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Applications of Mathematics, Vol. 66 (2021), No. 5, 673–699

Persistent URL: http://dml.cz/dmlcz/149078

UNIFIED ERROR ANALYSIS OF DISCONTINUOUS GALERKIN METHODS FOR PARABOLIC OBSTACLE PROBLEM

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Received April 20, 2020. Published online May 25, 2021.

Abstract. We introduce and study various discontinuous Galerkin (DG) finite element approximations for a parabolic variational inequality associated with a general obstacle problem in $\mathbb{R}^d$ $(d = 2, 3)$. For the fully-discrete DG scheme, we employ a piecewise linear finite element space for spatial discretization, whereas the time discretization is carried out with the implicit backward Euler method. We present a unified error analysis for all well known symmetric and non-symmetric DG fully discrete schemes, and derive error estimate of optimal order $\mathcal{O}(h+\Delta t)$ in an energy norm. Moreover, the analysis is performed without any assumptions on the speed of propagation of the free boundary and only the realistic regularity $u_t \in \mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))$ is assumed. Further, we present some numerical experiments to illustrate the performance of the proposed methods.

Keywords: finite element; discontinuous Galerkin method; parabolic obstacle problem

MSC 2020: 65N30, 65N15

1. Introduction

A parabolic obstacle problem is basically studied as a prototype model for parabolic variational inequalities. The numerical analysis of obstacle problems has become an effective and powerful technique for studying a wide class of problems arising in various branches of mathematical and engineering sciences in a unified and general framework. The numerical approximation of variational inequalities poses several challenges in handling the constraints, devising interpolation operators that obey the constraints, analysis with the limited regularity of the solution and implementations. Some enormous application of parabolic obstacle

The work was supported by the Council of Scientific and Industrial Research, India [RP03792G].

DOI: 10.21136/AM.2021.0030-20

problems are American option problem, Stefan problem and electrochemical machining problem, etc. The finite element method seems to be the most widely used numerical method in applied mathematics. In particular, there has been an active growth of discontinuous Galerkin (DG) methods in the past decades, due to the fact that DG methods disagree from the usual finite element methods. In DG methods, functions are allowed to be discontinuous over the element boundaries. Taking the full advantage of non requirement of inter element continuity, DG methods permit general meshes with hanging nodes and element of different shapes, and create more flexibility in mesh refinement. There exist several DG methods for the discretization of partial differential equations, variational inequalities, etc. Here we mention various well known DG methods, for example, SIPG method [1], [43], NIPG method [38], IIPG method [13], [24], [19], LDG method [12], [17], Bassi et al. [5], Brezzi et al. [10], Babuška-Zlámal [3], Brezzi et al. [11], WOPSIP [7], etc. These methods can be extended for solving parabolic obstacle problems.

In this article we derive a unified *a priori* error analysis for various symmetric and non-symmetric DG methods of parabolic variational inequality. Let $[0,T]\subset\mathbb{R}$ be the time interval and $\Omega\subset\mathbb{R}^d$ $(d=2,3)$ the bounded convex polygonal domain with boundary $\partial\Omega$. Convexity of the domain is essential in the analysis in order to have $H^2(\Omega)$ regularity of the solution of model problem which has been used in the error analysis. Let us denote by $\mathcal{H}^m(\Omega)$ the Sobolev space for $p=2$ and by $\mathcal{H}^1_0(\Omega)$ the subspace of $\mathcal{H}^1(\Omega)$ with zero trace. We denote by $\mathcal{H}^{-1}(\Omega)$ the dual space of $\mathcal{H}^1_0(\Omega)$. We use the notation $\langle\cdot,\cdot\rangle$ for the duality pairing between $\mathcal{H}^{-1}(\Omega)$ and $\mathcal{H}^1_0(\Omega)$. Also, the inner product in $\mathcal{L}^2(\Omega)$ is denoted by $(\cdot,\cdot)$. For $1\leqslant p\leqslant\infty$, let us consider $L^p(0,T;\mathcal{Y})$ as the spaces of all Lebesgue measurable functions $\Lambda\colon [0,T]\to\mathcal{Y}$ with bounded norm

$$\|\Lambda\|_{L^p(0,T;\mathcal{Y})} := \begin{cases} \left(\displaystyle\int_0^T \|\Lambda(t)\|^p_{\mathcal{Y}}\,\mathrm{d}t\right)^{1/p} & \text{if } 1\leqslant p<\infty,\\ \operatorname*{ess\,sup}_{t\in(0,T)} \|\Lambda(t)\|_{\mathcal{Y}} & \text{if } p=\infty.\end{cases}$$

Let $\mathcal{C}([0,T];\mathcal{Y})$ denote the space of all continuous functions $\Lambda\colon [0,T]\to\mathcal{Y}$. We also introduce the space $\mathcal{BV}(0,T;\mathcal{Y})$ of $\mathcal{Y}$-valued functions of bounded variation ([9], Definition A.2)

$$\mathrm{Var}_{\mathcal{Y}}\Phi := \sup_{\beta}\left\{\sum_{n=1}^{N}\|\Phi(\tau_n)-\Phi(\tau_{n-1})\|_{\mathcal{Y}}\right\}<\infty,$$

where the supremum is taken over all partitions $\beta=\{0=\tau_0<\ldots<\tau_n<\ldots<\tau_N=T\}$ of the time interval $[0,T]$. It is well known that if $\Phi\in\mathcal{BV}(0,T;\mathcal{Y})$, then at every point $s_0 \in [0,T)$ there exists the right limit $\Phi_+(s_0) = \lim\limits_{s\downarrow s_0} \Phi(s)$ (cf. [9], Definition A.2). Let us define a bilinear form a by $a(w,v) = (\nabla w, \nabla v)$. We consider the obstacle $\chi \in \mathcal{H}^2(\Omega)$ with $\chi|_{\partial\Omega} \leqslant 0$. Let us define the closed convex set by

$$\mathcal{K}_\chi := \{v \in \mathcal{H}_0^1(\Omega)\colon\ v \geqslant \chi \text{ a.e. in } \Omega\}.$$

For given initial condition $w_0 \in \mathcal{K}_\chi$ and $g \in \mathcal{L}^2(0,T;\mathcal{H}^{-1}(\Omega))$, the parabolic obstacle problem is to find $w\colon\ [0,T] \to \mathcal{K}_\chi$ such that

$$\Big\langle \frac{\partial w}{\partial t}(t), v - w(t) \Big\rangle + a(w(t), v - w(t)) \geqslant \langle g(t), v - w(t)\rangle \qquad \forall v \in \mathcal{K}_\chi,\ \text{a.e. on } (0,T), \tag{1.1}$$

$$w(0,x) = w_0(x) \quad x \in \Omega. \tag{1.2}$$

The obstacle problem (1.1)–(1.2) has a unique solution $w \in \mathcal{C}([0,T];\mathcal{L}^2(\Omega)) \cap \mathcal{L}^2(0,T;\mathcal{H}_0^1(\Omega))$. Further, if $g \in \mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))$, then $w \in \mathcal{C}([0,T];\mathcal{H}_0^1(\Omega)) \cap \mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))$ and $\frac{\partial w}{\partial t} \in \mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))$ (cf. [8]). The constraint $w(t) \geqslant \chi$ yields free boundary, which is the boundary of the contact set $\Omega^0(t) := \{x \in \Omega\colon\ w(t) = \chi\}$ and non-contact set $\Omega^+(t) := \{x \in \Omega\colon\ w(t) > \chi\}$.

The *a priori* error analysis for parabolic obstacle problem can be traced back to 1970's, see e.g. [6], [28]. Subsequently, there are substantial works on *apriori* error estimates for parabolic obstacle problems, see e.g. [6], [18], [34], [33], [35], [36], [39], [42], [44]. Furthermore, some *aposteriori* error analysis are studied for parabolic obstacle problems, see e.g. [32], [4]. Moreover, some related numerical analysis for parabolic variational inequalities and their solvers may be found in [27], [26]. Recently in [22], an error estimate of order

$$\mathcal{O}\Big(h + \Big(\log\frac{1}{\Delta t}\Big)^{1/4} \Delta t^{3/4}\Big)$$

is derived for conforming finite element approximation of the parabolic obstacle problem in two dimensions with general obstacle (non-affine) generalizing the analysis in [28] for zero obstacle problem. Also in [21], a conforming and discontinuous Galerkin methods for the parabolic obstacle problem with general obstacle are proposed and analyzed. In [21] we have derived an error estimate of order $\mathcal{O}(h + \Delta t)$ in an energy norm, and thus improved the analysis in [22]. For a zero obstacle problem, a similar error estimate is derived in [45] for conforming finite element approximation and utilizing the time-discrete analysis in [34], Theorem 3.20 by incorporating the obstacle constraints by using positivity preserving interpolation [14]. Further in [23], we have studied an error analysis with order of convergence $\mathcal{O}(h + \Delta t)$ in an energy norm for non-conforming Crouzeix-Raviart approximation in dimension two. The elements in Crouzeix-Raviart finite element space do not preserve the sign in the interior of each triangle even though they have right sign at the nods, i.e. the mid-points of the edges. But that is not the case for the conforming or the discontinuous Galerkin methods, which differs the analysis of the conforming or the discontinuous Galerkin methods from the analysis of the Crouzeix-Raviart non-conforming finite element method.

In this article, we improve upon the error analysis in [21] by making use of the time discrete analysis in [34], Theorem 3.20 or [33], [39] for time discretization and combining it with the spatial discretization using the discontinuous Galerkin methods for both two and three space dimensional parabolic obstacle problem. Particularly, the error analysis in [21] has been proposed only for all the well known symmetric DG methods. But the non-symmetric DG Methods are also equally important as the symmetric DG methods. For example, NIPG method ensures coercivity of the corresponding bilinear form of discrete problem, also a significant property of this method is that it is unconditionally stable with respect to the choice of the penalty parameter. However, the symmetric methods as well as NIPG method are not suitable for discretization of all problems as example quasilinear nonstationary convection-diffusion problems, Navier-stokes equations. For this a suitable DG method is IIPG method ([13], [24], [19] etc.), since some stabilization terms are missing, although IIPG has not the favorable properties as SIPG and NIPG. In this article we take care of all the well known symmetric and non-symmetric DG methods in a unified analysis. By taking full advantage of the regularity result in [40], Theorem 4 and the $\mathcal{H}^1$-stability and commutative property (with the time derivative) of the positive preserving interpolation (which was introduced in [14]), we derive error estimate of order $\mathcal{O}(h+\Delta t)$ for all the symmetric and non-symmetric fully-discrete DG schemes, in an energy norm defined precisely in the article. In our setting the obstacle χ is a general function in $\mathcal{H}^2(\Omega)$ and need not be zero or affine function.

The structure of the article is as follows. In Section 2, we introduce the model problem and the semi time-discrete problem. The fully discrete problems have been introduced in Section 3. In Section 4, we present a unified *a priori* error analysis for various symmetric and non-symmetric DG methods. Finally, Section 5 is devoted to discussing some numerical experiments to illustrate the theoretical results.

