{R}^n)$ will be needed. On this purpose, let us recall that a family $\mathcal{F} \subset PC([0,T],\mathbb{R}^n)$ is left equicontinuous (see [11]) if for every $\varepsilon > 0$ and $x \in [0,T]$ there exists $\delta > 0$ such that for every $f \in \mathcal{F}$,

$$|f(x) - f(y)| < \varepsilon \quad \forall\, y \in (x-\delta, x]$$

and

$$|f(x^+) - f(y)| < \varepsilon \quad \forall\, y \in (x, x+\delta).$$

In the following, we use a generalized Ascoli-Arzelà theorem whose prove is given in [11], Theorem 2 in a slightly different case, i.e., when the real valued functions are discontinuous from the left and are just continuous in each interval $[t_i, t_{i+1})$.

Proposition 2.1. *A family $\mathcal{F} \subset PC^1([0,T],\mathbb{R}^n)$ is compact if and only if it is bounded, left equicontinuous and the set $\{f'\colon f \in \mathcal{F}\}$ is left equicontinuous.*

We also need the following definitions and notions from multivalued theory in the sequel. We say that F is a *multivalued mapping* from X to Y (written $F\colon X \multimap Y$) if for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, i.e. the subset of $X \times Y$ defined by

$$\Gamma_F := \{(x,y) \in X \times Y;\ y \in F(x)\}.$$

The single valued function $f\colon X \to Y$ is called a *selection* of F if $\Gamma_f \subset \Gamma_F$, i.e., if $f(x) \in F(x)$ for every $x \in X$.

A multivalued mapping $F\colon X \multimap Y$ is called *upper semi-continuous* (shortly, u.s.c.) if for each open set $U \subset Y$, the set $\{x \in X;\ F(x) \subset U\}$ is open in X. A multivalued mapping $F\colon X \multimap Y$ is called *compact* if the set $F(X) = \bigcup_{x\in X} F(x)$ is contained in a compact subset of Y. Let us note that every u.s.c. mapping with closed values has a closed graph and that every compact multivalued mapping with closed graph is u.s.c.

Let Y be a metric space and $(\Omega, \mathcal{U}, \mu)$ be a *measurable space,* i.e., a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure μ on $\mathcal{U}$. A multivalued mapping $F\colon \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega;$ $F(\omega) \subset V\} \in \mathcal{U}$ for each open set $V \subset Y$.

We say that the mapping $F\colon J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $F(\cdot, x)\colon J \multimap \mathbb{R}^n$ is measurable for all $x \in \mathbb{R}^m$, the map $F(t, \cdot)\colon \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c. for a.a. $t \in J$, and the set $F(t, x)$ is compact and convex for all $(t, x) \in J \times \mathbb{R}^m$.

We employ the following selection result in the sequel, which was proved in [2], Proposition 6 in a quite general setting for continuous function q. Its proof can be easily extended to the piecewise continuous functions, so we omit it here.

Proposition 2.2. *Let $J \subset \mathbb{R}$ be a compact interval and $F\colon J \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that for every $r > 0$ there exists an integrable function $\mu_r\colon J \to [0, \infty)$ satisfying $|y| \leqslant \mu_r(t)$ for every $(t, x) \in J \times \mathbb{R}^m$, with $|x| \leqslant r$, and every $y \in F(t, x)$. Then the composition $F(t, q(t))$ admits, for every $q \in PC(J, \mathbb{R}^m)$, a measurable selection.*

Let $X \cap Y \neq \emptyset$ and $F\colon X \multimap Y$. We say that a point $x \in X \cap Y$ is a *fixed point* of F if $x \in F(x)$. The set of all fixed points of F is denoted by $\mathrm{Fix}(F)$, i.e.

$$\mathrm{Fix}(F) := \{x \in X;\ x \in F(x)\}.$$

For obtaining the existence of a solution, the following Kakutani-Ky Fan fixed-point theorem will be needed (see e.g. Theorem II.8.4. in [7]).

