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The variety of dual mock-Lie algebras

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Abstract. We classify all complex 7- and 8-dimensional dual mock-Lie algebras by the algebraic and geometric way. Also, we find all non-trivial complex 9-dimensional dual mock-Lie algebras.

Introduction

There are many results related to the algebraic and geometric classification of low dimensional algebras in the varieties of Jordan, Lie, Leibniz and Zinbiel algebras; for the algebraic classification see, for example, [1], [6], [7], [8], [9], [19], [22]; for the geometric classification and descriptions of degenerations see, for example, [1], [2], [3], [5], [11], [12], [13], [16], [17], [19], [21], [22], [23], [26]. Here we give the algebraic and geometric classification of complex dual mock-Lie algebras of small dimensions.

Commutative algebras satisfying the Jacobi identity are called *mock-Lie algebras*. Here, on the one side, we have a "commutative" analog of the Lie algebras and on the other side, we have Jordan algebras of nil index 3. The systematic study

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of mock-Lie and dual mock-Lie algebras was initiated in [30]; however, mock-Lie algebras appeared in many papers, such that [4], [28] (see [30] and the references therein for more information about it). As was noted in [30], dual mock-Lie algebras are in the intersection of anticommutative and antiassociative algebras (about antiassociative algebras, see in [24] and references therein). Let us give the main definition of the paper.

Definition 1. An n-dimensional dual mock-Lie algebra is an n-dimensional vector space $\mathfrak{D}$ together with a binary operation which is bilinear and satisfying the following relations:

$$xy = -yx \qquad \text{(anticommutativity)}$$
$$x(yz) = -(xy)z \qquad \text{(antiassociativity)}$$

Take arbitrary elements $a, b, c, d \in \mathfrak{D}$. We have

$$a(b(cd)) = -(ab)(cd),$$

on the other hand

$$\begin{aligned} a(b(cd)) &= -a(b(dc)) = a((dc)b) = -(a(dc))b \\ &= ((dc)a)b = -(dc)(ab) = -(ab)(dc) = (ab)(cd)\,, \end{aligned}$$

hence $(ab)(cd) = 0$, thus $(\mathfrak{D})^4 = 0$, i.e., any dual mock-Lie algebra is a 3-step nilpotent algebra.

The algebraic classification of nilpotent algebras will be achieved by the calculation of central extensions of algebras from the same variety which has a smaller dimension. Central extensions of algebras from various varieties were studied, for example, in [27], [29]. In this paper, we essentially follow the Skjelbred and Sund method adapted for dual mock-Lie algebras. Skjelbred and Sund [27] used central extensions of Lie algebras to classify nilpotent Lie algebras. Using the same method, all non-Lie central extensions of all 4-dimensional Malcev algebras [15] we found, and also, the method was adapted for many other non-associative algebras, such that Jordan, Novikov, Tortkara, etc. [1], [6], [7], [8], [9], [10], [14], [19].

Degenerations of algebras is an interesting subject, which has been studied in various papers. In particular, there are many results concerning degenerations of algebras of small dimensions in a variety defined by a set of identities. One of the important problems in this direction is a description of so-called rigid algebras. These algebras are of big interest since the closures of their orbits under the action of the generalized linear group form irreducible components of the variety under consideration (with respect to the Zariski topology). For example, rigid algebras in the varieties of all 4-dimensional Leibniz algebras [17], all 4-dimensional nilpotent Novikov algebras [19], all 6-dimensional nilpotent binary Lie algebras [1], all 6-dimensional nilpotent Tortkara algebras [11], and in some other varieties were classified. There are fewer works in which the full information about degenerations was given for a variety of algebras. This problem was solved for 2-dimensional pre-Lie algebras, for 2-dimensional terminal algebras, for 3-dimensional Novikov algebras, for 3-dimensional Jordan algebras, for 3-dimensional Leibniz algebras, for 3-dimensional anticommutative algebras, for 3-dimensional nilpotent algebras in [10], for 4-dimensional Lie algebras in [5], for 4-dimensional Zinbiel algebras, for 4-dimensional nilpotent Leibniz algebras, for 4-dimensional nilpotent commutative algebras in [10], for 5-dimensional nilpotent anticommutative algebras in [10], for 6-dimensional nilpotent Lie algebras in [26], [12], for 6-dimensional nilpotent Malcev algebras in [21], for 2-step nilpotent 7-dimensional Lie algebras [3], for all 2-dimensional algebras in [22], and so on.

1 The algebraic classification of dual mock-Lie algebras

1.1 The algebraic classification of (nilpotent) dual mock-Lie algebras

Let $\mathbf{A}$ and $\mathbf{V}$ be a dual mock-Lie algebra and a vector space, respectively, and $\mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A},\mathbf{V})$ denotes the space of skew-symmetric bilinear maps $\theta\colon \mathbf{A}\times\mathbf{A}\to\mathbf{V}$ satisfying

$$\theta(xy,z) = -\theta(x,yz)\,.$$

For $f\in \mathrm{Hom}(\mathbf{A},\mathbf{V})$, we introduce $\delta f\in \mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A},\mathbf{V})$ by the equality

$$\delta f(x,y) = f(xy)$$

and define

$$\mathrm{B}^2(\mathbf{A},\mathbf{V}) = \{\delta f \mid f\in \mathrm{Hom}(\mathbf{A},\mathbf{V})\}\,.$$

One can easily check that $\mathrm{B}^2(\mathbf{A},\mathbf{V})$ is a linear subspace of $\mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A},\mathbf{V})$. Let us define $\mathrm{H}^2_{\mathfrak{D}}(\mathbf{A},\mathbf{V})$ as the quotient space $\mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A},\mathbf{V})\big/\mathrm{B}^2(\mathbf{A},\mathbf{V})$. The equivalence class of $\theta\in \mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A},\mathbf{V})$ in $\mathrm{H}^2_{\mathfrak{D}}(\mathbf{A},\mathbf{V})$ is denoted by $[\theta]$.

Suppose now that $\dim\mathbf{A} = m < n$ and $\dim\mathbf{V} = n-m$. For any dual mock-Lie bilinear map $\theta\colon\mathbf{A}\times\mathbf{A}\to\mathbf{V}$, one can define on the space $\mathbf{A}_\theta := \mathbf{A}\oplus\mathbf{V}$ the dual mock-Lie bilinear product $[-,-]_{\mathbf{A}_\theta}$ by the equality

$$[x+x', y+y']_{\mathbf{A}_\theta} = xy + \theta(x,y)$$

for $x,y\in\mathbf{A}, x',y'\in\mathbf{V}$. The algebra $\mathbf{A}_\theta$ is called an *$(n-m)$-dimensional central extension* of $\mathbf{A}$ by $\mathbf{V}$. It is also clear that $\mathbf{A}_\theta$ is nilpotent if and only if so is $\mathbf{A}$. The algebra $\mathbf{A}_\theta$ is dual mock-Lie if and only if $\mathbf{A}$ is dual mock-Lie and θ is a cocycle.