2. Model problem and semi-discrete problem

For some positive integer N let $0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = T$ be a partition of $[0,T]$ with variable step-size $\Delta_n t = t_n - t_{n-1}$ for $n = 1, \ldots, N$. Denote by $\Delta t := \max\limits_{1\leqslant n\leqslant N} \Delta_n t$ the largest time-step. Given a function $g\colon [0,T] \to \mathcal{Y}$, we denote $g^n = g(t_n) \in \mathcal{Y}$. On a discrete set $\{g^0, g^1, g^2, \ldots, g^N\}$ of $N+1$ points we define

$$\partial g^n = \frac{g^n - g^{n-1}}{\Delta_n t} \quad \text{for } n = 1, 2, \ldots, N.$$

For any sequence $\{\mathcal{F}^n\}_{n=0}^N \in \mathcal{Y}$ we define the piecewise constant (in time) interpolant $\overline{\mathcal{F}} \in \mathcal{L}^\infty(0,T;\mathcal{Y})$ by

$$\overline{\mathcal{F}}(t) = \mathcal{F}^n \quad \forall\, t \in (t_{n-1}, t_n] \quad \text{for } n = 1, \ldots, N. \tag{2.1}$$

We also define the piecewise linear (in time) interpolant $\widetilde{\mathcal{F}} \in \mathcal{C}([0,T];\mathcal{Y})$ by

$$\widetilde{\mathcal{F}}(t) = \frac{t - t_{n-1}}{\Delta_n t}\mathcal{F}^n + \frac{t_n - t}{\Delta_n t}\mathcal{F}^{n-1} \quad \forall\, t \in [t_{n-1}, t_n] \text{ for } n = 1, \ldots, N. \tag{2.2}$$

We note that

$$\frac{\partial \widetilde{\mathcal{F}}}{\partial t}(t) = \partial \mathcal{F}^n \quad \forall\, t \in (t_{n-1}, t_n) \text{ for } n = 1, \ldots, N. \tag{2.3}$$

Moreover, for any sequence $\{\mathcal{F}^n\}_{n=0}^N \subset \mathcal{Y}$ we notice that

$$\|\overline{\mathcal{F}}\|_{L^p(0,T;\mathcal{Y})} = \Bigl(\sum_{n=1}^N \Delta_n t \|\mathcal{F}^n\|_{\mathcal{Y}}^p\Bigr)^{1/p} \quad \text{if } 1 \leqslant p < \infty,$$

and

$$\|\overline{\mathcal{F}}\|_{\mathcal{L}^\infty(0,T;\mathcal{Y})} = \|\widetilde{\mathcal{F}}\|_{\mathcal{L}^\infty(0,T;\mathcal{Y})} = \max_{n\in\{0,\ldots,N\}} \|\mathcal{F}^n\|_{\mathcal{Y}}.$$

Model problem: For the given initial condition $w_0 \in \mathcal{K}_\chi \cap \mathcal{H}^2(\Omega)$ and $g \in \mathcal{BV}(0,T\colon \mathcal{L}^2(\Omega))$, the model problem is to find $w\colon [0,T] \to \mathcal{K}_\chi$ such that

$$\Bigl(\frac{\partial w}{\partial t}(t), v - w(t)\Bigr) + a(w(t), v - w(t)) \geqslant (g(t), v - w(t)) \quad \forall v \in \mathcal{K}_\chi, \text{ a.e. on } (0,T), \tag{2.4}$$

$$w(x,0) = w_0(x), \quad x \in \Omega. \tag{2.5}$$

Here, we introduce the semi-discrete problem and state some well-known results, which are crucial for the subsequent analysis.

Semi-discrete problem: For any given $g \in \mathcal{BV}(0,T;\mathcal{L}^2(\Omega))$ we define $\mathcal{G}^n := g_+(t_n)$. Then, for $1 \leqslant n \leqslant N$ we find $\mathcal{W}^n \in \mathcal{K}_\chi$ such that

$$(\partial \mathcal{W}^n, v - \mathcal{W}^n) + a(\mathcal{W}^n, v - \mathcal{W}^n) \geqslant (\mathcal{G}^n, v - \mathcal{W}^n) \quad \forall v \in \mathcal{K}_\chi, \tag{2.6}$$

$$\mathcal{W}^0 = w_0 \in \mathcal{K}_\chi. \tag{2.7}$$

From Lions-Stampacchia theorem (cf. [30], [31]) the inequality (2.6)–(2.7) has a unique solution for any $1 \leqslant n \leqslant N$.

By applying integration by parts and using (2.6), we note that

$$a(\mathcal{W}^n, v - \mathcal{W}^n) \geqslant (\mathcal{G}^n - \partial \mathcal{W}^n, v - \mathcal{W}^n),$$
$$-(\Delta \mathcal{W}^n, v - \mathcal{W}^n) \geqslant (\mathcal{G}^n - \partial \mathcal{W}^n, v - \mathcal{W}^n).$$

Now by using the regularity results of elliptic obstacle problem, we deduce

$$\mathcal{W}^n \in \mathcal{H}^2(\Omega) \text{ and } \|\mathcal{W}^n\|_{\mathcal{H}^2(\Omega)} \leqslant C(\|\mathcal{G}^n - \partial \mathcal{W}^n\|_{\mathcal{L}^2(\Omega)} + \|\chi\|_{\mathcal{H}^2(\Omega)}). \tag{2.8}$$

Lemma 2.1. *Let $\mathcal{W}^n \in \mathcal{K}_\chi$ be the solution of* (2.6) *and* (2.7)*. Then $\mathcal{W}^n$ satisfies the following:*

$$\partial \mathcal{W}^n - \Delta \mathcal{W}^n - \mathcal{G}^n \geqslant 0 \quad \text{a.e. in } \Omega, \tag{2.9}$$

$$(\partial \mathcal{W}^n - \Delta \mathcal{W}^n - \mathcal{G}^n) = 0 \quad \text{a.e. in } \{x \in \Omega\colon \mathcal{W}^n > \chi\}. \tag{2.10}$$

Proof. The proofs of (2.9) and (2.10) follow from the theory of the elliptic obstacle problems (cf. [20], [29], [23]). □

The following lemma on convergence and regularity results (see [40], Theorem 4; [34], Theorem 3.20) for the semi-discrete solution of problem (2.6)–(2.7) will be crucial in the forthcoming error analysis.

Lemma 2.2 ([34], Theorem 3.20; [40], Theorem 4)**.** *For any $\mathcal{W}^0 \in \mathcal{H}^{-1}(\Omega)$, problem* (2.6) *has a unique solution $\{\mathcal{W}^n\}$ and $\mathcal{W}^n \in \mathcal{K}_\chi$ for $n = 1, \ldots, N$. If $\mathcal{W}^0 = w_0 \in \mathcal{K}_\chi$ and $g \in \mathcal{L}^1(0,T;\mathcal{L}^2(\Omega)) + \mathcal{L}^2(0,T;\mathcal{H}^{-1}(\Omega))$, then we have*

$$\widetilde{\mathcal{W}} \in \mathcal{L}^\infty(0,T;\mathcal{L}^2(\Omega)) \cap \mathcal{L}^2(0,T;\mathcal{H}^1_0(\Omega)),$$

where $\widetilde{\mathcal{W}}$ is the piecewise linear interpolant for the sequence $\{\mathcal{W}^n\}_{n=0}^N$. Further, if $g \in \mathcal{BV}(0,T;\mathcal{L}^2(\Omega))$, then we have $\frac{\partial \widetilde{\mathcal{W}}}{\partial t} \in \mathcal{L}^\infty(0,T;\mathcal{L}^2(\Omega)) \cap \mathcal{L}^2(0,T;\mathcal{H}^1_0(\Omega))$ and *there exists a constant C dependent on g and w_0 such that*

$$\max\Bigg\{\max_{0\leqslant t\leqslant T}\|w-\widetilde{\mathcal{W}}\|_{\mathcal{L}^2(\Omega)},\ \left(\int_0^T\|\nabla(w-\widetilde{\mathcal{W}})\|^2_{\mathcal{L}^2(\Omega)}\,\mathrm{d}t\right)^{1/2},$$
$$\left(\int_0^T\|\nabla(w-\overline{\mathcal{W}})\|^2_{\mathcal{L}^2(\Omega)}\,\mathrm{d}t\right)^{1/2}\Bigg\}\leqslant C\Delta t,$$

where $\overline{\mathcal{W}}$ is the piecewise constant interpolant for the sequence $\{\mathcal{W}^n\}_{n=0}^N$.

Let us emphasize that at several occasions we shall use the elementary inequality: for any real numbers r, s

$$rs\leqslant\frac{1}{2\varepsilon}r^2+\frac{\varepsilon}{2}s^2,\quad \varepsilon>0. \tag{2.11}$$

3. Discrete problem

In order to define fully-discrete DG schemes, we first introduce some notations and definitions. Let $\mathcal{T}_h$ be a shape regular d-simplicial triangulations of $\Omega\subset\mathbb{R}^d$ ($d=2,3$). Denote by K a d-simplex in $\mathcal{T}_h$ which is a triangle in dimension two and a tetrahedron in dimension three. Let h_K be the diameter of K, set $h:=\max\{h_K\colon\ K\in\mathcal{T}_h\}$ and let $|K|$ denote the d-dimensional Lebesgue measure of K. Given a simplex $K\in\mathcal{T}_h$, γ_K stands for its minimum angle of K; mesh regularity is equivalent to $\gamma_K\geqslant\gamma>0$. The elements of $\mathcal{T}_h$ are numbered by a fixed numbering for a particular h. The set of all vertices of K is denoted by $\mathcal{V}_K$. Let ω_e be the union of all d-simplices sharing the $(d-1)$-simplex e that is edge in dimension two and face in dimension three. The set of all $(d-1)$-simplices of K is denoted by $\mathcal{E}_h(K)$. Denote the set of all vertices of a d-simplex of $\mathcal{T}_h$ that are in Ω (or on $\partial\Omega$) by $\mathcal{V}_h^i$ (or $\mathcal{V}_h^b$). Set $\mathcal{V}_h=\mathcal{V}_h^i\cup\mathcal{V}_h^b$. Define $\mathcal{E}_h=\mathcal{E}_h^i\cup\mathcal{E}_h^b$ as the set of all $(d-1)$-simplices (edges/faces) in $\mathcal{T}_h$, where $\mathcal{E}_h^i$ denotes the set of all interior $(d-1)$-simplices and $\mathcal{E}_h^b$ denotes the set of all boundary $(d-1)$-simplices of $\mathcal{T}_h$.