Proposition 2.3. *Let C be a convex subset of a normed linear space and let $\varphi\colon C \multimap C$ be a compact u.s.c. mapping with compact and convex values. Then φ has a fixed point.*

3. Existence and localization result

In this section, let us formally rewrite the studied problem into the form

$$\begin{cases} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)) & \text{for a.a. } t \in [0, T], \\ x(0) = x(T) = 0, & \\ x(t_i^+) - x(t_i) = J_i(x(t_i)), & i = 1, \dots, p, \\ \dot{x}(t_i^+) - \dot{x}(t_i^-) = M_i(x(t_i)), & i = 1, \dots, p, \end{cases} \tag{3.1}$$

where

$$F(t, x(t), \dot{x}(t)) := h(t) - a\dot{x}(t) - b \sin x(t) - c \,\mathrm{Sgn}\, \dot{x}(t)$$

is an upper-Carathéodory mapping and $M_i\colon \mathbb{R} \to \mathbb{R}$ and $J_i\colon \mathbb{R} \to \mathbb{R}$ are continuous.

Theorem 3.1. *Let us consider the b.v.p.* (3.1), *where*

(1) *the impulse conditions J_i and M_i satisfies that $|J_i(x)| \leqslant k+l_1|x|$ and $|M_i(x)| \leqslant k+l_2|x|$ for all $i=1,\dots,p$ and $x \in \mathbb{R}$, where $k, l_1, l_2 > 0$ with $p < (2l_1+l_2)^{-1}$,*

(2) *the function $h\colon [0,T] \to \mathbb{R}$ playing the role of the forcing term satisfies $h \in L^\infty([0,T],\mathbb{R})$,*

(3)
$$\frac{\sqrt{(4|a|+pl_2)^2+16|a|(1-2pl_1-pl_2)}-4|a|-pl_2}{2|a|} > T.$$

Then the impulsive Dirichlet b.v.p. (3.1) *admits a solution $x(\cdot)$ such that*

$$\|x\|_E = \sup_{t\in[0,T]} |x(t)| + \sup_{t\in[0,T]} |\dot{x}(t)| \leqslant R,$$

where

$$R \geqslant \frac{(\|h\|_\infty+|b|+|c|)(T^2+4T)+12pk+Tpk}{4-|a|(T^2+4T)-8pl_1-pl_2(T+4)}. \tag{3.2}$$

Moreover, if $h(t)$ is not identically 0 *on* $[0,T]$, *then this solution is nontrivial.*

P r o o f. For the solvability of nonlinear problem (3.1), let us use the Schauder linearization device (see e.g. [3]). For this purpose, let us define the set Q of candidate solutions by the formula

$$Q := \Big\{ q \in PC^1([0,T],\mathbb{R}^n)\colon \sup_{t\in[0,T]} |q(t)| + \sup_{t\in[0,T]} |\dot{q}(t)| \leqslant R \Big\},$$

where R satisfies the formula (3.2). Let us note that the conditions put on p, l_1, l_2, and T guarantee that $R > 0$.

Then, for each $q \in Q$, let us consider the associated multivalued fully linearized problem with impulses

$$(P_q)\qquad \begin{cases} \ddot{x}(t) \in F_q(t) & \text{for a.a. } t \in [0,T], \\ x(0) = x(T) = 0, & \\ x(t_i^+) - x(t_i) = J_i(q(t_i)), & i = 1,\dots,p, \\ \dot{x}(t_i^+) - \dot{x}(t_i^-) = M_i(q(t_i)), & i = 1,\dots,p, \end{cases}$$

where $F_q(t) = F(t,q(t),\ \dot{q}(t)) = h(t) - a\dot{q}(t) - b\sin q(t) - c\,\mathrm{Sgn}\,\dot{q}(t)$. According to Proposition 2.2, there exists at least one measurable selection $f_q(\cdot)$ of the multivalued mapping $F_q(\cdot)$. The corresponding single-valued linear Dirichlet b.v.p. with impulses