For a dual mock-Lie bilinear form $\theta\colon\mathbf{A}\times\mathbf{A}\to\mathbf{V}$, the space

$$\theta^\perp = \{x\in\mathbf{A} \mid \theta(\mathbf{A},x) = 0\}$$

is called the *annihilator* of θ. For a dual mock-Lie algebra $\mathbf{A}$, the ideal

$$\mathrm{Ann}(\mathbf{A}) = \{x\in\mathbf{A} \mid \mathbf{A}x = 0\}$$

is called the *annihilator* of $\mathbf{A}$. One has

$$\mathrm{Ann}(\mathbf{A}_\theta) = (\theta^\perp\cap\mathrm{Ann}(\mathbf{A}))\oplus\mathbf{V}.$$

Any n-dimensional dual mock-Lie algebra with non-trivial annihilator can be represented in the form $\mathbf{A}_\theta$ for some m-dimensional dual mock-Lie algebra $\mathbf{A}$, an $(n-m)$--dimensional vector space $\mathbf{V}$ and $\theta \in \mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A}, \mathbf{V})$, where $m < n$ (see [15, Lemma 5]). Moreover, there is a unique such representation with

$$m = n - \dim \mathrm{Ann}(\mathbf{A})\,.$$

Note also that the last mentioned equality is equivalent to the condition

$$\theta^\perp \cap \mathrm{Ann}(\mathbf{A}) = 0\,.$$

Let us pick some $\phi \in \mathrm{Aut}(\mathbf{A})$, where $\mathrm{Aut}(\mathbf{A})$ is the automorphism group of $\mathbf{A}$. For $\theta \in \mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A}, \mathbf{V})$, let us define

$$(\phi\theta)(x, y) = \theta(\phi(x), \phi(y))\,.$$

Then we get an action of $\mathrm{Aut}(\mathbf{A})$ on $\mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A}, \mathbf{V})$ that induces an action of the same group on $\mathrm{H}^2_{\mathfrak{D}}(\mathbf{A}, \mathbf{V})$.

Definition 2. Let $\mathbf{A}$ be an algebra and I be a subspace of $\mathrm{Ann}(\mathbf{A})$. If $\mathbf{A} = \mathbf{A}_0 \oplus I$ then I is called an *annihilator component* of $\mathbf{A}$.

For a linear space $\mathbf{U}$, the *Grassmannian* $G_s(\mathbf{U})$ is the set of all k-dimensional linear subspaces of $\mathbf{U}$. For any $s \geq 1$, the action of $\mathrm{Aut}(\mathbf{A})$ on $\mathrm{H}^2_{\mathfrak{D}}(\mathbf{A}, \mathbb{C})$ induces an action of the same group on $G_s(\mathrm{H}^2_{\mathfrak{D}}(\mathbf{A}, \mathbb{C}))$. Let us define

$$\mathbf{T}_s(\mathbf{A}) = \Big\{ \mathbf{W} \in G_s(\mathrm{H}^2_{\mathfrak{D}}(\mathbf{A}, \mathbb{C})) \;\Big|\; \bigcap_{[\theta] \in W} \theta^\perp \cap \mathrm{Ann}(\mathbf{A}) = 0 \Big\}.$$

Note that $\mathbf{T}_s(\mathbf{A})$ is stable under the action of $\mathrm{Aut}(\mathbf{A})$.

Let us fix a basis $e_1, \ldots, e_s$ of $\mathbf{V}$, and $\theta \in \mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A}, \mathbf{V})$. Then there are unique

$$\theta_i \in \mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A}, \mathbb{C}) \quad (1 \leq i \leq s)$$

such that

$$\theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y) e_i$$

for all $x, y \in \mathbf{A}$. Note that $\theta^\perp = \theta_1^\perp \cap \theta_2^\perp \cdots \cap \theta_s^\perp$ in this case. If $\theta^\perp \cap \mathrm{Ann}(\mathbf{A}) = 0$, then by [15, Lemma 13] the algebra $\mathbf{A}_\theta$ has a nontrivial annihilator component if and only if $[\theta_1], [\theta_2], \ldots, [\theta_s]$ are linearly dependent in $\mathrm{H}^2_{\mathfrak{D}}(\mathbf{A}, \mathbb{C})$. Thus, if

$$\theta^\perp \cap \mathrm{Ann}(\mathbf{A}) = 0$$

and the annihilator component of $\mathbf{A}_\theta$ is trivial, then $\langle [\theta_1], \ldots, [\theta_s] \rangle$ is an element of $\mathbf{T}_s(\mathbf{A})$. Now, if $\vartheta \in \mathrm{Z}^2_{\mathfrak{D}}(\mathbf{A}, \mathbf{V})$ is such that $\vartheta^\perp \cap \mathrm{Ann}(\mathbf{A}) = 0$ and the annihilator component of $\mathbf{A}_\vartheta$ is trivial, then by [15, Lemma 17] one has $\mathbf{A}_\vartheta \cong \mathbf{A}_\theta$ if and only if

$$\langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle, \langle [\vartheta_1], [\vartheta_2], \ldots, [\vartheta_s] \rangle \in \mathbf{T}_s(\mathbf{A})$$

belong to the same orbit under the action of $\mathrm{Aut}(\mathbf{A})$, where

$$\vartheta(x, y) = \sum_{i=1}^{s} \vartheta_i(x, y) e_i\,.$$

Hence, there is a one-to-one correspondence between the set of $\mathrm{Aut}(\mathbf{A})$-orbits on $\mathbf{T}_s(\mathbf{A})$ and the set of isomorphism classes of central extensions of $\mathbf{A}$ by $\mathbf{V}$ with s-dimensional annihilator and trivial annihilator component. Consequently, to construct all n-dimensional central extensions with s-dimensional annihilator and trivial annihilator component of a given $(n-s)$-dimensional algebra $\mathbf{A}$, one has to describe $\mathbf{T}_s(\mathbf{A})$, $\mathrm{Aut}(\mathbf{A})$ and the action of $\mathrm{Aut}(\mathbf{A})$ on $\mathbf{T}_s(\mathbf{A})$ and then for each orbit under the action of $\mathrm{Aut}(\mathbf{A})$ on $\mathbf{T}_s(\mathbf{A})$ pick a representative and construct the algebra corresponding to it.

We will use the following auxiliary notation during the construction of central extensions. Let $\mathbf{A}$ be a dual mock-Lie algebra with the basis $e_1, e_2, \ldots, e_n$.

$$\Delta_{ij} \colon \mathbf{A} \times \mathbf{A} \to \mathbb{C}$$

denotes the dual mock-Lie bilinear form defined by the equalities

$$\Delta_{ij}(e_i, e_j) = -\Delta_{ij}(e_j, e_i) = 1$$

and

$$\Delta_{ij}(e_l, e_m) = 0$$

for $\{l, m\} \neq \{i, j\}$. In this case Δ_{ij} with $1 \leq i < j \leq n$ form a basis of the space of dual mock-Lie bilinear forms on $\mathbf{A}$. We also denote by $\mathfrak{D}^i_j$ the jth i-dimensional dual mock-Lie algebra.

1.2 The algebraic classification of low dimensional dual mock-Lie algebras

Thanks to [20], we have the classification of all 6-dimensional nilpotent anticommutative algebras and choosing only dual mock-Lie algebras from the list of algebras presented in [20] we have the classification of all low dimensional dual mock-Lie algebras. By the straightforward verification, it follows that only $\mathbb{M}_{01}$, $\mathbb{M}_{03}$, $\mathbb{M}_{04}$, $\mathbb{M}_{23}$, $\mathbb{M}_{24}$, and $\mathbb{M}_{26}$ satisfy the antiassociativity law. We thus have the following table:

$\mathfrak{D}^6_{01}$	:	$e_1e_2 = e_3$		
$\mathfrak{D}^6_{02}$	:	$e_1e_2 = e_5$	$e_3e_4 = e_5$	
$\mathfrak{D}^6_{03}$	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	
$\mathfrak{D}^6_{04}$	:	$e_1e_3 = e_5$	$e_2e_4 = e_6$	
$\mathfrak{D}^6_{05}$	:	$e_1e_2 = e_5$	$e_1e_3 = e_6$	$e_3e_4 = e_5$
$\mathfrak{D}^6_{06}$	:	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_3 = e_6$.