Let us define a broken Sobolev space associated with the triangulation $\mathcal{T}_h$ as

$$\mathcal{H}^1(\Omega,\mathcal{T}_h):=\{v\in\mathcal{L}^2(\Omega)\colon\ v|_K\in\mathcal{H}^1(K)\ \forall\,K\in\mathcal{T}_h\}.$$

For any $(d-1)$-simplices $e_{pq}\in\mathcal{E}_h^i$ shared by two d-simplices K_p and K_q (see Figure 1) such that $e_{pq}=\partial K_p\cap\partial K_q$, if $p\geqslant q$, then ν is the unit normal of e_{pq} pointing from K_p to K_q, and $\nu=\nu_p=-\nu_q$. For any $v\in\mathcal{H}^1(\Omega,\mathcal{T}_h)$ there are two traces of v along e_{pq}. We define the jump and mean of v on e_{pq} by

$$[\![v]\!]=v|_p\nu|_p+v|_q\nu|_q\ \text{and}\ \{\!\{v\}\!\}=\frac{1}{2}(v|_p+v|_q),\ \text{respectively.}$$

Similarly, define for $v \in \mathcal{H}^1(\Omega, \mathcal{T}_h)^2$ the jump and mean of v on $e_{pq} \in \mathcal{E}^i_{pq}$ by

$$[\![v]\!] = v|_p \cdot \nu_p + v|_q \cdot \nu_q \quad \text{and} \quad \{\!\{v\}\!\} = \frac{1}{2}(v|_p + v|_q), \text{ respectively.}$$

For any $e \in \mathcal{E}^b_h$ there exists a $K \in \mathcal{T}_h$ such that $e = \partial K \cap \partial\Omega$. Let ν_e be the unit normal of e that points outside K. Also for any $v \in \mathcal{H}^1(\Omega, \mathcal{T}_h)$ we set on $e \in \mathcal{E}^b_h$,

$$[\![v]\!] = v\nu_e \quad \text{and} \quad \{\!\{v\}\!\} = v$$

and for $v \in \mathcal{H}^1(\Omega, \mathcal{T}_h)^2$

$$[\![v]\!] = v \cdot \nu_e \quad \text{and} \quad \{\!\{v\}\!\} = v.$$

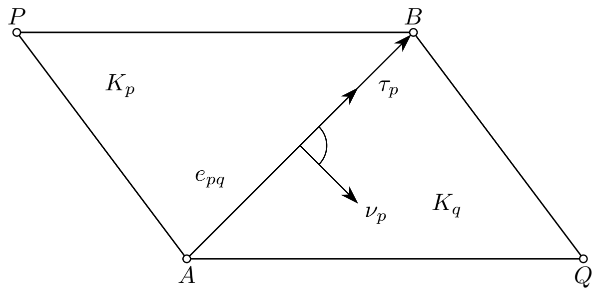

Figure 1. For the case $d = 2$, two neighboring triangles K_p and K_q that share the edge e_{pq} with initial node A and end node B and unit normal ν. The orientation of $\nu = \nu_p = -\nu_q$ equals the outer normal of K_p, hence, points into K_q.

The finite element spaces are defined by

$$V_h := \{v_h \in \mathcal{C}(\overline{\Omega})\colon\ v_h|_K \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h\},$$
$$V_{\mathrm{DG}}(\mathcal{T}_h) := \{v_{\mathrm{DG}} \in \mathcal{L}^2(\Omega)\colon\ v_{\mathrm{DG}}|_K \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h\}.$$

Positive preserving interpolation ([14])**:** Now we define the positive preserving interpolation which was introduced in [14]. We denote the interior nodes of triangulation $\mathcal{T}_h$ by $\{z_i\}_{i=1}^I$. Let $\varphi_{z_i} \in V_h$ be the ith canonical basis function, i.e. $\varphi_{z_i}(z_j) = \delta_{ij}$ for $1 \leqslant i,\ j \leqslant I$, where δ_{ij} is the Kronecker delta function. For each $1 \leqslant i \leqslant I$, let $\omega_i := \cup\{K \in \mathcal{T}_h\colon\ \mathrm{supp}(\varphi_{z_i}) \cap K \neq \emptyset\}$ be the star surrounding z_i. Let Δ_{z_i} be the maximal ball centered at z_i such that $\Delta_{z_i} \subset \omega_i$. Let h_i denote the diameter of ω_i and ϱ_i the radius of Δ_{z_i}. Then mesh regularity implies the existence of a constant C independent of h such that $Ch_i \leqslant \varrho_i \leqslant h_i$ (cf. [14]). Define the interpolation $\Pi_h\colon\ L^1(\Omega) \to V_h$ by

$$\Pi_h v(x) := \sum_{i=1}^I \frac{1}{|\Delta_{z_i}|} \int_{\Delta_{z_i}} v(x)\varphi_{z_i}(x),$$

where $|\Delta_{z_i}|$ is the d dimensional Lebesgue measure of the ball Δ_{z_i}. For any $K \in \mathcal{T}_h$ we denote the union of elements surrounding K by ω_K:

$$\omega_K := \bigcup_{K' \in \mathcal{T}_h} \{K' \colon K' \cap K \neq \emptyset\}.$$

The interpolation operator satisfies the following stability estimates (cf. [14], Lemma 3.1): For any $K \in \mathcal{T}_h$,

$$\|\Pi_h v\|_{\mathcal{L}^2(K)} \leqslant C|v|_{\mathcal{L}^2(\omega_K)} \quad \forall v \in \mathcal{L}^2(\Omega), \tag{3.1}$$

$$\|\nabla \Pi_h v\|_{\mathcal{L}^2(K)} \leqslant C|\nabla v|_{\mathcal{L}^2(\omega_K)} \quad \forall v \in \mathcal{H}_0^1(\Omega), \tag{3.2}$$

where h_K is the diameter of K.

The interpolation operator satisfies the following approximation properties, cf. [14], Lemma 3.2: For any $K \in \mathcal{T}_h$, $l = 0, 1$ and $m = 1, 2$,

$$\|v - \Pi_h v\|_{H^l(K)} \leqslant C h_K^{m-l} |v|_{\mathcal{H}^m(\omega_K)} \quad \forall v \in \mathcal{H}^m(\Omega) \cap \mathcal{H}_0^1(\Omega), \tag{3.3}$$

where h_K is the diameter of K.

At this point we define the discrete analogue of $\mathcal{K}$ by

$$\mathcal{K}_h := \{v_h \in V_h \colon v_h(z) \geqslant \chi(z) \ \forall\, z \in \mathcal{V}_h\},$$
$$\mathcal{K}_{\rm DG} := \{V_{\rm DG}(\mathcal{T}_h) \colon v_{\rm DG}|_K(z) \geqslant \Pi_h \chi(z) \ \forall\, z \in \mathcal{V}_K \ \forall\, K \in \mathcal{T}_h\}.$$

Here we introduce the DG methods. Define

$$\mathcal{A}_h(p,q) := a_h(p,q) + b_h(p,q),$$

where

$$a_h(p,q) := \sum_{K \in \mathcal{T}_h} \int_K \nabla p \cdot \nabla q \,\mathrm{d}x,$$

and the bilinear form b_h consists of all consistency and stability terms. Set the notation $\|\nabla_h p_{\rm DG}\|^2_{\mathcal{L}^2(\Omega)} := a_h(p_{\rm DG}, p_{\rm DG})$ for all $p_{\rm DG} \in V_{\rm DG}(\mathcal{T}_h)$. Also, recall that for any given $g \in \mathcal{BV}(0,T;\mathcal{L}^2(\Omega))$ we define $\mathcal{G}^n := g_+(t_n)$.

Fully-discrete DG scheme: The fully-discrete DG scheme consists of finding $\mathcal{W}^n_{\rm DG} \in \mathcal{K}_{\rm DG}$ for $1 \leqslant n \leqslant N$ such that

$$(\partial \mathcal{W}^n_{\rm DG}, v_{\rm DG} - \mathcal{W}^n_{\rm DG}) + \mathcal{A}_h(\mathcal{W}^n_{\rm DG}, v_{\rm DG} - \mathcal{W}^n_{\rm DG}) \geqslant (\mathcal{G}^n, v_{\rm DG} - \mathcal{W}^n_{\rm DG}) \quad \forall\, v_{\rm DG} \in \mathcal{K}_{\rm DG}, \tag{3.4}$$

$$\mathcal{W}^0_{\rm DG} = \Pi_h w_0 \in \mathcal{K}_{\rm DG}. \tag{3.5}$$

The choice of bilinear from $b_h(\cdot,\cdot)$ for several DG methods is such that corresponding discrete bilinear form $\mathcal{A}_h$ is coercive and bounded with respect to some norm on $V_{\rm DG}(\mathcal{T}_h)$. The existence and uniqueness of a solution to the inequality (3.4)–(3.5) follows from the Lions-Stampacchia theorem (cf. [30], [31]) for any $1 \leqslant n \leqslant N$. As an example, for all the choices of $b_h(\cdot,\cdot)$ listed in [25], the corresponding discrete bilinear form $\mathcal{A}_h$ is coercive and bounded with respect to some norm on $V_{\rm DG}(\mathcal{T}_h)$. For the sake of completeness, we recall some bilinear forms $b_h(\cdot,\cdot)$ which have been listed in [25].

SIPG method [1], [43]**:** For $\gamma \geqslant \gamma_0 > 0$,

$$b_h(p,q) = -\sum_{e\in\mathcal{E}_h}\int_e (\{\{\nabla p\}\}[[q]] + \{\{\nabla q\}\}[[p]])\,{\rm d}\sigma + \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e}[[p]]\,[[q]]\,{\rm d}\sigma. \tag{3.6}$$

NIPG method [38]**:** For $\gamma > 0$,

$$b_h(p,q) = -\sum_{e\in\mathcal{E}_h}\int_e (\{\{\nabla p\}\}[[q-]]\{\{\nabla q\}\}[[p]])\,{\rm d}\sigma + \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e}[[p]]\,[[q]]\,{\rm d}\sigma. \tag{3.7}$$

IIPG method [13], [24], [19]**:** For $\gamma \geqslant \gamma_0 > 0$,

$$b_h(p,q) = -\sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla p\}\}[[q]]\,{\rm d}\sigma + \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e}[[p]]\,[[q]]\,{\rm d}\sigma. \tag{3.8}$$

LDG method [12], [17]**:** For $\gamma > 0$ and $\beta \in \mathbb{R}^2$,

$$\begin{aligned} b_h(p,q) = &-\sum_{e\in\mathcal{E}_h}\int_e (\{\{\nabla p\}\}[[q]] + \{\{\nabla q\}\}[[p]])\,{\rm d}\sigma \\ &+\sum_{e\in\mathcal{E}_h^i}\int_e (\beta\cdot[[p]]\,[[\nabla q]] + [[\nabla p]]\beta\cdot[[q]])\,{\rm d}\sigma \\ &+\sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e}[[p]]\,[[q]]\,{\rm d}\sigma \\ &+\int_\Omega [r([[p]]) + l(\beta\cdot[[p]])]\cdot[r([[q]]) + l(\beta\cdot[[q]])]\,{\rm d}x. \end{aligned} \tag{3.9}$$

Particular choices of β can be found in [16] for superconvergence.

Bassi et al. [5]**:** For $\gamma > 3$,

$$b_h(p,q) = -\sum_{e\in\mathcal{E}_h}\int_e (\{\{\nabla p\}\}[[q]] + \{\{\nabla q\}\}[[p]])\,{\rm d}\sigma + \sum_{e\in\mathcal{E}_h}\int_\Omega \gamma r_e([[p]])r_e([[q]])\,{\rm d}x. \tag{3.10}$$

Brezzi et al. [10]**:** For $\gamma > 0$,

$$b_h(p,q) = -\sum_{e\in\mathcal{E}_h}\int_e (\{\{\nabla p\}\}[\![q]\!] + \{\{\nabla q\}\}[\![p]\!])\,\mathrm{d}\sigma + \int_\Omega r([\![p]\!])\cdot r([\![q]\!])\,\mathrm{d}x + \sum_{e\in\mathcal{E}_h}\int_\Omega \gamma r_e([\![p]\!]) r_e([\![q]\!])\,\mathrm{d}x. \tag{3.11}$$

Babuška-Zlámal [3]**:** For $\gamma > 0$,

$$b_h(p,q) = \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e^3}[\![p]\!][\![q]\!]\,\mathrm{d}\sigma. \tag{3.12}$$

Brezzi et al. [11]**:** For $\gamma > 0$,

$$b_h(p,q) = \sum_{e\in\mathcal{E}_h}\int_\Omega \frac{\gamma}{h_e^2} r_e([\![p]\!]) r_e([\![q]\!])\,\mathrm{d}x. \tag{3.13}$$

WOPSIP [7]**:** For $\gamma > 0$,

$$b_h(p,q) = \sum_{e\in\mathcal{E}_h}\int_e \frac{\gamma}{h_e^3}\pi_e([\![p]\!])\pi_e([\![q]\!])\,\mathrm{d}\sigma. \tag{3.14}$$

Here r and l denote the global lifting operators and r_e denotes the local lifting operator which is defined in [2] and $\pi_e\colon \mathcal{L}^2(e) \to P_0(e)$ is the $\mathcal{L}^2$-projection.