$$\begin{cases} \ddot{x}(t) = f_q(t) & \text{for a.a. } t \in [0,T], \\ x(0) = x(T) = 0, & \\ x(t_i^+) - x(t_i) = J_i(q(t_i)), & i = 1, \dots, p, \\ \dot{x}(t_i^+) - \dot{x}(t_i^-) = M_i(q(t_i)), & i = 1, \dots, p, \end{cases}$$

admits a solution which can be, for a.a. $t \in [0,T]$, expressed (see e.g. [13]) by

$$x(t) = \int_0^T G(t,s) f_q(s)\, \mathrm{d}s + \sum_{i=1}^{p} \frac{\partial G}{\partial s}(t,t_i) J_i(q(t_i)) + \sum_{i=1}^{p} G(t,t_i) M_i(q(t_i)),$$

where

$$G(t,s) = \begin{cases} \dfrac{(s-T)t}{T} & \text{for } 0 \leqslant t \leqslant s \leqslant T, \\ \dfrac{(t-T)s}{T} & \text{for } 0 \leqslant s \leqslant t \leqslant T \end{cases}$$

is the Green function of the corresponding homogeneous problem $\ddot{x}(t) = 0$, $x(0) = x(T) = 0$, $x(t_i^+) - x(t_i) = 0$, $\dot{x}(t_i^+) - \dot{x}(t_i^-) = 0$, $i = 1, \dots, p$. Therefore, for each $q \in Q$, the set of solutions of (P_q) is nonempty.

Let us show that for each $q \in Q$, the set of solutions of (P_q) is convex. For this purpose, let $q \in Q$ be arbitrary and let $x_1(\cdot)$, $x_2(\cdot)$ be two solutions of the problem (P_q). Then there exist measurable selections $f_1(\cdot)$, $f_2(\cdot)$ of $F_q(\cdot)$ such that for a.a. $t \in [0,T]$,

$$x_j(t) = \int_0^T G(t,s) f_j(s)\, \mathrm{d}s + \sum_{i=1}^{p} \frac{\partial G}{\partial s}(t,t_i) J_i(q(t_i)) + \sum_{i=1}^{p} G(t,t_i) M_i(q(t_i)), \quad j = 1,2.$$

Therefore, for all $\lambda \in (0,1)$,

$$\begin{aligned} \lambda x_1(t) + (1-\lambda) x_2(t) = & \int_0^T G(t,s)(\lambda f_1(s) + (1-\lambda) f_2(s))\, \mathrm{d}s \\ & + \sum_{i=1}^{p} \frac{\partial G}{\partial s}(t,t_i) J_i(q(t_i)) + \sum_{i=1}^{p} G(t,t_i) M_i(q(t_i)). \end{aligned}$$

Since $F_q(\cdot)$ has convex values, $\lambda f_1(\cdot) + (1-\lambda) f_2(\cdot)$ is a measurable selection of $F_q(\cdot)$ as well, and so $\lambda x_1(\cdot) + (1-\lambda) x_2(\cdot)$ is a solution of (P_q). Thus, the set of solutions of (P_q) is, for all $q \in Q$, convex as required.

Let us show now that all solutions of the associated problems (P_q), $q \in Q$, lie in Q. For this purpose, let $q \in Q$ be arbitrary and let $x(\cdot)$ be a solution of the problem (P_q). Then

$$\begin{aligned}
\|x\|_E =& \sup_{t\in[0,T]} |x(t)| + \sup_{t\in[0,T]} |\dot{x}(t)| \\
=& \sup_{t\in[0,T]} \left| \int_0^T G(t,s) f_q(s)\,\mathrm{d}s + \sum_{i=1}^p \frac{\partial G}{\partial s}(t,t_i) J_i(q(t_i)) + \sum_{i=1}^p G(t,t_i) M_i(q(t_i)) \right| \\
&+ \sup_{t\in[0,T]} \left| \int_0^T \frac{\partial G}{\partial t}(t,s) f_q(s)\,\mathrm{d}s \right. \\
&\left. + \sum_{i=1}^p \frac{\partial}{\partial t}\Big(\frac{\partial G}{\partial s}(t,t_i)\Big) J_i(q(t_i)) + \sum_{i=1}^p \frac{\partial G}{\partial t}(t,t_i) M_i(q(t_i)) \right| \\
\leqslant& \frac{T^2}{4}(\|h\|_\infty + |a|R + |b| + |c|) + p(k + l_1 R) + \frac{T}{4} p(k + l_2 R) \\
&+ T(\|h\|_\infty + |a|R + |b| + |c|) + p(k + l_1 R) + p(k + l_2 R) \\
=& \frac{1}{4}[R(T^2|a| + 4|a|T + 8pl_1 + Tpl_2 + 4pl_2) \\
&+ (T^2 + 4T)(\|h\|_\infty + |b| + |c|) + 12pk + Tpk] \leqslant R,
\end{aligned}$$