1.3 The algebraic classification of 7-dimensional dual mock-Lie algebras

Thanks to [3] we have the classification of all indecomposable 7-dimensional 2-step nilpotent dual mock-Lie algebras.

$\mathfrak{D}^7_{07}$	:	$e_1e_2 = e_7$	$e_3e_4 = e_7$	$e_5e_6 = e_7$	
$\mathfrak{D}^7_{08}$	:	$e_1e_2 = e_6$	$e_1e_4 = e_7$	$e_3e_5 = e_7$	
$\mathfrak{D}^7_{09}$	:	$e_1e_2 = e_6$	$e_1e_5 = e_7$	$e_3e_4 = e_6$	$e_2e_3 = e_7$
$\mathfrak{D}^7_{10}$	:	$e_1e_2 = e_5$	$e_2e_3 = e_6$	$e_2e_4 = e_7$	
$\mathfrak{D}^7_{11}$	:	$e_1e_2 = e_5$	$e_2e_3 = e_6$	$e_3e_4 = e_7$	
$\mathfrak{D}^7_{12}$	:	$e_1e_2 = e_5$	$e_2e_3 = e_6$	$e_2e_4 = e_7$	$e_3e_4 = e_5$
$\mathfrak{D}^7_{13}$	:	$e_1e_2 = e_5$	$e_1e_3 = e_6$	$e_2e_4 = e_7$	$e_3e_4 = e_5$

The key tool in the classification of dual mock-Lie algebras will be the following obvious Lemma.

Lemma 1. *If the i-dimensional algebra $\mathfrak{D}^i_j$ does not have nontrivial dual mock-Lie central extension, then for every $k \in \mathbb{N}$ the $(i+k)$-dimensional algebra $\mathfrak{D}^{i+k}_j$ does not having nontrivial dual mock-Lie central extensions.*

Hence, to find non-2-step nilpotent 7-dimensional dual mock-Lie algebras we need to calculate all non-split 2-dimensional central extensions of all 5-dimensional dual mock-Lie algebras and all non-split 1-dimensional central extensions of all 6-dimensional dual mock-Lie algebras. By an easy calculation, we have the cohomology spaces of these algebras.

$\mathfrak{D}^5$	Multiplication table	$\mathrm{H}^2_{\mathfrak{D}}(\mathfrak{D}^5)$
$\mathfrak{D}^5_{01}$	$e_1e_2 = e_3$	$\langle [\Delta_{14}], [\Delta_{15}], [\Delta_{24}], [\Delta_{25}], [\Delta_{45}] \rangle$
$\mathfrak{D}^5_{02}$	$e_1e_2 = e_5,\ e_3e_4 = e_5$	$\langle [\Delta_{13}], [\Delta_{14}], [\Delta_{23}], [\Delta_{24}] \rangle$
$\mathfrak{D}^5_{03}$	$e_1e_2 = e_4,\ e_1e_3 = e_5$	$\langle [\Delta_{23}] \rangle$

$\mathfrak{D}^6$	Multiplication table	$\mathrm{H}^2_{\mathfrak{D}}(\mathfrak{D}^6)$
$\mathfrak{D}^6_{04}$	$e_1e_3 = e_5,\ e_2e_4 = e_6$	$\langle [\Delta_{12}], [\Delta_{14}], [\Delta_{23}], [\Delta_{34}] \rangle$
$\mathfrak{D}^6_{05}$	$e_1e_2 = e_5,\ e_1e_3 = e_6,\ e_3e_4 = e_5$	$\langle [\Delta_{14}], [\Delta_{23}], [\Delta_{24}] \rangle$
$\mathfrak{D}^6_{06}$	$e_1e_2 = e_4,\ e_1e_3 = e_5,\ e_2e_3 = e_6$	$\langle [\Delta_{16}] - [\Delta_{25}] + [\Delta_{34}] \rangle$

Analyzing the cohomology spaces of these algebras, we should conclude that only the algebra $\mathfrak{D}^6_{06}$ has a non-split dual mock-Lie central extension. Now, we have a new 7-dimensional dual mock-Lie algebra

$$\mathfrak{D}^7_{14} : \quad e_1e_2 = e_4, \quad e_1e_3 = e_5, \quad e_1e_6 = e_7, \quad e_2e_3 = e_6, \quad e_2e_5 = -e_7, \quad e_3e_4 = e_7.$$

1.4 The algebraic classification of 8-dimensional dual mock-Lie algebras

It is easy to see that the algebra $\mathfrak{D}^7_{14}$ has no non-trivial dual mock-Lie central extensions. Hence, we will consider the cohomology space only for the following algebras.

$\mathfrak{D}^7$	Multiplication table	$\mathrm{H}^2_{\mathfrak{D}}(\mathfrak{D}^7)$
$\mathfrak{D}^7_{06}$	$e_1e_2=e_4,\ e_1e_3=e_5,\ e_2e_3=e_6$	$\left\langle \begin{array}{c}[\Delta_{16}]-[\Delta_{25}]+[\Delta_{34}],\\ [\Delta_{17}],[\Delta_{27}],[\Delta_{37}]\end{array}\right\rangle$
$\mathfrak{D}^7_{07}$	$e_1e_2=e_7,\ e_3e_4=e_7,\ e_5e_6=e_7$	$\left\langle \begin{array}{c}[\Delta_{13}],[\Delta_{14}],[\Delta_{15}],[\Delta_{16}],\\ [\Delta_{23}],[\Delta_{24}],[\Delta_{25}],[\Delta_{26}],\\ [\Delta_{35}],[\Delta_{36}],[\Delta_{45}],[\Delta_{46}]\end{array}\right\rangle$
$\mathfrak{D}^7_{08}$	$e_1e_2=e_6,\ e_1e_4=e_7,$ $e_3e_5=e_7$	$\left\langle \begin{array}{c}[\Delta_{13}],[\Delta_{14}],[\Delta_{15}],[\Delta_{23}],\\ [\Delta_{24}],[\Delta_{25}],[\Delta_{34}],[\Delta_{45}]\end{array}\right\rangle$
$\mathfrak{D}^7_{09}$	$e_1e_2=e_6,\ e_1e_5=e_7,$ $e_3e_4=e_6,\ e_2e_3=e_7$	$\left\langle \begin{array}{c}[\Delta_{13}],[\Delta_{14}],[\Delta_{24}],[\Delta_{25}],\\ [\Delta_{35}],[\Delta_{45}]\end{array}\right\rangle$
$\mathfrak{D}^7_{10}$	$e_1e_2=e_5,\ e_2e_3=e_6,\ e_2e_4=e_7$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{34}]\rangle$
$\mathfrak{D}^7_{11}$	$e_1e_2=e_5,\ e_2e_3=e_6,\ e_3e_4=e_7$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{24}]\rangle$
$\mathfrak{D}^7_{12}$	$e_1e_2=e_5,\ e_2e_3=e_6,$ $e_2e_4=e_7,\ e_3e_4=e_5$	$\langle[\Delta_{13}],[\Delta_{14}]\rangle$
$\mathfrak{D}^7_{13}$	$e_1e_2=e_5,\ e_1e_3=e_6,$ $e_2e_4=e_7,\ e_3e_4=e_5$	$\langle[\Delta_{14}],[\Delta_{23}]\rangle$