For the first six DG methods (3.6)–(3.11), the discrete norm $\|\cdot\|_h$ on $V_{\mathrm{DG}}(\mathcal{T}_h)$ is defined by

$$\|v_{\mathrm{DG}}\|_h^2 = \sum_{K\in\mathcal{T}_h}\|\nabla v_{\mathrm{DG}}\|^2_{\mathcal{L}^2(K)} + \sum_{e\in\mathcal{E}_h}\frac{\gamma}{h_e}\|[\![v_{\mathrm{DG}}]\!]\|^2_{\mathcal{L}^2(e)},$$

and for the last three DG methods (3.12)–(3.14), the discrete norm $\|\cdot\|_h$ on $V_{\mathrm{DG}}(\mathcal{T}_h)$ is defined by

$$\|v_{\mathrm{DG}}\|_h^2 = \mathcal{A}_h(v_{\mathrm{DG}}, v_{\mathrm{DG}}).$$

Here we emphasize that for all the methods in (3.6)–(3.14) the corresponding discrete form $\mathcal{A}_h$ is coercive and bounded over $V_{\mathrm{DG}}(\mathcal{T}_h)$ with respect to the norm $\|\cdot\|_h$ (see [2], [25]). Therefore, for all the discrete forms $\mathcal{A}_h$ there exist positive constants α (coercitivity constant) and C independent of h such that

$$\mathcal{A}_h(v_{\mathrm{DG}}, v_{\mathrm{DG}}) \geqslant \alpha\|v_{\mathrm{DG}}\|_h^2 \quad \forall\, v_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T}_h),$$
$$\mathcal{A}_h(v_{\mathrm{DG}}, w_{\mathrm{DG}}) \leqslant C\|v_{\mathrm{DG}}\|_h\|w_{\mathrm{DG}}\|_h \quad \forall\, v_{\mathrm{DG}},\ w_{\mathrm{DG}} \in V_{\mathrm{DG}}(\mathcal{T}_h).$$

Henceforth, we use the notation C for a generic constant whose value can change at each occurrence but is independent of the parameters Δt and h. Also at several occasions we shall use the following inequalities: the trace inequality (cf. [15], [37])

$$\|v\|_{\mathcal{L}^2(\partial K)} \leqslant C(h_K^{-1/2}\|v\|_{\mathcal{L}^2(K)} + h_K^{1/2}\|\nabla v\|_{\mathcal{L}^2(K)}) \quad \forall v \in \mathcal{H}^1(K), \tag{3.15}$$

and for any $v \in \mathbb{P}_1(K)$, the inverse inequality (cf. [15], [37])

$$\|\nabla v\|_{\mathcal{L}^2(K)} \leqslant C h_K^{-1}\|v\|_{\mathcal{L}^2(K)}. \tag{3.16}$$

4. Error estimate for fully-discrete DG scheme

In this section, we furnish a unified *a priori* error analysis for all the symmetric and non-symmetric fully-discrete DG schemes (which have been listed in Section 4). We show that the error in a certain norm converges with optimal order $\mathcal{O}(h + \Delta t)$.

Theorem 4.1. *Let w and $\{\mathcal{W}_{\rm DG}^n\}_{n=0}^N$ be the solutions of equations* (2.4)–(2.5) *and* (3.4)–(3.5), *respectively. Then there exists a constant C independent of h and Δt such that*

$$\max_{n\in\{1,\ldots,N\}} \|w^n - \mathcal{W}_{\rm DG}^n\|_{\mathcal{L}^2(\Omega)}^2 + \int_0^T \|w - \overline{\mathcal{W}}_{\rm DG}\|_h^2 \,{\rm d}t \leqslant C(h^2 + (\Delta t)^2),$$

where $\overline{\mathcal{W}}_{\rm DG}$ is the piecewise constant (in time) interpolant for the sequence $\{\mathcal{W}_{\rm DG}^n\}_{n=0}^N$.

Proof. Recall that $\{\mathcal{W}^n\}_{n=0}^N$ is the solution of semi-discrete problem (2.6)–(2.7), and $\overline{\mathcal{W}}$ and $\widetilde{\mathcal{W}}$ are the piecewise constant interpolant and the piecewise linear interpolant for the sequence $\{\mathcal{W}^n\}_{n=0}^N$ as defined in (2.1) and (2.2), respectively. Also define $\widetilde{\mathcal{W}}_{\rm DG}$ to be the piecewise linear interpolant for the sequence $\{\mathcal{W}_{\rm DG}^n\}_{n=0}^N$ as it is defined in (2.2). By using the identity $r(r-s) = r^2/2 - s^2/2 + (r-s)^2/2$ for any real numbers r, s, we have

$$\begin{aligned}
&\int_0^{t_M} \Big(\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \widetilde{\mathcal{W}}_{\rm DG}), \overline{\mathcal{W}} - \overline{\mathcal{W}}_{\rm DG}\Big)\,{\rm d}t \\
&\qquad = \sum_{n=1}^{M}((\mathcal{W}^n - \mathcal{W}_{\rm DG}^n) - (\mathcal{W}^{n-1} - \mathcal{W}_{\rm DG}^{n-1}), (\mathcal{W}^n - \mathcal{W}_{\rm DG}^n)) \\
&\qquad = \frac{1}{2}\sum_{n=1}^{M} \|(\mathcal{W}^n - \mathcal{W}_{\rm DG}^n) - (\mathcal{W}^{n-1} - \mathcal{W}_{\rm DG}^{n-1})\|_{\mathcal{L}^2(\Omega)}^2 \\
&\qquad\quad + \frac{1}{2}\|\mathcal{W}^M - \mathcal{W}_{\rm DG}^M\|_{\mathcal{L}^2(\Omega)}^2 - \frac{1}{2}\|\mathcal{W}^0 - \mathcal{W}_{\rm DG}^0\|_{\mathcal{L}^2(\Omega)}^2 \quad \text{for } M = 1, \ldots, N.
\end{aligned}$$

Therefore,

$$\text{(4.1)}\quad \max_{n\in\{1,\dots,N\}} \|\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}\|^2_{\mathcal{L}^2(\Omega)} \leqslant 2\int_0^{t_M} \Big(\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \widetilde{\mathcal{W}}_{\mathrm{DG}}), \overline{\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\Big)\,\mathrm{d}t + \|\mathcal{W}^0 - \mathcal{W}^0_{\mathrm{DG}}\|^2_{\mathcal{L}^2(\Omega)}.$$

Next we choose a piecewise constant function in time $\overline{\Pi_h \mathcal{W}}$ such that

$$\overline{\Pi_h \mathcal{W}}(t) = \Pi_h \mathcal{W}(t_n) \quad \forall\, t \in (t_{n-1}, t_n],\ n = 1, \dots, N.$$

Then we expand

$$\int_0^T \Big(\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \widetilde{\mathcal{W}}_{\mathrm{DG}}), \overline{\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\Big)\,\mathrm{d}t = \int_0^T \Big(\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \widetilde{\mathcal{W}}_{\mathrm{DG}}), \overline{\mathcal{W}} - \overline{\Pi_h \mathcal{W}}\Big)\,\mathrm{d}t + \int_0^T \Big(\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \widetilde{\mathcal{W}}_{\mathrm{DG}}), \overline{\Pi_h \mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\Big)\,\mathrm{d}t.$$

Let us define

$$\mathcal{E}_{\mathrm{DG}}(t) := \Big(\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \widetilde{\mathcal{W}}_{\mathrm{DG}}), \overline{\Pi_h \mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\Big) + \alpha \|\overline{\Pi_h \mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\|^2_h.$$

Thus from (4.1) we arrive at

$$\text{(4.2)}\quad \max_{n\in\{1,\dots,N\}} \|\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}\|^2_{\mathcal{L}^2(\Omega)} + 2\alpha \int_0^T \|\overline{\Pi_h \mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\|^2_h\,\mathrm{d}t \leqslant 2\int_0^T \mathcal{E}_{\mathrm{DG}}(t)\,\mathrm{d}t + 2\int_0^T \Big(\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \widetilde{\mathcal{W}}_{\mathrm{DG}}), \overline{\mathcal{W}} - \overline{\Pi_h \mathcal{W}}\Big)\,\mathrm{d}t + \|\mathcal{W}^0 - \mathcal{W}^0_{\mathrm{DG}}\|^2_{\mathcal{L}^2(\Omega)}.$$

By integrating $\mathcal{E}_{\mathrm{DG}}(t)$ over 0 to T, we get

$$\begin{aligned}\int_0^T \mathcal{E}_{\mathrm{DG}}(t)\,\mathrm{d}t &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \Big(\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \widetilde{\mathcal{W}}_{\mathrm{DG}}), \overline{\Pi_h \mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\Big)\,\mathrm{d}t \\ &\quad + \alpha \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|(\overline{\Pi_h \mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}})\|^2_h\,\mathrm{d}t \\ &= \sum_{n=1}^N (\partial(\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}), \Pi_h \mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t \\ &\quad + \alpha \sum_{n=1}^N \|\Pi_h \mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}\|^2_h \Delta_n t.\end{aligned}$$

Next, by using the coercivity of $\mathcal{A}_h$, we have

$$\begin{aligned}\int_0^T \mathcal{E}_{\mathrm{DG}}(t)\,\mathrm{d}t \leqslant\ & \sum_{n=1}^{N}(\partial(\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}), \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ & + \sum_{n=1}^{N}\mathcal{A}_h(\Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ =\ & \sum_{n=1}^{N}(\partial\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ & + \sum_{n=1}^{N}\mathcal{A}_h(\Pi_h\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ & - \sum_{n=1}^{N}(\partial\mathcal{W}^n_{\mathrm{DG}}, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ & - \sum_{n=1}^{N}\mathcal{A}_h(\mathcal{W}^n_{\mathrm{DG}}, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t.\end{aligned}$$

Then, by taking $v_{\mathrm{DG}} = \Pi_h\mathcal{W}^n$ in (3.4), we arrive at

$$\begin{aligned}\int_0^T \mathcal{E}_{\mathrm{DG}}(t)\,\mathrm{d}t \leqslant\ & \sum_{n=1}^{N}(\partial\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ & + \sum_{n=1}^{N}\mathcal{A}_h(\Pi_h\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ & - \sum_{n=1}^{N}(\mathcal{G}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t.\end{aligned}$$