according to (3.2). Therefore, all solutions of the associated problems (P_q) lie in Q. Moreover, there exist constants $M_0 \geqslant 0$, $M_1 \geqslant 0$ such that $|x(0)| \leqslant M_0$ and $|\dot{x}(0)| \leqslant M_1$ for all solutions of the associated problems.

If we denote by $\mathfrak{T}\colon Q \multimap E$ the solution mapping which assigns to each $q \in Q$ the set of solutions of (P_q), then it follows from the previous reasoning that $\mathfrak{T}$ has convex values, $\mathfrak{T}(Q)$ is bounded and that $\mathfrak{T}(Q) \subset Q$. Moreover, it is obvious that each fixed point of the solution mapping $\mathfrak{T}$ is the solution of the original problem (3.1) belonging to Q. Therefore, the problem of the existence of a solution of problem (3.1) can be transformed into the fixed point problem $q \in \mathfrak{T}(q)$.

The existence of the desired fixed point will be obtained using Kakutani-Ky Fan fixed-point theorem which is stated in the paper in the form of Proposition 2.3. In order to apply Proposition 2.3, let us at first show that $\mathfrak{T}$ has a closed graph. Let $\{(q_k, x_k)\} \subset \Gamma_{\mathfrak{T}}$ be arbitrary and such that $(q_k, x_k) \to (q, x)$, $q \in Q$. Then, since $x_k \in Q$, $x_k \to x$ and Q is closed, it holds that $x \in Q$. Moreover, x_k is a solution of (P_{q_k}), and so, according to Proposition 2.2, we get the existence of the selection $f_k(\cdot) \in F_{q_k}(\cdot)$ such that $\dot{x}_k(t_{i+1}) - \dot{x}_k(t) = \int_t^{t_{i+1}} f_k(s)\,\mathrm{d}s$ for every $t \in (t_i, t_{i+1}]$ and $i = 0, \dots, p$. Furthermore,

$$|f_k(t)| \leqslant \|h\|_\infty + |a|R + |b| + |c| \quad \text{for a.a. } [0,T].$$

This implies that $\{f_k\}$ is bounded in $L^1([0,T],\mathbb{R})$, and so it has a weakly converging subsequence, for the sake of simplicity still denoted as the sequence, which converges to a function f. In particular, $\int_t^{t_{i+1}} f_k(s)\,\mathrm{d}s \to \int_t^{t_{i+1}} f(s)\,\mathrm{d}s$ for every $t \in (t_i, t_{i+1}]$ and $i = 0,\dots,p$. Hence,

$$\dot{x}(t_{i+1}) - \dot{x}(t) = \lim_{k\to\infty} [\dot{x}_k(t_{i+1}) - \dot{x}_k(t)] = \lim_{k\to\infty} \int_t^{t_{i+1}} f_k(s)\,\mathrm{d}s = \int_t^{t_{i+1}} f(s)\,\mathrm{d}s$$

for $t \in (t_i, t_{i+1}]$ and $i = 0,\dots,p$. Therefore, there exists $\ddot{x}(t) = f(t)$ for a.a. $t \in [0,T]$. It remains to prove that $f(\cdot) \in F_q(\cdot)$. Since F_q is upper-Carathéodory, there exists, for every $\varepsilon > 0$ and a.a. $t \in [0,T]$, a positive number δ such that if $|(\alpha,\beta) - (q(t),\dot{q}(t))| \leqslant \delta$, then