From here, only the algebra $\mathfrak{D}^7_{06}$ may have a non-trivial dual mock-Lie central extension. Let us find it. The automorphism group $\mathrm{Aut}(\mathfrak{D}^7_{06})$ consists of invertible matrices of the form

$$\varphi=\begin{pmatrix} a & b & c & 0 & 0 & 0 & 0\\ d & e & f & 0 & 0 & 0 & 0\\ g & h & k & 0 & 0 & 0 & 0\\ l & m & n & ae-db & af-dc & bf-ec & p\\ q & r & s & ah-gb & ak-gc & bk-hc & i\\ j & t & u & dh-ge & dk-gf & ek-hf & v\\ w & x & y & 0 & 0 & 0 & z \end{pmatrix}.$$

Let us use the notations

$$\nabla_1:=[\Delta_{16}]-[\Delta_{25}]+[\Delta_{34}],\quad \nabla_2:=[\Delta_{17}],\quad \nabla_3:=[\Delta_{27}],\quad \nabla_4:=[\Delta_{37}].$$

Take $\theta=\sum_{i=1}^4\alpha_i\nabla_i\in \mathrm{H}^2_{\mathfrak{D}}(\mathfrak{D}^7_{06},\mathbb{C})$. If $\varphi\in\mathrm{Aut}(\mathfrak{D}^7_{06})$, then

$$\varphi^T\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \alpha_1 & \alpha_2\\ 0 & 0 & 0 & 0 & -\alpha_1 & 0 & \alpha_3\\ 0 & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_4\\ 0 & 0 & -\alpha_1 & 0 & 0 & 0 & 0\\ 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0\\ -\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0\\ -\alpha_2 & -\alpha_3 & -\alpha_4 & 0 & 0 & 0 & 0 \end{pmatrix}\varphi$$

$$= \begin{pmatrix} 0 & \beta_1^* & \beta_2^* & 0 & 0 & \alpha_1^* & \alpha_2^*\\ -\beta_1^* & 0 & \beta_3^* & 0 & -\alpha_1^* & 0 & \alpha_3^*\\ -\beta_2^* & -\beta_3^* & 0 & \alpha_1^* & 0 & 0 & \alpha_4^*\\ 0 & 0 & -\alpha_1^* & 0 & 0 & 0 & 0\\ 0 & \alpha_1^* & 0 & 0 & 0 & 0 & 0\\ -\alpha_1^* & 0 & 0 & 0 & 0 & 0 & 0\\ -\alpha_2^* & -\alpha_3^* & -\alpha_4^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} \alpha_1^* &= -(ceg - bfg - cdh + afh + bdk - aek)\alpha_1\,,\\ \alpha_2^* &= (-di + gp + av)\alpha_1 + az\alpha_2 + dz\alpha_3 + gz\alpha_4\,,\\ \alpha_3^* &= (-ei + hp + bv)\alpha_1 + bz\alpha_2 + ez\alpha_3 + hz\alpha_4\,,\\ \alpha_4^* &= (-fi + kp + cv)\alpha_1 + cz\alpha_2 + fz\alpha_3 + kz\alpha_4\,. \end{aligned}$$

Hence, $\phi\langle\theta\rangle = \langle\theta^*\rangle$, where $\theta^* = \sum_{i=1}^{4}\alpha_i^*\nabla_i$. We are interested in elements with $\alpha_1 \neq 0$ and $(\alpha_2, \alpha_3, \alpha_4) \neq (0, 0, 0)$. Without loss of generality, we can suppose that $\alpha_4 \neq 0$. So, by choosing the following non-zero elements $d = h = a = e = k = 1$ and

$$v = 1 - \frac{\alpha_2}{\alpha_4}\,, \quad i = 1 + \frac{\alpha_3}{\alpha_4}\,, \quad z = \frac{\alpha_1}{\alpha_4}\,, \quad c = -1 + \frac{1}{\alpha_1}$$

we get the representative $\langle\nabla_1 + \nabla_4\rangle$. Now we have the new 8-dimensional dual mock-Lie algebra $\mathfrak{D}_{36}^8$ constructed from $\mathfrak{D}_{06}^7$:

$$\mathfrak{D}_{36}^8 \quad : \quad \begin{array}{lll} e_1e_2 = e_4\,, & e_1e_3 = e_5\,, & e_2e_3 = e_6\,,\\ e_1e_6 = e_8\,, & e_2e_5 = -e_8\,, & e_3e_4 = e_8\,,\\ e_3e_7 = e_8\,. & & \end{array}$$

Thanks to [2] we have the list of all 8-dimensional 2-step nilpotent indecomposable Lie algebras:

$\mathfrak{D}^8_{15}$	:	$e_1e_2 = e_4\,,$	$e_3e_2 = e_5\,,$	$e_6e_7 = e_8\,,$	
$\mathfrak{D}^8_{16}$	:	$e_1e_2 = e_5\,,$	$e_3e_4 = e_5\,,$	$e_6e_7 = e_8\,,$	
$\mathfrak{D}^8_{17}$	:	$e_1e_2 = e_7\,,$	$e_3e_4 = e_8\,,$	$e_5e_6 = e_7 + e_8\,,$	
$\mathfrak{D}^8_{18}$	:	$e_1e_2 = e_7\,,$	$e_4e_5 = e_7\,,$	$e_1e_3 = e_8\,,$	$e_4e_6 = e_8\,,$
$\mathfrak{D}^8_{19}$	:	$e_1e_2 = e_7\,,$	$e_4e_5 = e_7\,,$	$e_3e_4 = e_8\,,$	$e_5e_6 = e_8\,,$
$\mathfrak{D}^8_{20}$	:	$e_1e_2 = e_7\,,$	$e_3e_4 = e_7\,,$	$e_5e_6 = e_7\,,$	$e_4e_5 = e_8\,,$
$\mathfrak{D}^8_{21}$	:	$e_1e_2 = e_7\,,$	$e_3e_4 = e_7\,,$	$e_5e_6 = e_7\,,$	$e_2e_3 = e_8\,,$
		$e_4e_5 = e_8\,,$			
$\mathfrak{D}^8_{22}$	:	$e_1e_2 = e_6\,,$	$e_4e_5 = e_6\,,$	$e_2e_3 = e_7\,,$	$e_1e_3 = e_8\,,$
$\mathfrak{D}^8_{23}$	:	$e_1e_2 = e_6\,,$	$e_4e_5 = e_6\,,$	$e_2e_3 = e_7\,,$	$e_3e_4 = e_8\,,$
$\mathfrak{D}^8_{24}$	:	$e_1e_2 = e_6\,,$	$e_2e_3 = e_7\,,$	$e_4e_5 = e_7\,,$	$e_3e_4 = e_8\,,$
$\mathfrak{D}^8_{25}$	:	$e_1e_2 = e_6\,,$	$e_2e_3 = e_7\,,$	$e_4e_5 = e_7\,,$	$e_3e_4 = e_8\,,$
		$e_5e_1 = e_8\,,$			
$\mathfrak{D}^8_{26}$	:	$e_1e_2 = e_6\,,$	$e_1e_3 = e_7\,,$	$e_1e_4 = e_8\,,$	$e_2e_5 = e_7\,,$
$\mathfrak{D}^8_{27}$	:	$e_1e_2 = e_6\,,$	$e_1e_3 = e_7\,,$	$e_1e_4 = e_8\,,$	$e_2e_3 = e_8\,,$
		$e_4e_5 = e_7\,,$			
$\mathfrak{D}^8_{28}$	:	$e_1e_2 = e_6\,,$	$e_1e_3 = e_7\,,$	$e_1e_5 = e_8\,,$	$e_2e_4 = e_8\,,$
		$e_3e_4 = e_6\,,$			
$\mathfrak{D}^8_{29}$	:	$e_1e_2 = e_6\,,$	$e_1e_3 = e_7\,,$	$e_2e_3 = e_8\,,$	$e_1e_4 = e_8\,,$
		$e_2e_5 = e_7\,,$			
$\mathfrak{D}^8_{30}$	:	$e_1e_2 = e_6\,,$	$e_1e_3 = e_7\,,$	$e_2e_3 = e_8\,,$	$e_1e_4 = e_8\,,$
		$e_2e_5 = e_7\,,$	$e_4e_5 = e_6\,,$		
$\mathfrak{D}^8_{31}$	:	$e_1e_2 = e_6\,,$	$e_2e_3 = e_7\,,$	$e_3e_4 = e_7\,,$	$e_4e_5 = e_8\,,$
$\mathfrak{D}^8_{32}$	:	$e_1e_2 = e_6\,,$	$e_2e_3 = e_7\,,$	$e_3e_4 = e_8\,,$	$e_4e_5 = e_7\,,$
		$e_5e_1 = e_7\,,$			
$\mathfrak{D}^8_{33}$	:	$e_1e_2 = e_5\,,$	$e_2e_3 = e_6\,,$	$e_3e_4 = e_7\,,$	$e_4e_1 = e_8\,,$
$\mathfrak{D}^8_{34}$	:	$e_1e_2 = e_5\,,$	$e_1e_3 = e_6\,,$	$e_2e_3 = e_7\,,$	$e_1e_4 = e_8\,,$
$\mathfrak{D}^8_{35}$	:	$e_1e_2 = e_5\,,$	$e_1e_3 = e_6\,,$	$e_2e_4 = e_6\,,$	$e_2e_3 = e_7\,,$
		$e_1e_4 = e_8\,.$			