After that, we expand the right-hand side of the inequality in the following way:

$$\begin{aligned}\int_0^T \mathcal{E}_{\mathrm{DG}}(t)\,\mathrm{d}t \leqslant\ & \sum_{n=1}^{N}(\partial\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ & + \sum_{n=1}^{N}\mathcal{A}_h(\Pi_h\mathcal{W}^n - \mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ & + \sum_{n=1}^{N}\mathcal{A}_h(\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t\\ & - \sum_{n=1}^{N}(\mathcal{G}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}})\Delta_n t.\end{aligned}$$

Then by using boundedness of $\mathcal{A}_h$, we obtain

$$\begin{aligned}(4.3)\qquad \int_0^T \mathcal{E}_{\rm DG}(t)\,{\rm d}t \leqslant C &\sum_{n=1}^N \|\Pi_h\mathcal{W}^n - \mathcal{W}^n\|_h \|\Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG}\|_h \Delta_n t \\ &+ \sum_{n=1}^N (\partial\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG})\Delta_n t \\ &+ \sum_{n=1}^N \mathcal{A}_h(\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG})\Delta_n t \\ &- \sum_{n=1}^N (\mathcal{G}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG})\Delta_n t.\end{aligned}$$

Here we define $\Upsilon^n = \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG}$ for $n = 1, \ldots, N$, to make convenience of the presentation. Thus, it is easy to see (cf. [21], pages 13–14) that

$$a_h(\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG}) = -\int_\Omega \Delta\mathcal{W}^n \Upsilon^n\,{\rm d}x + \sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla\mathcal{W}^n\}\}[[\Upsilon^n]]\,{\rm d}\sigma.$$

Therefore,

$$\begin{aligned}(4.4)\qquad &\mathcal{A}_h(\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG}) \\ &\quad = -\int_\Omega \Delta\mathcal{W}^n \Upsilon^n\,{\rm d}x + \sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla\mathcal{W}^n\}\}[[\Upsilon^n]]\,{\rm d}\sigma + b_h(\mathcal{W}^n, \Upsilon^n).\end{aligned}$$

We first consider SIPG, NIPG, IIPG, LDG, Bassi et al. and Brezzi et al. methods in (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11), respectively. In that case we have

$$(4.5)\qquad \mathcal{A}_h(\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG}) = -\int_\Omega \Delta\mathcal{W}^n(\Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG})\,{\rm d}x.$$

Now we consider Babuška-Zlámal, Brezzi et al. and WOPSIP methods in (3.12), (3.13) and (3.14), respectively. In that case we have

$$\begin{aligned}(4.6)\qquad \mathcal{A}_h(\mathcal{W}^n, \Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG}) = &-\int_\Omega \Delta\mathcal{W}^n(\Pi_h\mathcal{W}^n - \mathcal{W}^n_{\rm DG})\,{\rm d}x \\ &+ \sum_{e\in\mathcal{E}_h}\int_e \{\{\nabla\mathcal{W}^n\}\}[[\Upsilon^n]]\,{\rm d}\sigma.\end{aligned}$$

For the methods SIPG, NIPG, IIPG, LDG, Bassi et al. and Brezzi et al. in (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11), respectively, from (4.2), (4.3) and (4.5) we deduce

$$(4.7)\qquad \max_{n\in\{1,\ldots,N\}} \|\mathcal{W}^n - \mathcal{W}^n_{\rm DG}\|^2_{\mathcal{L}^2(\Omega)} + 2\alpha\int_0^T \|\overline{\Pi_h\mathcal{W}} - \overline{\mathcal{W}}_{\rm DG}\|_h^2\,{\rm d}t \leqslant 2\sum_{i=1}^3 \mathcal{E}^i_{\rm DG},$$

and for the methods Babuška-Zlámal, Brezzi et al. and WOPSIP in (3.12), (3.13) and (3.14), respectively, from (4.2), (4.3) and (4.6) we deduce

$$\max_{n\in\{1,\dots,N\}}\|\mathcal{W}^n-\mathcal{W}^n_{\rm DG}\|^2_{\mathcal{L}^2(\Omega)}+2\alpha\int_0^T\|\overline{\Pi_h\mathcal{W}}-\overline{\mathcal{W}}_{\rm DG}\|^2_h\,{\rm d}t\leqslant 2\sum_{i=1}^4\mathcal{E}^i_{\rm DG}, \tag{4.8}$$

where

$$\begin{aligned}
\mathcal{E}^1_{\rm DG}&=\|\mathcal{W}^0-\mathcal{W}^0_{\rm DG}\|^2_{\mathcal{L}^2(\Omega)}+\int_0^T\Big(\frac{\partial}{\partial t}(\widetilde{\mathcal{W}}-\widetilde{\mathcal{W}}_{\rm DG}),\overline{\mathcal{W}}-\overline{\Pi_h\mathcal{W}}\Big)\,{\rm d}t,\\
\mathcal{E}^2_{\rm DG}&=C\sum_{n=1}^N\|\Pi_h\mathcal{W}^n-\mathcal{W}^n\|_h\|\Pi_h\mathcal{W}^n-\mathcal{W}^n_{\rm DG}\|_h\Delta_nt,\\
\mathcal{E}^3_{\rm DG}&=\sum_{n=1}^N(\partial\mathcal{W}^n-\Delta\mathcal{W}^n-\mathcal{G}^n,\Pi_h\mathcal{W}^n-\mathcal{W}^n_{\rm DG})\Delta_nt,\\
\mathcal{E}^4_{\rm DG}&=\sum_{e\in\mathcal{E}_h}\int_e\{\{\nabla\mathcal{W}^n\}\}[[\Upsilon^n]]\,{\rm d}\sigma.
\end{aligned}$$

Estimation of $\mathcal{E}^1_{\rm DG}$: By using (2.3), we have precisely

$$\begin{aligned}
\mathcal{E}^1_{\rm DG}&=\|\mathcal{W}^0-\mathcal{W}^0_{\rm DG}\|^2_{\mathcal{L}^2(\Omega)}+\sum_{n=1}^N(\partial(\mathcal{W}^n-\mathcal{W}^n_{\rm DG}),\mathcal{W}^n-\Pi_h\mathcal{W}^n)\Delta_nt\\
&=\|\mathcal{W}^0-\mathcal{W}^0_{\rm DG}\|^2_{\mathcal{L}^2(\Omega)}\\
&\quad+\sum_{n=1}^N((\mathcal{W}^n-\mathcal{W}^n_{\rm DG})-(\mathcal{W}^{n-1}-\mathcal{W}^{n-1}_{\rm DG}),\mathcal{W}^n-\Pi_h\mathcal{W}^n)\\
&=\|\mathcal{W}^0-\mathcal{W}^0_{\rm DG}\|^2_{\mathcal{L}^2(\Omega)}+\sum_{n=1}^N(\mathcal{W}^{n-1}-\mathcal{W}^{n-1}_{\rm DG},\mathcal{W}^{n-1}-\Pi_h\mathcal{W}^{n-1})\\
&\quad+(\mathcal{W}^N-\mathcal{W}^N_{\rm DG},\mathcal{W}^N-\Pi_h\mathcal{W}^N)-(\mathcal{W}^0-\mathcal{W}^0_{\rm DG},\mathcal{W}^0-\Pi_h\mathcal{W}^0)\\
&\quad-\sum_{n=1}^N(\mathcal{W}^{n-1}-\mathcal{W}^{n-1}_{\rm DG},\mathcal{W}^n-\Pi_h\mathcal{W}^n)\\
&=\|\mathcal{W}^0-\mathcal{W}^0_{\rm DG}\|^2_{\mathcal{L}^2(\Omega)}+(\mathcal{W}^N-\mathcal{W}^N_{\rm DG},\mathcal{W}^N-\Pi_h\mathcal{W}^N)\\
&\quad-(\mathcal{W}^0-\mathcal{W}^0_{\rm DG},\mathcal{W}^0-\Pi_h\mathcal{W}^0)\\
&\quad-\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\Big(\mathcal{W}^{n-1}-\mathcal{W}^{n-1}_{\rm DG},\frac{(\mathcal{W}^n-\Pi_h\mathcal{W}^n)-(\mathcal{W}^{n-1}-\Pi_h\mathcal{W}^{n-1})}{\Delta_nt}\Big)\,{\rm d}t\\
&=\|\mathcal{W}^0-\mathcal{W}^0_{\rm DG}\|^2_{\mathcal{L}^2(\Omega)}+(\mathcal{W}^N-\mathcal{W}^N_{\rm DG},\mathcal{W}^N-\Pi_h\mathcal{W}^N)\\
&\quad-(\mathcal{W}^0-\mathcal{W}^0_{\rm DG},\mathcal{W}^0-\Pi_h\mathcal{W}^0)\\
&\quad-\sum_{n=1}^N\int_{t_{n-1}}^{t_n}(\mathcal{W}^{n-1}-\mathcal{W}^{n-1}_{\rm DG},\frac{\partial}{\partial t}(\widetilde{\mathcal{W}}-\Pi_h\widetilde{\mathcal{W}}))\,{\rm d}t.
\end{aligned}$$

By applying the Cauchy-Schwarz inequality and Young's inequality (2.11), we arrive at

$$\mathcal{E}_{\rm DG}^1 \leqslant \varepsilon\|(\mathcal{W}^N - \mathcal{W}_{\rm DG}^N)\|^2_{\mathcal{L}^2(\Omega)} + \frac{1}{\varepsilon}\|(\mathcal{W}^N - \Pi_h\mathcal{W}^N)\|^2_{\mathcal{L}^2(\Omega)} + 2\|(\mathcal{W}^0 - \Pi_h\mathcal{W}^0)\|^2_{\mathcal{L}^2(\Omega)} + \varepsilon\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\|\mathcal{W}^{n-1} - \mathcal{W}_{\rm DG}^{n-1}\|^2 + \frac{1}{\varepsilon}\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\Big\|\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \Pi_h\widetilde{\mathcal{W}})\Big\|^2_{\mathcal{L}^2(\Omega)}.$$

Since the interpolation Π_h is $\mathcal{H}^1$-stable and it commutes with $\frac{\partial}{\partial t}$, this implies

$$\begin{aligned}\frac{\partial}{\partial t}\Pi_h\widetilde{\mathcal{W}}(t) &= \frac{\partial}{\partial t}\sum_{i=1}^I\Big(\frac{1}{|\Delta_{z_i}|}\int_{\Delta_{z_i}}\widetilde{\mathcal{W}}(t)\,{\rm d}x\Big)\varphi_{z_i}\\ &= \sum_{i=1}^I\Big(\frac{1}{|\Delta_{z_i}|}\int_{\Delta_{z_i}}\frac{\partial}{\partial t}\widetilde{\mathcal{W}}(t)\,{\rm d}x\Big)\varphi_{z_i}\\ &= \Pi_h\frac{\partial}{\partial t}\widetilde{\mathcal{W}}(t).\end{aligned}$$