$$F(t,\alpha,\beta) \subset F(t,q(t),\dot{q}(t)) + B_0^{\varepsilon}.$$

Recalling that the convergence in $PC^1([0,T],\mathbb{R}^n)$ of q_k to q implies the pointwise convergence of both sequences and of the sequences of their derivatives to the same limits, we get that for every $t \in [0,T]$ and $\delta > 0$ there exists $\overline{k}$ such that for $k \geqslant \overline{k}$, $|(q_k(t),\dot{q}_k(t)) - (q(t),\dot{q}(t))| \leqslant \delta$. Therefore, for every $\varepsilon > 0$ and a.a. $t \in [0,T]$ there exists $\overline{k}$ such that if $k \geqslant \overline{k}$, then

$$f_k(t) \in F(t,q_k(t),\dot{q}_k(t)) \subset F(t,q(t),\dot{q}(t)) + B_0^{\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, we get that $f(t) \in F(t,q(t),\dot{q}(t))$ for a.a. $t \in [0,T]$, i.e., that $\mathfrak{T}$ has a closed graph. Recalling that a compact mapping with closed graph is u.s.c. and has compact values, it only remains to prove that $\mathfrak{T}$ is compact. According to Proposition 2.1, we need to prove that $\mathfrak{T}(Q)$ is left equicontinuous, and has left equicontinuous set of derivatives.

Let $x \in \mathfrak{T}(q)$. Then there exists $f(\cdot) \in F(\cdot,q(\cdot),\dot{q}(\cdot))$ such that for every $\bar{t},\tilde{t} \in (t_i,t_{i+1}]$ with $\bar{t} > \tilde{t}$ and $i = 0,\dots,p$,

$$\dot{x}(\bar{t}) = \dot{x}(\tilde{t}) + \int_{\tilde{t}}^{\bar{t}} f(s)\,\mathrm{d}s \tag{3.3}$$

and consequently,

$$x(\bar{t}) = x(\tilde{t}) + \dot{x}(\tilde{t})(\bar{t}-\tilde{t}) + \int_{\tilde{t}}^{\bar{t}} (\bar{t}-s)f(s)\,\mathrm{d}s. \tag{3.4}$$

It follows from (3.3) and (3.4) that for every $\bar{t},\tilde{t} \in (t_i,t_{i+1}]$ with $\bar{t} > \tilde{t}$ and $i = 0,\dots,p$,

$$|\dot{x}(\bar{t}) - \dot{x}(\tilde{t})| = \left|\int_{\tilde{t}}^{\bar{t}} f(s)\,\mathrm{d}s\right| \leqslant \int_{\tilde{t}}^{\bar{t}} (\|h\|_\infty + |a|R + |b| + |c|)\,\mathrm{d}s$$

and

$$|x(\bar{t}) - x(\tilde{t})| \leqslant R|\bar{t} - \tilde{t}| + \int_{\tilde{t}}^{\bar{t}} (\bar{t} - s)(\|h\|_\infty + |a|R + |b| + |c|)\,\mathrm{d}s.$$

Thus, if $t \neq t_1, \dots, t_p$, one can take δ sufficiently small such that $(t - \delta, t + \delta) \cap \{t_1, \dots, t_p\} = \emptyset$ and conclude (from the absolute continuity of the Lebesgue integral) that the functions x and $\dot{x}$ are equicontinuous at t. The left equicontinuity can be deduced similarly for $t \in \{t_1, \dots, t_p\}$.

So, we have proved that $\mathfrak{T}(Q)$ is compact, and hence, it follows from Proposition 2.3 that there exists a fixed point of $\mathfrak{T}(\cdot)$ in Q. This fixed point is then the solution of the original problem (3.1). $\square$

Remark 3.1. It follows from the proof of Theorem 3.1 that it is possible to replace in (3.1) the impulses in the form $\dot{x}(t_i^+) - \dot{x}(t_i^-) = M_i(x(t_i))$, $i = 1, \dots, p$, by $\dot{x}(t_i^+) - \dot{x}(t_i^-) = M_i(\dot{x}(t_i))$, $i = 1, \dots, p$, where $M_i\colon \mathbb{R} \to \mathbb{R}$ are continuous for all $i = 1, \dots, p$.