1.5 The algebraic classification of 9-dimensional dual mock-Lie algebras

As far as a description of 2-step nilpotent Lie algebras is concerned, there is only some particular classification of these algebras [25]. Here, we give the classification of all complex 9-dimensional non-Lie dual mock-Lie algebras. Analyzing the dimension of cohomology spaces of i-dimensional 2-step nilpotent Lie algebras ($i = 3, 4, 5, 6, 7$), we conclude that only $\mathfrak{D}^7_{06}$ maybe give some non-trivial $(9-i)$-dimensional dual mock-Lie cenrtal extensions. Hence, we will calculate 2-dimensional dual mock-Lie central extensions of $\mathfrak{D}^7_{06}$ and 1-dimensional dual mock-Lie extensions of 8-dimensional 2-step nilpotent Lie algebras.

1.5.1 2-dimensional dual mock-Lie central extensions of 7-dimensional 2-step nilpotent Lie algebras

Here we are considering 2-dimensional dual mock-Lie central extensions of $\mathfrak{D}^7_{06}$. Consider the vector space generated by the following two cocycles

$$\theta_1 = \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3 + \alpha_4\nabla_4\,,$$
$$\theta_2 = \beta_2\nabla_2 + \beta_3\nabla_3 + \beta_4\nabla_4\,.$$

It is easy to see, that we can suppose that $\alpha_1\beta_2 \neq 0$. Then by choosing the following nonzero elements

$$a = -\frac{\beta_3}{\alpha_1}, \quad b = -\frac{\beta_4}{\beta_2}, \quad c = \frac{1}{\beta_2},$$
$$d = \frac{\beta_2}{\alpha_1}, \quad h = 1, \quad i = \frac{\alpha_3}{\alpha_1},$$
$$p = -\frac{\alpha_4}{\alpha_1}, \quad v = -\frac{\alpha_2}{\alpha_1}, \quad z = 1,$$

we have the representative $\langle\nabla_1, \nabla_4\rangle$ which gives the following 9-dimensional algebra:

$$\mathfrak{D}^9_{37} : \quad e_1e_2 = e_4\,, \quad e_1e_3 = e_5\,, \quad e_2e_3 = e_6\,,$$
$$e_1e_6 = e_8\,, \quad e_2e_5 = -e_8\,, \quad e_3e_4 = e_8\,,$$
$$e_3e_7 = -e_9\,.$$

1.5.2 1-dimensional dual mock-Lie central extensions of 8-dimensional 2-step nilpotent Lie algebras

By Lemma 1 and [2, Theorem 3.8, 3.9], we have the following dual mock-Lie algebras having nontrivial dual mock-Lie extensions.

$\mathfrak{D}^8$	$\mathrm{H}^2_{\mathfrak{D}}(\mathfrak{D}^8)$
$\mathfrak{D}^8_{06}$	$\langle[\Delta_{16}]-[\Delta_{25}]+[\Delta_{34}],[\Delta_{17}],[\Delta_{18}],[\Delta_{27}],[\Delta_{28}],[\Delta_{37}],[\Delta_{38}],[\Delta_{78}]\rangle$
$\mathfrak{D}^8_{15}$	$\langle[\Delta_{13}],[\Delta_{16}],[\Delta_{17}],[\Delta_{26}],[\Delta_{27}],[\Delta_{36}],[\Delta_{37}]\rangle$
$\mathfrak{D}^8_{16}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{16}],[\Delta_{17}],[\Delta_{23}],[\Delta_{24}],[\Delta_{26}],[\Delta_{27}],[\Delta_{36}],[\Delta_{37}],[\Delta_{46}],[\Delta_{47}]\rangle$
$\mathfrak{D}^8_{17}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{15}],[\Delta_{16}],[\Delta_{23}],[\Delta_{24}],[\Delta_{25}],[\Delta_{26}],[\Delta_{35}],[\Delta_{36}],[\Delta_{45}],[\Delta_{46}]\rangle$
$\mathfrak{D}^8_{18}$	$\langle[\Delta_{14}],[\Delta_{15}],[\Delta_{16}],[\Delta_{23}],[\Delta_{24}],[\Delta_{25}],[\Delta_{26}],[\Delta_{34}],[\Delta_{35}],[\Delta_{36}],[\Delta_{56}]\rangle$
$\mathfrak{D}^8_{19}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{15}],[\Delta_{16}],[\Delta_{23}],[\Delta_{24}],[\Delta_{25}],[\Delta_{26}],[\Delta_{35}],[\Delta_{36}],[\Delta_{46}]\rangle$
$\mathfrak{D}^8_{20}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{15}],[\Delta_{16}],[\Delta_{23}],[\Delta_{24}],[\Delta_{25}],[\Delta_{13}],[\Delta_{26}],[\Delta_{35}],[\Delta_{36}],[\Delta_{46}]\rangle$
$\mathfrak{D}^8_{21}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{15}],[\Delta_{16}],[\Delta_{24}],[\Delta_{25}],[\Delta_{26}],[\Delta_{35}],[\Delta_{36}],[\Delta_{46}]\rangle$
$\mathfrak{D}^8_{22}$	$\langle[\Delta_{14}],[\Delta_{15}],[\Delta_{24}],[\Delta_{25}],[\Delta_{34}],[\Delta_{35}]\rangle$
$\mathfrak{D}^8_{23}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{15}],[\Delta_{24}],[\Delta_{25}],[\Delta_{35}]\rangle$
$\mathfrak{D}^8_{24}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{15}],[\Delta_{24}],[\Delta_{25}],[\Delta_{35}]\rangle$
$\mathfrak{D}^8_{25}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{24}],[\Delta_{25}],[\Delta_{35}]\rangle$
$\mathfrak{D}^8_{26}$	$\langle[\Delta_{15}],[\Delta_{23}],[\Delta_{24}],[\Delta_{34}],[\Delta_{35}],[\Delta_{45}]\rangle$
$\mathfrak{D}^8_{27}$	$\langle[\Delta_{15}],[\Delta_{24}],[\Delta_{15}],[\Delta_{34}],[\Delta_{35}]\rangle$
$\mathfrak{D}^8_{28}$	$\langle[\Delta_{14}],[\Delta_{23}],[\Delta_{25}],[\Delta_{35}],[\Delta_{45}]\rangle$
$\mathfrak{D}^8_{29}$	$\langle[\Delta_{15}],[\Delta_{24}],[\Delta_{34}],[\Delta_{35}],[\Delta_{45}]\rangle$
$\mathfrak{D}^8_{30}$	$\langle[\Delta_{15}],[\Delta_{24}],[\Delta_{34}],[\Delta_{35}]\rangle$
$\mathfrak{D}^8_{31}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{15}],[\Delta_{24}],[\Delta_{25}],[\Delta_{35}]\rangle$
$\mathfrak{D}^8_{32}$	$\langle[\Delta_{13}],[\Delta_{14}],[\Delta_{24}],[\Delta_{25}],[\Delta_{35}]\rangle$
$\mathfrak{D}^8_{33}$	$\langle[\Delta_{13}],[\Delta_{24}]\rangle$
$\mathfrak{D}^8_{34}$	$\langle[\Delta_{15}],[\Delta_{24}],[\Delta_{25}],[\Delta_{34}],[\Delta_{35}],[\Delta_{45}]\rangle$
$\mathfrak{D}^8_{35}$	$\langle[\Delta_{34}]\rangle$