Then by using the interpolation error estimates (3.3) and the regularity result $\frac{\partial\widetilde{\mathcal{W}}}{\partial t} \in \mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))$ from Lemma 2.2, we have

$$\begin{aligned}\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\Big\|\frac{\partial}{\partial t}(\widetilde{\mathcal{W}} - \Pi_h\widetilde{\mathcal{W}})\Big\|^2_{\mathcal{L}^2(\Omega)}\,{\rm d}t &= \sum_{n=1}^N\int_{t_{n-1}}^{t_n}\Big\|\frac{\partial\widetilde{\mathcal{W}}}{\partial t}(t) - \frac{\partial}{\partial t}\Pi_h\widetilde{\mathcal{W}}(t)\Big\|^2_{\mathcal{L}^2(\Omega)}\,{\rm d}t\\ &= \sum_{n=1}^N\int_{t_{n-1}}^{t_n}\Big\|\frac{\partial\widetilde{\mathcal{W}}}{\partial t}(t) - \Pi_h\frac{\partial\widetilde{\mathcal{W}}}{\partial t}(t)\Big\|^2_{\mathcal{L}^2(\Omega)}\,{\rm d}t\\ &\leqslant Ch^2\Big\|\frac{\partial\widetilde{\mathcal{W}}}{\partial t}\Big\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))}.\end{aligned}\tag{4.9}$$

Also, by using the interpolation error estimates (3.3), we find

$$\|\mathcal{W}^0 - \Pi_h\mathcal{W}^0\|_{\mathcal{L}^2(\Omega)} = \|w_0 - \Pi_h w_0\|_{\mathcal{L}^2(\Omega)} \leqslant Ch^2\|w_0\|_{\mathcal{H}^2(\Omega)}$$

and

$$\|(\mathcal{W}^N - I_h\mathcal{W}^N)\|_{\mathcal{L}^2(\Omega)} \leqslant Ch^2\|\mathcal{W}^N\|_{\mathcal{H}^2(\Omega)}.$$

Therefore,

$$\begin{aligned}\mathcal{E}_{\rm DG}^1 \leqslant \varepsilon\max_{n\in\{1,\dots,N\}}\|\mathcal{W}^n - \mathcal{W}_{\rm DG}^n\|^2_{\mathcal{L}^2(\Omega)} + Ch^2\Big\|\frac{\partial\widetilde{\mathcal{W}}}{\partial t}\Big\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))}\\ + Ch^4(\|\mathcal{W}^N\|^2_{\mathcal{H}^2(\Omega)} + \|w_0\|^2_{\mathcal{H}^2(\Omega)}).\end{aligned}\tag{4.10}$$

Estimation of $\mathcal{E}_{\mathrm{DG}}^2$: By using Young's inequality (2.11), we find

$$\mathcal{E}_{\mathrm{DG}}^2 \leqslant \frac{1}{4\varepsilon}\sum_{n=1}^{N}\|\Pi_h\mathcal{W}^n-\mathcal{W}^n\|_h^2\Delta_n t+\varepsilon\sum_{n=1}^{N}\|\Pi_h\mathcal{W}^n-\mathcal{W}_{\mathrm{DG}}^n\|_h^2\Delta_n t. \tag{4.11}$$

By employing interpolation error estimates (3.3), we obtain

$$\begin{aligned}\mathcal{E}_{\mathrm{DG}}^2 &\leqslant Ch^2\sum_{n=1}^{N}\|\mathcal{W}^n\|_{\mathcal{H}^2(\Omega)}^2\Delta_n t+\varepsilon\sum_{n=1}^{N}\|\Pi_h\mathcal{W}^n-\mathcal{W}_{\mathrm{DG}}^n\|_h^2\Delta_n t\\ &\leqslant Ch^2\|\overline{\mathcal{W}}\|_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))}^2+\varepsilon\sum_{n=1}^{N}\|\Pi_h\mathcal{W}^n-\mathcal{W}_{\mathrm{DG}}^n\|_h^2\Delta_n t.\end{aligned} \tag{4.12}$$

Estimation of $\mathcal{E}_{\mathrm{DG}}^3$: Recall that

$$\mathcal{E}_{\mathrm{DG}}^3=\sum_{n=1}^{N}(\partial\mathcal{W}^n-\Delta\mathcal{W}^n-\mathcal{G}^n,\Pi_h\mathcal{W}^n-\mathcal{W}_{\mathrm{DG}}^n)\Delta_n t.$$

For convenience of presentation, we define $\mathcal{Z}^n:=\partial\mathcal{W}^n-\Delta\mathcal{W}^n-\mathcal{G}^n$. Next we define three subsets of the triangulation $\mathcal{T}_h$ as follows:

$$\begin{aligned}\mathcal{T}_h^+(n)&:=\{K\in\mathcal{T}_h:\ \mathcal{W}^n>\chi \text{ in } K\},\\ \mathcal{T}_h^0(n)&:=\{K\in\mathcal{T}_h:\ \mathcal{W}^n=\chi \text{ in } K\},\\ \mathcal{T}_h^F(n)&:=\mathcal{T}_h\setminus(\mathcal{T}_h^+(n)\cup\mathcal{T}_h^0(n)).\end{aligned}$$

Thus,

$$\sum_{K\in\mathcal{T}_h}\int_K\mathcal{Z}^n(\Pi_h\mathcal{W}^n-\mathcal{W}_{\mathrm{DG}}^n)\,\mathrm{d}x=\mathcal{I}_1^n+\mathcal{I}_2^n+\mathcal{I}_3^n, \tag{4.13}$$

where

$$\mathcal{I}_1^n:=\sum_{K\in\mathcal{T}_h^+(n)}\int_K\mathcal{Z}^n(\Pi_h\mathcal{W}^n-\mathcal{W}_{\mathrm{DG}}^n)\,\mathrm{d}x,\quad \mathcal{I}_2^n:=\sum_{K\in\mathcal{T}_h^0(n)}\int_K\mathcal{Z}^n(\Pi_h\mathcal{W}^n-\mathcal{W}_{\mathrm{DG}}^n)\,\mathrm{d}x,$$

and

$$\mathcal{I}_3^n:=\sum_{K\in\mathcal{T}_h^{\mathcal{G}}(n)}\int_K\mathcal{Z}^n(\Pi_h\mathcal{W}^n-\mathcal{W}_{\mathrm{DG}}^n)\,\mathrm{d}x.$$

Therefore,

$$\mathcal{E}_{\mathrm{DG}}^3=\sum_{n=1}^{N}\big(\mathcal{I}_1^n+\mathcal{I}_2^n+\mathcal{I}_3^n\big)\Delta_n t. \tag{4.14}$$

If $K \in \mathcal{T}_h^+(n)$, then by (2.10), we have $\mathcal{Z}^n = 0$, which implies $\mathcal{I}_1^n = 0$. Next, for any $K \in \mathcal{T}_h^0(n)$ we have $\mathcal{W}^n = \chi$ in K. Thus, $\Pi_h \mathcal{W}^n = \Pi_h \chi \leqslant \mathcal{W}_{\rm DG}^n$ in K. Since $\mathcal{Z}^n \geqslant 0$ a.e. in Ω (by (2.9)), we have

$$\mathcal{I}_2^n := \sum_{K \in \mathcal{T}_h^0(n)} \int_K \mathcal{Z}^n (\Pi_h \chi - \mathcal{W}_{\rm DG}^n)\,{\rm d}x \leqslant 0.$$

Now we turn to estimate $\mathcal{I}_3^n$. For any $K \in \mathcal{T}_h^F(n)$,

$$\int_K \mathcal{Z}^n (\Pi_h \mathcal{W}^n - \mathcal{W}_{\rm DG}^n)\,{\rm d}x = \int_K \mathcal{Z}^n (\Pi_h \mathcal{W}^n - \mathcal{W}^n + \mathcal{W}^n - \chi + \chi - \Pi_h \chi + \Pi_h \chi - \mathcal{W}_{\rm DG}^n)\,{\rm d}x.$$

Since $\int_K \mathcal{Z}^n(\mathcal{W}^n - \chi)\,{\rm d}x = 0$ and $\int_K \mathcal{Z}^n(\Pi_h \chi - \mathcal{W}_{\rm DG}^n)\,{\rm d}x \leqslant 0$, we have

$$\int_K \mathcal{Z}^n (\Pi_h \mathcal{W}^n - \mathcal{W}_{\rm DG}^n)\,{\rm d}x \leqslant \int_K \mathcal{Z}^n (\Pi_h (\mathcal{W}^n - \chi) - (\mathcal{W}^n - \chi))\,{\rm d}x.$$

We see in (2.8) that $\mathcal{W}^n \in \mathcal{H}^2(\Omega)$. Now from Lemma 2.2 and (2.3) it is easy to see that $\mathcal{Z}^n = \partial \mathcal{W}^n - \Delta \mathcal{W}^n - \mathcal{G}^n \in \mathcal{L}^2(\Omega)$. Therefore by using the Cauchy-Schwarz inequality and interpolation error estimates (3.3), we arrive at

$$\int_K \mathcal{Z}^n (\Pi_h \mathcal{W}^n - \mathcal{W}_{\rm DG}^n)\,{\rm d}x \leqslant C h_K^2 \|\mathcal{Z}^n\|_{\mathcal{L}^2(K)} \|\mathcal{W}^n - \chi\|_{\mathcal{H}^2(\omega_K)}.$$

By using Young's inequality (2.11) and inequality $(x-y)^2 \leqslant 2x^2 + 2y^2$, we derive

$$\mathcal{I}_3^n \leqslant C h^2 \sum_{K \in \mathcal{T}_h^F(n)} \|\mathcal{Z}^n\|_{\mathcal{L}^2(K)}^2 + C h^2 \sum_{K \in \mathcal{T}_h^F(n)} \|\mathcal{W}^n\|_{\mathcal{H}^2(\omega_K)}^2 + C h^2 \sum_{K \in \mathcal{T}_h^F(n)} \|\chi\|_{\mathcal{H}^2(\omega_K)}^2.$$

Since $\|\mathcal{Z}^n\|_{\mathcal{L}^2(\Omega)}^2 \leqslant \|\partial \mathcal{W}^n\|_{\mathcal{L}^2(\Omega)}^2 + \|\Delta \mathcal{W}^n\|_{\mathcal{L}^2(\Omega)}^2 + \|\mathcal{G}^n\|_{\mathcal{L}^2(\Omega)}^2$, we have

$$\mathcal{I}_3^n \leqslant C h^2 \|\partial \mathcal{W}^n\|_{\mathcal{L}^2(\Omega)}^2 + C h^2 \|\mathcal{W}^n\|_{\mathcal{H}^2(\Omega)}^2 + C h^2 \|\mathcal{G}^n\|_{\mathcal{L}^2(\Omega)}^2 + C h^2 \|\chi\|_{\mathcal{H}^2(\Omega)}^2.$$

Therefore, from (2.3) and (4.14) we deduce

$$\begin{aligned} \text{(4.15)} \qquad \mathcal{E}_{\rm DG}^3 \leqslant \sum_{n=1}^N \mathcal{I}_3^n \Delta_n t \leqslant C h^2 \Big\| \frac{\partial \widetilde{\mathcal{W}}}{\partial t} \Big\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))}^2 &+ C h^2 \|\overline{\mathcal{W}}\|_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))}^2 \\ &+ C h^2 \|\overline{\mathcal{G}}\|_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))}^2 + C h^2 \|\chi\|_{\mathcal{H}^2(\Omega)}^2. \end{aligned}$$