Remark 3.2. Let us note that by the same way, the existence result can be deduced also for different types of boundary conditions. In the case that the corresponding homogenous problem is trivially solvable, we would apply in the proof of Theorem 3.1 the corresponding Green function and obtain the analogous result (with different estimates of the solutions and their derivatives) also in this case.

Example 3.1. Let us consider the forced (mathematical) pendulum equation with viscous damping and dry friction terms together with three impulses and Dirichlet boundary conditions

$$(3.5)\qquad \begin{cases} \ddot{x}(t) + \dfrac{1}{10}\dot{x}(t) + 5\sin x(t) + 2\,\mathrm{sgn}\,\dot{x}(t) = \sin\Big(\dfrac{\pi}{2}t\Big) & \text{for a.a. } t \in [0,1], \\ x(0) = x(1) = 0, & \\ x\Big(\dfrac{i}{4}^+\Big) - x\Big(\dfrac{i}{4}\Big) = 6 + \dfrac{5}{100}x\Big(\dfrac{i}{4}\Big), & i = 1,2,3, \\ \dot{x}\Big(\dfrac{i}{4}^+\Big) - \dot{x}\Big(\dfrac{i}{4}^-\Big) = 6 + \dfrac{1}{10}\dot{x}\Big(\dfrac{i}{4}^-\Big), & i = 1,2,3. \end{cases}$$

If we consider, because of discontinuity at $y = 0$ in $\mathrm{sgn}\,y$, the corresponding multivalued problem, where the function $\mathrm{sgn}\,y$ is replaced by the multivalued mapping $\mathrm{Sgn}\,y$, then we can apply Theorem 3.1 for $T = 1$, $a = \frac{1}{10}$, $b = 5$, $c = 2$, $h(t) = \sin(\frac{\pi}{2}t)$, $p = 3$, $k = 6$, $l_1 = \frac{5}{100}$, $l_2 = \frac{1}{10}$, and obtain the existence of Filippov solutions $x(\cdot)$ of (3.5). The numerical solution of (3.5) obtained by Matlab software is presented in Figure 1.

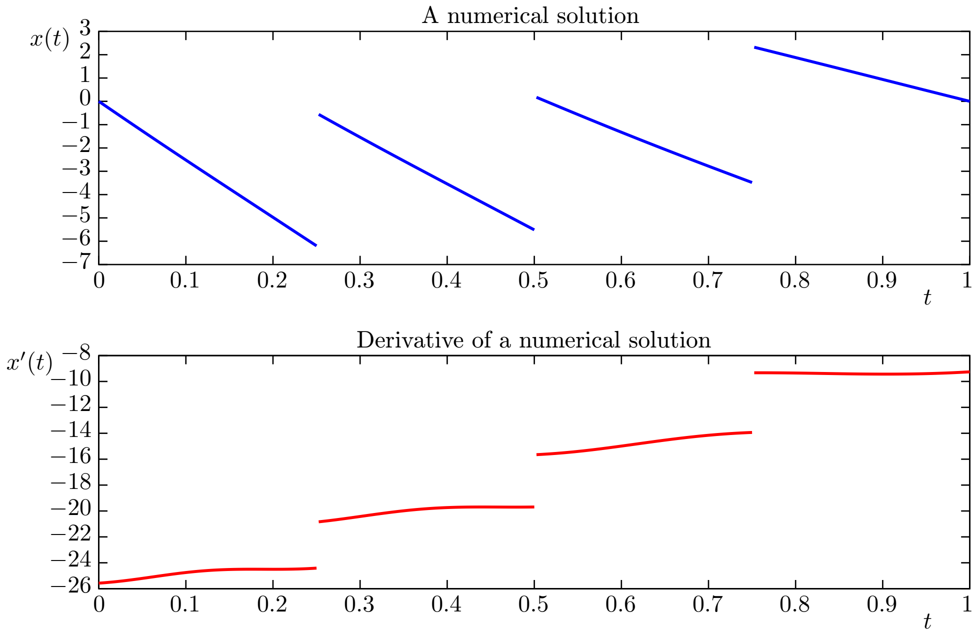

Figure 1. A numerical solution of (3.5) and its derivative.