From here, only the algebra $\mathfrak{D}^8_{06}$ maybe have a non-trivial dual mock-Lie central extension. We will find it. The automorphism group $\mathrm{Aut}(\mathfrak{D}^8_{06})$ consists of invertible matrices of the form

$$\varphi=\begin{pmatrix} a & b & c & 0 & 0 & 0 & 0 & 0\\ d & e & f & 0 & 0 & 0 & 0 & 0\\ g & h & k & 0 & 0 & 0 & 0 & 0\\ l & m & n & ae-db & af-dc & bf-ec & p_1 & p_2\\ q & r & s & ah-gb & ak-gc & bk-hc & i_1 & i_2\\ j & t & u & dh-ge & dk-gf & ek-hf & v_1 & v_2\\ w_1 & x_1 & y_1 & 0 & 0 & 0 & z_1 & z_2\\ w_2 & x_2 & y_2 & 0 & 0 & 0 & z_3 & z_4 \end{pmatrix}.$$

Let us use the notations

$$\begin{aligned}
\nabla_1 &:= [\Delta_{16}] - [\Delta_{25}] + [\Delta_{34}], & \nabla_2 &:= [\Delta_{17}],\\
\nabla_3 &:= [\Delta_{18}], & \nabla_4 &:= [\Delta_{27}],\\
\nabla_5 &:= [\Delta_{28}], & \nabla_6 &:= [\Delta_{37}],\\
\nabla_7 &:= [\Delta_{38}], & \nabla_8 &:= [\Delta_{78}].
\end{aligned}$$

Take $\theta = \sum_{i=1}^{8} \alpha_i \nabla_i \in \mathrm{H}^2_{\mathfrak{D}}(\mathfrak{D}^8_{06}, \mathbb{C})$. If $\varphi \in \mathrm{Aut}(\mathfrak{D}^8_{06})$, then

$$\varphi^T \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \alpha_1 & \alpha_2 & \alpha_3\\
0 & 0 & 0 & 0 & -\alpha_1 & 0 & \alpha_4 & \alpha_5\\
0 & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_6 & \alpha_7\\
0 & 0 & -\alpha_1 & 0 & 0 & 0 & 0 & 0\\
0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0\\
-\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\alpha_2 & -\alpha_4 & -\alpha_6 & 0 & 0 & 0 & 0 & \alpha_8\\
-\alpha_3 & -\alpha_5 & -\alpha_7 & 0 & 0 & 0 & -\alpha_8 & 0
\end{pmatrix} \varphi$$

$$= \begin{pmatrix}
0 & \beta_1^* & \beta_2^* & 0 & 0 & \alpha_1^* & \alpha_2^* & \alpha_3^*\\
-\beta_1^* & 0 & \beta_3^* & 0 & -\alpha_1^* & 0 & \alpha_4^* & \alpha_5^*\\
-\beta_2^* & -\beta_3^* & 0 & \alpha_1^* & 0 & 0 & \alpha_6^* & \alpha_7 *\\
0 & 0 & -\alpha_1^* & 0 & 0 & 0 & 0 & 0\\
0 & \alpha_1^* & 0 & 0 & 0 & 0 & 0 & 0\\
-\alpha_1^* & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-\alpha_2^* & -\alpha_4^* & -\alpha_6^* & 0 & 0 & 0 & 0 & \alpha_8^*\\
-\alpha_3^* & -\alpha_5^* & -\alpha_7^* & 0 & 0 & 0 & -\alpha_8^* & 0
\end{pmatrix},$$

where

$$\begin{aligned}
\alpha_1^* &= -(ceg - bfg - cdh + afh + bdk - aek)\alpha_1\\
\alpha_2^* &= (-di_1 + gp_1 + av_1)\alpha_1 + (a\alpha_2 + d\alpha_4 + g\alpha_6 - w_2\alpha_8)z_1\\
&\quad + (a\alpha_3 + d\alpha_5 + g\alpha_7 + w_1\alpha_8)z_3\\
\alpha_3^* &= (-di_2 + gp_2 + av_2)\alpha_1 + (a\alpha_2 + d\alpha_4 + g\alpha_6 - w_2\alpha_8)z_2\\
&\quad + (a\alpha_3 + d\alpha_5 + g\alpha_7 + w_1\alpha_8)z_4\\
\alpha_4^* &= (-ei_1 + hp_1 + bv_1)\alpha_1 + (b\alpha_2 + e\alpha_4 + h\alpha_6 - x_2\alpha_8)z_1\\
&\quad + (b\alpha_3 + e\alpha_5 + h\alpha_7 + x_1\alpha_8)z_3\\
\alpha_5^* &= (ei_2 + hp_2 + bv_2)\alpha_1 + (b\alpha_2 + e\alpha_4 + h\alpha_6 - x_2\alpha_8)z_2\\
&\quad + (b\alpha_3 + e\alpha_5 + h\alpha_7 + x_1\alpha_8)z_4\\
\alpha_6^* &= (-fi_1 + kp_1 + cv_1)\alpha_1 + (c\alpha_2 + f\alpha_4 + k\alpha_6 - y_1\alpha_8)z_1\\
&\quad + (c\alpha_3 + f\alpha_5 + k\alpha_7 + y_1\alpha_8)z_3\\
\alpha_7^* &= (-fi_2 + kp_2 + cv_2)\alpha_1 + (c\alpha_2 + f\alpha_4 + k\alpha_6 - y_1\alpha_8)z_2\\
&\quad + (c\alpha_3 + f\alpha_5 + k\alpha_7 + y_1\alpha_8)z_4\\
\alpha_8^* &= (-z_2z_3 + z_1z_4)\alpha_8 .
\end{aligned}$$

Hence, $\phi\langle\theta\rangle = \langle\theta^*\rangle$, where $\theta^* = \sum_{i=1}^{8} \alpha_i^* \nabla_i$. Here, we have the following situations:

1. $\alpha_1, \alpha_8 \neq 0$, then by choosing the following nonzero elements

$$c = 1\,, \quad e = 1\,, \quad h = 1\,, \quad k = 1\,,$$
$$g = -\frac{1}{\alpha_1}\,, \quad z_1 = \frac{1}{\alpha_8}\,, \quad z_2 = 2\,, \quad z_4 = 1\,,$$

$$y_1 = \frac{-\alpha_3 + \alpha_5}{\alpha_8}\,, \qquad x_2 = \frac{-\alpha_2 - \alpha_3 + \alpha_4 + \alpha_5}{\alpha_8}\,,$$
$$p_1 = -\frac{\alpha_2 + \alpha_3 - \alpha_5 + \alpha_6}{\alpha_1 \alpha_8}\,, \qquad p_2 = -\frac{2\alpha_2 + 2\alpha_3 - \alpha_5 + 2\alpha_6 + \alpha_7}{\alpha_1}\,,$$
$$w_1 = -\frac{\alpha_5}{\alpha_1 \alpha_8}\,, \qquad w_2 = \frac{\alpha_2 + \alpha_3 - \alpha_5}{\alpha_1 \alpha_8}\,,$$

we have the representative $\langle\nabla_1 + \nabla_8\rangle$. Now we have the new 9-dimensional dual mock-Lie algebra:

$$\mathfrak{D}^9_{38} \quad : \quad e_1e_2 = e_4\,, \quad e_1e_3 = e_5\,, \quad e_2e_3 = e_6\,, \quad e_1e_6 = e_9\,,$$
$$e_2e_5 = -e_9\,, \quad e_3e_4 = e_9\,, \quad e_7e_8 = e_9\,.$$

2. $\alpha_1 \neq 0, \alpha_8 = 0$, then by choosing the following nonzero elements

$$a = \frac{1}{\alpha_1}\,, \quad e = 1\,, \quad k = 1\,,$$
$$v_1 = -\frac{\alpha_2}{\alpha_1}\,, \quad v_2 = -\frac{\alpha_3}{\alpha_1}\,, \quad p_1 = -\frac{\alpha_6}{\alpha_1}\,,$$
$$p_2 = -\frac{\alpha_7}{\alpha_1}\,, \quad i_1 = \frac{\alpha_4}{\alpha_1}\,, \quad i_2 = \frac{\alpha_5}{\alpha_1}\,,$$

we have the representative $\langle\nabla_1\rangle$ and it is a split algebra.

3. if $\alpha_1, \alpha_8 = 0$, then we can suppose that $\alpha_7 \neq 0$ and by choosing the following nonzero elements

$$a = 1\,, \quad e = 1\,, \quad k = 1\,,$$
$$f = 1\,, \quad z_1 = 1\,, \quad z_4 = 1\,,$$
$$g = -\frac{\alpha_2}{\alpha_6}\,, \quad h = -\frac{\alpha_4}{\alpha_6}\,, \quad k = -\frac{\alpha_4}{\alpha_6}\,,$$

then we have a representative from $\langle\nabla_2, \nabla_4, \nabla_6\rangle$, which gives a split algebra.

Summarizing, we have the following theorem.

Theorem 1. *Let $\mathfrak{D}$ be a complex 9-dimensional indecomposable non-Lie dual mock--Lie algebra, then $\mathfrak{D}$ is isomorphic to $\mathfrak{D}^9_{37}$ or $\mathfrak{D}^9_{38}$:*

$$\mathfrak{D}^9_{37} \quad : \quad e_1e_2 = e_4\,, \quad e_1e_3 = e_5\,, \quad e_2e_3 = e_6\,, \quad e_1e_6 = e_8\,,$$
$$e_2e_5 = -e_8\,, \quad e_3e_4 = e_8\,, \quad e_3e_7 = -e_9\,,$$
$$\mathfrak{D}^9_{38} \quad : \quad e_1e_2 = e_4\,, \quad e_1e_3 = e_5\,, \quad e_2e_3 = e_6\,, \quad e_1e_6 = e_9\,,$$
$$e_2e_5 = -e_9\,, \quad e_3e_4 = e_9\,, \quad e_7e_8 = e_9\,.$$

2 Degenerations of dual mock-Lie algebras

2.1 Degenerations of algebras

Given an n-dimensional vector space $\mathbf{V}$, the set $\mathrm{Hom}(\mathbf{V}\otimes\mathbf{V},\mathbf{V}) \cong \mathbf{V}^*\otimes\mathbf{V}^*\otimes\mathbf{V}$ is a vector space of dimension n^3. This space has a structure of the affine variety $\mathbb{C}^{n^3}$. Indeed, let us fix a basis $e_1,\dots,e_n$ of $\mathbf{V}$. Then any $\mu \in \mathrm{Hom}(\mathbf{V}\otimes\mathbf{V},\mathbf{V})$ is determined by n^3 structure constants $c_{i,j}^k \in \mathbb{C}$ such that

$$\mu(e_i \otimes e_j) = \sum_{k=1}^{n} c_{i,j}^k e_k\,.$$

A subset of $\mathrm{Hom}(\mathbf{V}\otimes\mathbf{V},\mathbf{V})$ is *Zariski-closed* if it can be defined by a set of polynomial equations in the variables $c_{i,j}^k$ $(1 \le i,j,k \le n)$.

Let T be a set of polynomial identities. All algebra structures on $\mathbf{V}$ satisfying polynomial identities from T form a Zariski-closed subset of the variety

$$\mathrm{Hom}(\mathbf{V}\otimes\mathbf{V},\mathbf{V})\,.$$

We denote this subset by $\mathbb{L}(T)$. The general linear group $\mathrm{GL}(\mathbf{V})$ acts on $\mathbb{L}(T)$ by conjugation:

$$(g * \mu)(x\otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x,y \in \mathbf{V}$, $\mu \in \mathbb{L}(T) \subset \mathrm{Hom}(\mathbf{V}\otimes\mathbf{V},\mathbf{V})$ and $g \in \mathrm{GL}(\mathbf{V})$. Thus, $\mathbb{L}(T)$ is decomposed into $\mathrm{GL}(\mathbf{V})$-orbits that correspond to the isomorphism classes of algebras. Let $O(\mu)$ denote the $\mathrm{GL}(\mathbf{V})$-orbit of $\mu \in \mathbb{L}(T)$ and $\overline{O(\mu)}$ its Zariski closure.

Let $\mathbf{A}$ and $\mathbf{B}$ be two n-dimensional algebras satisfying identities from T and $\mu,\lambda \in \mathbb{L}(T)$ represent $\mathbf{A}$ and $\mathbf{B}$ respectively. We say that $\mathbf{A}$ *degenerates to* $\mathbf{B}$ and write $\mathbf{A} \to \mathbf{B}$ if $\lambda \in \overline{O(\mu)}$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of μ and λ. If $\mathbf{A} \not\cong \mathbf{B}$, then the assertion $\mathbf{A} \to \mathbf{B}$ is called a *proper degeneration*. We write $\mathbf{A} \not\to \mathbf{B}$ if $\lambda \notin \overline{O(\mu)}$.

Let $\mathbf{A}$ be represented by $\mu \in \mathbb{L}(T)$. Then $\mathbf{A}$ is *rigid* in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called *irreducible* if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an *irreducible component*. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra $\mathbf{A}$ is rigid in $\mathbb{L}(T)$ if and only if $\overline{O(\mu)}$ is an irreducible component of $\mathbb{L}(T)$.