Estimation of $\mathcal{E}_{\rm DG}^4$: The estimation of $\mathcal{E}_{\rm DG}^4$ is similar to the estimation of Q in [21], pages 16–17. For the sake of completeness, we provide the following details. First we consider the estimation of $\mathcal{E}_{\rm DG}^4$ for Babuška-Zlámal and Brezzi et al. methods in (3.12) and (3.13), respectively. By applying Cauchy-Schwarz inequality and Young's inequality (2.11) for both Babuška-Zlámal and Brezzi et al. methods in (3.12) and (3.13), respectively, we find

$$\mathcal{E}_{\mathrm{DG}}^{4} \leqslant \sum_{n=1}^{N}\Bigg(\frac{1}{4\varepsilon\gamma}\sum_{e\in\mathcal{E}_h} h_e^3\|\{\{\nabla\mathcal{W}^n\}\}\|_{\mathcal{L}^2(e)}^2 + \varepsilon\sum_{e\in\mathcal{E}_h}\frac{\gamma}{h_e^3}\|[[\Upsilon^n]]\|_{\mathcal{L}^2(e)}^2\Bigg)\Delta_n t.$$

Further, the by using the trace inequality (3.15), we obtain for Babuška-Zlámal method in (3.12):

$$\begin{aligned}\mathcal{E}_{\mathrm{DG}}^{4} &\leqslant Ch^2\|\overline{\mathcal{W}}\|_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))}^2 + Ch^4\|\overline{\mathcal{W}}\|_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))}^2\\ &\quad + \varepsilon\int_0^T\Bigg(\sum_{e\in\mathcal{E}_h}\frac{\gamma}{h_e^3}\|[[\overline{\Pi_h\mathcal{W}}-\overline{\mathcal{W}}_{\mathrm{DG}}]]\|_{\mathcal{L}^2(e)}^2\Bigg)\,\mathrm{d}t.\end{aligned}\tag{4.16}$$

For Brezzi et al. method in (3.13), by using

$$C_1\|r_e(\varphi)\|_{\mathcal{L}^2(\Omega)}^2 \leqslant \frac{1}{h_e}\|\varphi\|_{\mathcal{L}^2(e)}^2 \leqslant C_2\|r_e(\varphi)\|_{\mathcal{L}^2(\Omega)}^2 \quad \forall\,\varphi\in[\mathbb{P}_1(e)]^2$$

(cf. [2]) and the trace inequality (3.15), we deduce

$$\begin{aligned}\mathcal{E}_{\mathrm{DG}}^{4} &\leqslant Ch^2\|\overline{\mathcal{W}}\|_{\mathcal{L}^2(0,T;\mathcal{H}^1(\Omega))}^2 + Ch^4\|\overline{\mathcal{W}}\|_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))}^2\\ &\quad + \varepsilon\int_0^T\Bigg(\sum_{e\in\mathcal{E}_h}\frac{\gamma}{h_e^2}\|r_e([[\overline{\Pi_h\mathcal{W}}-\overline{\mathcal{W}}_{\mathrm{DG}}]])\|_{\mathcal{L}^2(e)}^2\Bigg)\,\mathrm{d}t.\end{aligned}\tag{4.17}$$

For WOPSIP method in (3.14) we split $\mathcal{E}_{\mathrm{DG}}^{4}$ into two parts as follows:

$$\begin{aligned}\mathcal{E}_{\mathrm{DG}}^{4} &= \sum_{n=1}^{N}\Bigg(\sum_{e\in\mathcal{E}_h}\int_e\{\{\nabla(\mathcal{W}^n-\Pi_h\mathcal{W}^n)\}\}[[\Upsilon^n]]\,\mathrm{d}\sigma\\ &\quad + \sum_{e\in\mathcal{E}_h}\int_e\{\{\nabla\Pi_h\mathcal{W}^n\}\}[[\Upsilon^n]]\,\mathrm{d}\sigma\Bigg)\Delta_n t = \sum_{n=1}^{N}(\mathcal{B}_1^n+\mathcal{B}_2^n)\Delta_n t.\end{aligned}\tag{4.18}$$

Now by applying the Cauchy-Schwarz inequality and the triangle inequality, we find

$$\begin{aligned}\mathcal{B}_1^n &\leqslant \sum_{e\in\mathcal{E}_h}\|\{\{\nabla(\mathcal{W}^n-\Pi_h\mathcal{W}^n)\}\}\|_{\mathcal{L}^2(e)}\|[[\Upsilon^n]]-\pi_e([[\Upsilon^n]])\|_{\mathcal{L}^2(e)}\\ &\quad + \sum_{e\in\mathcal{E}_h}\|\{\{\nabla(\mathcal{W}^n-\Pi_h\mathcal{W}^n)\}\}\|_{\mathcal{L}^2(e)}\|\pi_e([[\Upsilon^n]])\|_{\mathcal{L}^2(e)}.\end{aligned}$$

Since $\|[\![\Upsilon^n]\!] - \pi_e([\![\Upsilon^n]\!])\|_{\mathcal{L}^2(e)} \leqslant Ch_e^{1/2}\|\nabla_h\Upsilon^n\|_{\mathcal{L}^2(\omega_e)}$ (here ω_e is the union of all $K \in \mathcal{T}_h$, which share e), by using the trace inequality (3.15), interpolation error estimates (3.3) and Young's inequality (2.11), we deduce

$$\begin{aligned}\mathcal{B}_1^n &\leqslant C\sum_{e\in\mathcal{E}_h} h_e^{1/2}\|\mathcal{W}^n\|_{\mathcal{H}^2(\mathcal{S}_{\omega_e})} h_e^{1/2}\|\nabla_h\Upsilon^n\|_{\mathcal{L}^2(\omega_e)}\\ &\quad+\sum_{e\in\mathcal{E}_h} h_e^{1/2}\|\mathcal{W}^n\|_{\mathcal{H}^2(\mathcal{S}_{\omega_e})}\|\pi_e([\![\Upsilon^n]\!])\|_{\mathcal{L}^2(e)}\\ &\leqslant Ch^2\|\mathcal{W}^n\|^2_{\mathcal{H}^2(\Omega)} + \varepsilon\|\nabla_h\Upsilon^n\|^2_{\mathcal{L}^2(\Omega)} + Ch^4\|\mathcal{W}^n\|^2_{\mathcal{H}^2(\Omega)}\\ &\quad+\varepsilon\sum_{e\in\mathcal{E}_h}\frac{\gamma}{h_e^3}\|\pi_e([\![\Upsilon^n]\!])\|^2_{\mathcal{L}^2(e)},\end{aligned}$$

where $\mathcal{S}_{\omega_e} = \bigcup_{K\in\omega_e}\omega_K$ and $\omega_K := \bigcup_{K'\in\mathcal{T}_h}\{K' \colon K'\cap K \neq \emptyset\}$. Therefore,

$$\begin{aligned}\sum_{n=1}^N \mathcal{B}_1^n\Delta_n t &\leqslant Ch^2\|\overline{\mathcal{W}}\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))} + \varepsilon\int_0^T \|\nabla_h(\overline{\Pi_h\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}})\|^2_{\mathcal{L}^2(\Omega)}\,\mathrm{d}t\\ &\quad+Ch^4\|\overline{\mathcal{W}}\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))}\\ &\quad+\varepsilon\int_0^T\Big(\sum_{e\in\mathcal{E}_h}\frac{\gamma}{h_e^3}\|\pi_e([\![\overline{\Pi_h\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}]\!])\|^2_{\mathcal{L}^2(e)}\Big)\,\mathrm{d}t.\end{aligned}\tag{4.19}$$

Since $\pi_e([\![v]\!]) = |e|^{-1}\int_e [\![v]\!]\,\mathrm{d}\sigma$ (here $|e|$ is the Lebesgue measure of e), by using the Cauchy-Schwarz inequality and Young's inequality (2.11), we obtain

$$\begin{aligned}\mathcal{B}_2^n &= \sum_{e\in\mathcal{E}_h}\int_e |e|\{\{\nabla\Pi_h\mathcal{W}^n\}\}\pi_e([\![\Upsilon^n]\!])\,\mathrm{d}\sigma\\ &\leqslant \sum_{e\in\mathcal{E}_h}\Big(\frac{|e|^2h_e^3}{4\gamma\varepsilon}\|\{\{\nabla\Pi_h\mathcal{W}^n\}\}\|^2_{\mathcal{L}^2(e)} + \varepsilon\frac{\gamma}{h_e^3}\|\pi_e([\![\Upsilon^n]\!])\|^2_{\mathcal{L}^2(e)}\Big).\end{aligned}$$

Then by using the trace inequality (3.15) and since $\mathcal{W}^n \in \mathcal{H}^2(\Omega)$, by $\mathcal{H}^1$-stability of interpolation Π_h in (3.2), we arrive at

$$\begin{aligned}\sum_{n=1}^N \mathcal{B}_2^n\Delta_n t &\leqslant Ch^4\|\overline{\mathcal{W}}\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))}\\ &\quad+\varepsilon\int_0^T\Big(\sum_{e\in\mathcal{E}_h}\frac{\gamma}{h_e^3}\|\pi_e([\![\overline{\Pi_h\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}]\!])\|^2_{\mathcal{L}^2(e)}\Big)\,\mathrm{d}t.\end{aligned}\tag{4.20}$$

Therefore, from (4.18), (4.19) and (4.20) we get

$$\begin{aligned}\mathcal{E}_{\mathrm{DG}}^4 &\leqslant Ch^2\|\overline{\mathcal{W}}\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))} + \varepsilon\int_0^T \|\nabla_h(\overline{\Pi_h\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}})\|^2_{\mathcal{L}^2(\Omega)}\,\mathrm{d}t \\ &\quad + Ch^4\|\overline{\mathcal{W}}\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))} \\ &\quad + \varepsilon\int_0^T\bigg(\sum_{e\in\mathcal{E}_h}\frac{\gamma}{h_e^3}\|\pi_e([\![\overline{\Pi_h\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}]\!])\|^2_{\mathcal{L}^2(e)}\bigg)\,\mathrm{d}t.\end{aligned} \tag{4.21}$$

For the methods SIPG, NIPG, IIPG, LDG, Bassi et al. and Brezzi et al. in (3.6), (3.7), (3.8), (3.9), (3.10) and (3.11), respectively, from (4.7), (4.10), (4.12) and (4.15), by choosing ε small, we obtain

$$\begin{aligned}\max_{n\in\{1,\dots,N\}}&\|\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}\|^2_{\mathcal{L}^2(\Omega)} + \int_0^T \|\overline{\Pi_h\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\|^2_h\,\mathrm{d}t \\ &\leqslant Ch^4\|\mathcal{W}^N\|^2_{\mathcal{H}^2(\Omega)} + Ch^2\Big\|\frac{\partial\widetilde{\mathcal{W}}}{\partial t}\Big\|^2_{\mathcal{L}^2(\mathcal{J};\mathcal{H}^1(\Omega))} \\ &\quad + Ch^4\|w_0\|^2_{\mathcal{H}^2(\Omega)} + Ch^2\|\overline{\mathcal{W}}\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))} \\ &\quad + Ch^2(\|\overline{\mathcal{G}}\|^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))} + \|\chi\|^2_{\mathcal{H}^2(\Omega)}).\end{aligned} \tag{4.22}$$

For the methods Babuška-Zlámal, Brezzi et al. and WOPSIP in (3.12), (3.13) and (3.14), respectively, from (4.8), (4.10), (4.12), (4.15), (4.16), (4.17) and (4.21), by choosing ε small, we get the following:

$$\begin{aligned}\max_{n\in\{1,\dots,N\}}&\|\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}\|^2_{\mathcal{L}^2(\Omega)} + \int_0^T \|\overline{\Pi_h\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\|^2_h\,\mathrm{d}t \\ &\leqslant Ch^4\|\mathcal{W}^N\|^2_{\mathcal{H}^2(\Omega)} + Ch^2\Big\|\frac{\partial\widetilde{\mathcal{W}}}{\partial t}\Big\|^2_{\mathcal{L}^2(\mathcal{J};\mathcal{H}^1(\Omega))} \\ &\quad + Ch^4\|w_0\|^2_{\mathcal{H}^2(\Omega)} + Ch^2\|\overline{\mathcal{W}}\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))} \\ &\quad + Ch^2(\|\overline{\mathcal{G}}\|^2_{\mathcal{L}^2(0,T;\mathcal{L}^2(\Omega))} + \|\chi\|^2_{\mathcal{H}^2(\Omega)}) \\ &\quad + Ch^4\|\overline{\mathcal{W}}\|^2_{\mathcal{L}^2(0,T;\mathcal{H}^2(\Omega))}.\end{aligned} \tag{4.23}$$

Therefore, for all the methods in (3.6)–(3.14), the inequality

$$\max_{n\in\{1,\dots,N\}}\|\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}\|^2_{\mathcal{L}^2(\Omega)} + \int_0^T \|\overline{\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\|^2_h\,\mathrm{d}t \leqslant Ch^2 \tag{4.24}$$

follows from (4.22) and (4.23).