4. Conclusion

Impulsive differential equations are suitable for the mathematical simulation of rapid changes in evolution processes. In this paper, the impulsive problem to the pendulum equation with a nonzero viscous damping coefficient as well as with a nonzero friction coefficient was studied together with Dirichlet boundary conditions. Because of discontinuity in dry friction part it was appropriate to implement the tools from multivalued analysis, and subsequently obtain the existence result by applying the Kakutani-Ky Fan fixed point theorem. The theory was illustrated by an example whose numerical solution was obtained by Matlab software. The present result generalized previous works, where only the impulsive problems with zero friction coefficient were under consideration or where the problems without impulses were studied. Some interesting questions dealing with this topic deserve further investigation. It will be for example suitable to study the problem with different types of boundary conditions or to investigate the case of state-dependent impulses which generalize the impulses at fixed moments considered in this paper.

References

[1] *J. Andres, H. Machů*: Dirichlet boundary value problem for differential equations involving dry friction. Bound. Value Probl. *2015* (2015), Article ID 106, 17 pages. zbl MR doi

[2] *I. Benedetti, V. Obukhovskii, V. Taddei*: On noncompact fractional order differential inclusions with generalized boundary condition and impulses in a Banach space. J. Funct. Spaces *2015* (2015), Article ID 651359, 10 pages. zbl MR doi

[3] *M. Cecchi, M. Furi, M. Marini*: On continuity and compactness of some nonlinear operators associated with differential equations in noncompact intervals. Nonlinear Anal., Theory Methods Appl. *9* (1985), 171–180. zbl MR doi

[4] *H. Chen, J. Li, Z. He*: The existence of subharmonic solutions with prescribed minimal period for forced pendulum equations with impulses. Appl. Math. Modelling *37* (2013), 4189–4198. zbl MR doi

[5] *A. F. Filippov*: Differential Equations with Discontinuous Right-Hand Sides. Mathematics and Its Applications: Soviet Series 18. Kluwer Academic, Dordrecht, 1988. zbl MR

[6] *S. Fučík*: Solvability of Nonlinear Equations and Boundary Value Problems. Mathematics and Its Applications 4. D. Reidel, Dordrecht, 1980. zbl MR

[7] *A. Granas, J. Dugundji*: Fixed Point Theory. Springer Monographs in Mathematics. Springer, Berlin, 2003. zbl MR doi

[8] *G. Hamel*: Über erzwungene Schwingungen bei endlichen Amplituden. Math. Ann. *86* (1922), 1–13. (In German.) zbl doi

[9] *F. Kong*: Subharmonic solutions with prescribed minimal period of a forced pendulum equation with impulses. Acta Appl. Math. *158* (2018), 125–137. zbl MR doi

[10] *J. Mawhin*: Global results for the forced pendulum equation. Handbook of Differential Equations: Ordinary Differential Equations. Vol. 1. Elsevier, Amsterdam, 2004, pp. 533–589. zbl MR doi

[11] *J. Meneses, R. Naulin*: Ascoli-Arzelá theorem for a class of right continuous functions. Ann. Univ. Sci. Budap. Eötvös, Sect. Math. *38* (1995), 127–135. zbl MR

[12] *M. Pavlačková*: A Scorza-Dragoni approach to Dirichlet problem with an upper-Carathéodory right-hand side. Topol. Methods Nonlinear Anal. *44* (2014), 239–247. zbl MR doi

[13] *I. Rachůnková, J. Tomeček*: Second order BVPs with state dependent impulses via lower and upper functions. Cent. Eur. J. Math. *12* (2014), 128–140. zbl MR doi

[14] *J. Xie, Z. Luo*: Subharmonic solutions with prescribed minimal period of an impulsive forced pendulum equation. Appl. Math. Lett. *52* (2016), 169–175. zbl MR doi

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