In the present work, we use the methods applied to Lie algebras in [5], [26], [12], [13]. First of all, if $\mathbf{A} \to \mathbf{B}$ and $\mathbf{A} \not\cong \mathbf{B}$, then $\dim \mathfrak{Der}(\mathbf{A}) < \dim \mathfrak{Der}(\mathbf{B})$, where $\mathfrak{Der}(\mathbf{A})$ is the Lie algebra of derivations of $\mathbf{A}$. We will compute the dimensions of algebras of derivations and will check the assertion $\mathbf{A} \to \mathbf{B}$ only for such $\mathbf{A}$ and $\mathbf{B}$ that $\dim \mathfrak{Der}(\mathbf{A}) < \dim \mathfrak{Der}(\mathbf{B})$. Secondly, if $\mathbf{A} \to \mathbf{C}$ and $\mathbf{C} \to \mathbf{B}$ then $\mathbf{A} \to \mathbf{B}$. If there is no $\mathbf{C}$ such that $\mathbf{A} \to \mathbf{C}$ and $\mathbf{C} \to \mathbf{B}$ are proper degenerations, then the assertion $\mathbf{A} \to \mathbf{B}$ is called a *primary degeneration*. It is easy to see that any algebra degenerates to the algebra with zero multiplication. From now on we use this fact without mentioning it.

To prove primary degenerations, we will construct families of matrices parametrized by t. Namely, let $\mathbf{A}$ and $\mathbf{B}$ be two algebras represented by the structures μ and λ from $\mathbb{L}(T)$ respectively. Let $e_1, \dots, e_n$ be a basis of $\mathbf{V}$ and $c_{i,j}^k$ $(1 \le i,j,k \le n)$ be the structure constants of λ in this basis. If there exist $a_i^j(t) \in \mathbb{C}$ $(1 \le i,j \le n$, $t \in \mathbb{C}^*)$ such that

$$E_i^t = \sum_{j=1}^{n} a_i^j(t) e_j \,, \quad 1 \le i \le n,$$

form a basis of $\mathbf{V}$ for any $t \in \mathbb{C}^*$, and the structure constants $c_{i,j}^k(t)$ of μ in the basis $E_1^t, \dots, E_n^t$ satisfy

$$\lim_{t \to 0} c_{i,j}^k(t) = c_{i,j}^k \,,$$

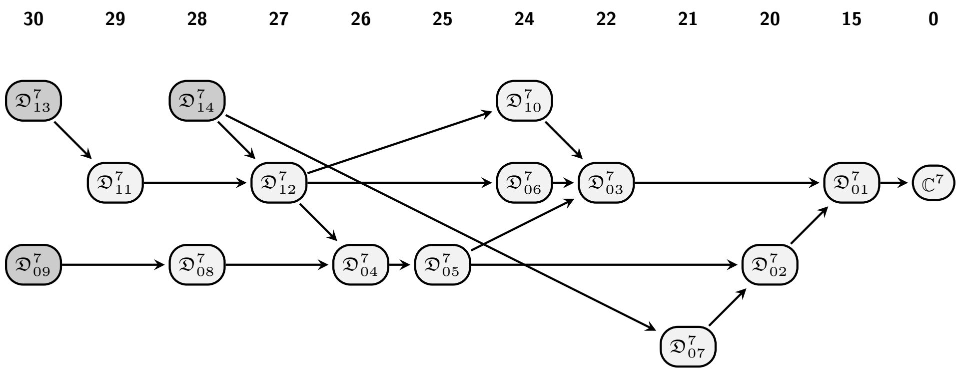
then $\mathbf{A} \to \mathbf{B}$. In this case $E_1^t, \dots, E_n^t$ is called a *parametric basis* for $\mathbf{A} \to \mathbf{B}$.

If the number of orbits under the action of $\mathrm{GL}(\mathbf{V})$ on $\mathbb{L}(T)$ is finite, then the graph of primary degenerations gives the whole picture. In particular, the description of rigid algebras and irreducible components can be easily obtained.

2.2 The geometric classification of dual mock-Lie algebras

2.2.1 Degenerations of 7-dimensional dual mock-Lie algebras

Theorem 2. *The variety of complex 7-dimensional dual mock-Lie algebras is 30-dimensional and has three irreducible components defined by rigid algebras $\mathfrak{D}_{09}^7, \mathfrak{D}_{13}^7$ and $\mathfrak{D}_{14}^7$. The complete graph of degenerations in the given variety is presented below.*



Proof. Thanks to [3] we have all degenerations in the variety of all 7-dimensional 2-step nilpotent Lie algebras. By an easy calculation, we have that the dimension of the algebra of derivation of the algebra $\mathfrak{D}_{14}^7$ is 21. Hence, it can not degenerate to $\mathfrak{D}_{08}^7, \mathfrak{D}_{09}^7, \mathfrak{D}_{11}^7, \mathfrak{D}_{13}^7$.

The degeneration $\mathfrak{D}_{14}^7 \to \mathfrak{D}_{12}^7$ is obtained by the following parametric basis

$$E_1^t = te_4\,, \quad E_2^t = t^2e_2 - e_3\,, \quad E_3^t = te_3 + te_5 + t^3e_6\,, \quad E_4^t = e_1 + e_2 + t^2e_4 - e_5\,,$$
$$E_5^t = te_7\,, \quad E_6^t = t^3e_6\,, \quad E_7^t = e_5 + e_6\,.$$

The degeneration $\mathfrak{D}^7_{14} \to \mathfrak{D}^7_{07}$ is obtained by the following parametric basis

$$E_1^t = te_1\,, \quad E_2^t = e_6\,, \quad E_3^t = e_2\,, \quad E_4^t = -te_5\,,$$
$$E_5^t = te_3\,, \quad E_6^t = e_4\,, \quad E_7^t = te_7\,.$$

□

Remark 1. Note that the graph of primary degenerations of 7-dimensional 2-step nilpotent Lie algebras from [3] is not correct. We gave the corrected graph of degenerations of 7-dimensional 2-step nilpotent Lie algebras from [3].

2.2.2 The geometric classification of 8-dimensional dual mock-Lie algebras

Thanks to [2] we have that the variety of 8-dimensional 2-step nilpotent Lie algebras has three rigid algebras: $\mathfrak{D}^8_{17}, \mathfrak{D}^8_{30}$ and $\mathfrak{D}^8_{33}$. It is easy to see, that the algebra $\mathfrak{D}^8_{36}$ is satisfying the following invariant conditions $A_4A_5 = 0$ and $A_1A_4 \subseteq A_8$, but the cited algebras do not satisfy it. It follows that there are no degenerations $\mathfrak{D}^8_{36} \to \mathfrak{D}^8_{17}, \mathfrak{D}^8_{30}, \mathfrak{D}^8_{33}$.

The degeneration $\mathfrak{D}^8_{36} \to \mathfrak{D}^8_{14}$ is obtained by the following parametric basis

$$E_1^t = e_1\,, \quad E_2^t = e_2\,, \quad E_3^t = e_3\,, \quad E_4^t = e_4\,,$$
$$E_5^t = e_5\,, \quad E_6^t = e_6\,, \quad E_7^t = e_8\,, \quad E_8^t = te_7\,.$$

Hence, we have the following

Theorem 3. *The variety of complex* 8*-dimensional dual mock-Lie algebras has four irreducible components defined by rigid algebras* $\mathfrak{D}^8_{17}, \mathfrak{D}^8_{30}, \mathfrak{D}^8_{33}$ *and* $\mathfrak{D}^8_{36}$.

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