By the triangle inequality we have

$$\begin{aligned}
&\|w^n - \mathcal{W}_{\mathrm{DG}}^n\|^2_{\mathcal{L}^2(\Omega)} + \int_0^T \|w - \overline{\mathcal{W}}_{\mathrm{DG}}\|_h^2 \,\mathrm{d}t \\
&\leqslant \max_{n\in\{1,\dots,N\}} \|w^n - \mathcal{W}^n\|^2_{\mathcal{L}^2(\Omega)} + \max_{n\in\{1,\dots,N\}} \|\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}\|^2_{\mathcal{L}^2(\Omega)} \\
&\quad + \int_0^T \|w - \overline{\mathcal{W}}\|_h^2 \,\mathrm{d}t + \int_0^T \|\overline{\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\|_h^2 \,\mathrm{d}t \\
&= \max_{n\in\{1,\dots,N\}} \|w^n - \mathcal{W}^n\|^2_{\mathcal{L}^2(\Omega)} + \max_{n\in\{1,\dots,N\}} \|\mathcal{W}^n - \mathcal{W}^n_{\mathrm{DG}}\|^2_{\mathcal{L}^2(\Omega)} \\
&\quad + \int_0^T \|\nabla(w - \overline{\mathcal{W}})\|^2_{\mathcal{L}^2(\Omega)} \,\mathrm{d}t + \int_0^T \|\overline{\mathcal{W}} - \overline{\mathcal{W}}_{\mathrm{DG}}\|_h^2 \,\mathrm{d}t
\end{aligned}$$

for any $n = 1, \dots, N$.

Hence, from (4.24) and by Lemma 2.2, we finally arrive at

$$\max_{n\in\{1,\dots,N\}} \|w^n - \mathcal{W}_{\mathrm{DG}}^n\|^2_{\mathcal{L}^2(\Omega)} + \int_0^T \|w - \overline{\mathcal{W}}_{\mathrm{DG}}\|_h^2 \,\mathrm{d}t \leqslant C(h^2 + (\Delta t)^2)$$

for all the methods in (3.6)–(3.14). This completes the proof. □

5. Numerical experiments

In this section, some numerical experiments are discussed to illustrate the performance of the proposed method derived in Theorem 4.1. We consider $2d$ oscillating moving circle from [32] as a model example. We implement the experiments by using MATLAB and algorithms (the primal-dual active set method) developed in the article [26] and some of our in house algorithms. In the experiments, we consider the four DG formulations SIPG (3.6), NIPG (3.7), IIPG (3.8) and LDG (3.9). We chose the penalty parameter $\gamma = 10$ for SIPG and IIPG and $\gamma = 1$ for NIPG and LDG. In the LDG formulation (3.9), we take $\beta = (0, 0)$.

$2d$ **oscillating moving circle:** Let Ω be the square $[-1, 1] \times [-1, 1]$ and $\mathcal{J}$ be the time interval $[0, 0.25]$, and let the non-contact and contact sets be

$$\Omega^+(t) := \{(x, y) \in \Omega \colon\ R(t) > R_0(t)\} \text{ and } \Omega^0(t) := \{(x, y) \in \Omega \colon\ R(t) \leqslant R_0(t)\},$$

respectively, where $R(t) = \{(x - R_1\cos(a\pi t))^2 + (y - R_1\sin(a\pi t))^2\}^{1/2}, R_0(t) = \frac{1}{3} + 0.3\sin(4a\pi t)$, $R_1 = \frac{1}{3}$ and $a = 4$. The exact solution u is given by

$$w(x, y, t) = \begin{cases} \frac{1}{2}(R^2(t) - R_0^2(t))^2 & \text{if } (x, y) \in \Omega^+(t), \\ 0 & \text{if } (x, y) \in \Omega^0(t). \end{cases}$$

The initial and boundary conditions are computed from w. The obstacle is $\chi := 0$ and forcing function g is given by

$$g(x,y,t) = \begin{cases} 4(R_0^2(t) - 2R^2(t) - \frac{1}{2}(R^2(t) - R_0^2(t))(P(t) + R_0(t)R_0'(t))) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } (x,y) \in \Omega^+(t), \\ -4R_0^2(t)(1 - R^2(t) + R_0^2(t)) \qquad\qquad\qquad \text{if } (x,y) \in \Omega^0(t), \end{cases}$$

where $P(t) = (x - C_1(t))C_1'(t) + (y - C_2(t))C_2'(t)$, $C_1(t) = R_1 \cos(a\pi t)$ and $C_2(t) = R_1 \sin(a\pi t)$. The free boundary is an oscillating circle with radius $R_0(t)$ and center $(C_1(t), C_2(t))$ moving anticlockwise along the circle of radius R_1 centered at the origin. We define

$$\text{total error}_{\text{DG}} := \max_{n\in\{1,\ldots,N\}} \|w^n - \mathcal{W}_{\text{DG}}^n\|_{\mathcal{L}^2(\Omega)} + \left(\int_{\mathcal{J}} \|(w - \overline{\mathcal{W}}_{\text{DG}})(t)\|_h^2 \, \mathrm{d}t\right)^{1/2},$$

$$\text{DOF} := \text{number of degrees of freedom in space, where}$$

$$N = \text{number of time steps.}$$

According to our analysis, we expect the convergence rate to be $\mathcal{O}(\Delta t)$ (in the norm defined above) with uniform time-step, and to be $\mathcal{O}(h)$ with spatial mesh refinement. We present the computed errors and orders of convergence in Table 1 and Table 2, for NIPG and IIPG methods, respectively. We get the same computed errors and orders of convergence for SIPG and LDG methods as in [21], Table 7.2 and [21], Table 7.3, respectively.

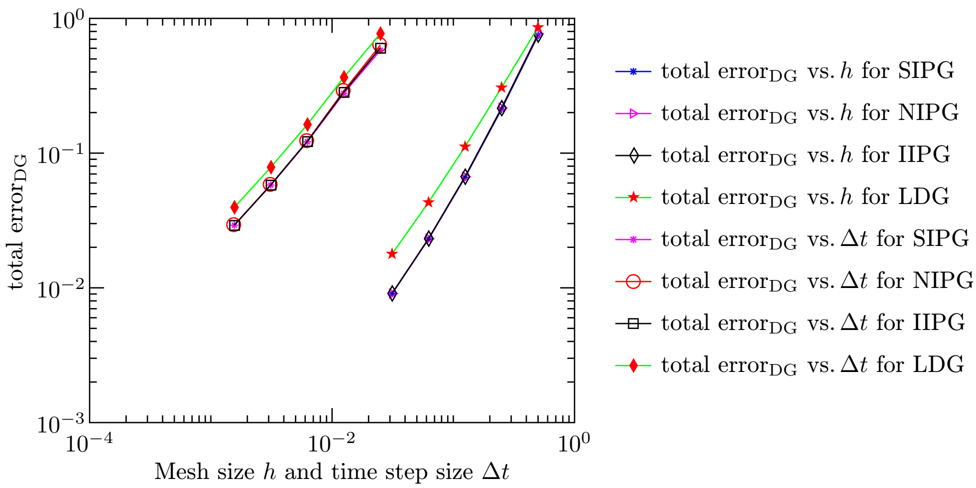

Figure 2. Mesh size h, time step size Δt versus total error$_{\text{DG}}$ for various DG methods.

N	h	DOF	total error$_{\mathrm{DG}}$	order
50	$\frac{1}{4}$	48	0.76826192	—
100	$\frac{1}{8}$	192	0.21680061	1.82522940
200	$\frac{1}{16}$	768	0.06676175	1.69927516
400	$\frac{1}{32}$	3072	0.02314769	1.52815328
800	$\frac{1}{64}$	12288	0.00906717	1.35214297

N	h	DOF	total error$_{\mathrm{DG}}$	order
10	$\frac{1}{4}$	192	0.63439329	—
20	$\frac{1}{8}$	768	0.28981282	1.13025609
40	$\frac{1}{16}$	3072	0.12279353	1.23888685
80	$\frac{1}{32}$	12288	0.05814057	1.07861742
160	$\frac{1}{64}$	49152	0.02920752	0.99320529

Table 1. Errors and orders of convergence in h (left table) & errors and orders of convergence in Δt (right table) for NIPG.

N	h	DOF	total error$_{\mathrm{DG}}$	order
50	$\frac{1}{4}$	48	0.75822986	—
100	$\frac{1}{8}$	192	0.21547514	1.81511384
200	$\frac{1}{16}$	768	0.06682247	1.68911621
400	$\frac{1}{32}$	3072	0.02320090	1.52615206
800	$\frac{1}{64}$	12288	0.00908344	1.35287043

N	h	DOF	total error$_{\mathrm{DG}}$	order
10	$\frac{1}{4}$	192	0.60338834	—
20	$\frac{1}{8}$	768	0.28368477	1.08879809
40	$\frac{1}{16}$	3072	0.12235533	1.21321171
80	$\frac{1}{32}$	12288	0.05811903	1.07399427
160	$\frac{1}{64}$	49152	0.02920893	0.99260097

Table 2. Errors and orders of convergence in h (left table) & errors and orders of convergence in Δt (right table) for IIPG.

In the left table of Table 1 (or Table 2) we show the order of convergence for space variable h with the quotient $\Delta t/h = 0.01$, and in the right table of Table 1 (or Table 2) we show the order of convergence for space variable Δt with the quotient $\Delta t/h = 0.1$ for NIPG method (or IIPG method). These experiments establish the correctness of the expected convergence rates for all the methods considered here. In Figure 2, we present the convergence history of the four DG methods.

Acknowledgements. The author would like to thank anonymous referee and editor for their helpful and constructive comments that lead to the improvement of this article.

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