Communications in Mathematics

Mohamed Laghzal; Abdelouahed El Khalil; Abdelfattah Touzani

A Weighted Eigenvalue Problems Driven by both $p(\cdot)$-Harmonic and $p(\cdot)$-Biharmonic Operators

Communications in Mathematics, Vol. 29 (2021), No. 3, 443–455

Persistent URL: http://dml.cz/dmlcz/149328

Communications in Mathematics 29 (2021) 443–455
DOI: 10.2478/cm-2020-0011

A Weighted Eigenvalue Problems Driven by both $p(\cdot)$-Harmonic and $p(\cdot)$-Biharmonic Operators

Mohamed Laghzal, Abdelouahed El Khalil, Abdelfattah Touzani

Abstract. The existence of at least one non-decreasing sequence of positive eigenvalues for the problem driven by both $p(\cdot)$-Harmonic and $p(\cdot)$-biharmonic operators

$$\Delta^2_{p(x)} u - \Delta_{p(x)} u = \lambda w(x)|u|^{q(x)-2}u \quad \text{in } \Omega,$$
$$u \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)\,,$$

is proved by applying a local minimization and the theory of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$.

1 Introduction and Assumptions

We are concerned with the following nonhomogeneous eigenvalue problem:

$$\begin{gathered} \Delta(|\Delta u|^{p(x)-2}\Delta u) - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda w(x)|u|^{q(x)-2}u \quad \text{in } \Omega, \\ u \in W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega), \end{gathered} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 1$ is a bounded smooth domain, $\lambda > 0$ is a real parameter, $p(\cdot), q(\cdot) \colon \overline{\Omega} \to (1, +\infty)$ are continuous functions, and w is a nonnegative function satisfying conditions which will be stated later.

2020 MSC: 58E05, 35J35, 47J10.
Key words: Palais-Smale condition, Ljusternick-Schnirelmann, Variational methods, $p(\cdot)$-biharmonic operator, $p(\cdot)$-harmonic operator, Variable exponent.
Affiliation:
Mohamed Laghzal – University Sidi Mohamed Ben Abdellah, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, P.O. Box 1796 Atlas Fez Morocco
E-mail: mohamed.laghzal@usmba.ac.ma
Abdelouahed El Khalil – Imam Mohammad Ibn Saud Islamic University (IMSIU), College of Science, Department of Mathematics and Statistics, P.O. Box 90950, 11623 Riyadh KSA
E-mail: alakhalil@imamu.edu.sa
Abdelfattah Touzani – University Sidi Mohamed Ben Abdellah, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, P.O. Box 1796 Atlas Fez Morocco
E-mail: atouzani07@gmail.com

$\Delta^2_{p(\cdot)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$, is the $p(\cdot)$-biharmonic operator which is a natural generalization of the p-biharmonic (when $p(\cdot)$ is a positive constant), and $\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(\cdot)$-harmonic operator, that is

$$\Delta_{p(x)}u := \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \sum_{i=1}^{N}\left(|\nabla u|^{p(x)-2}\frac{\partial u}{\partial x_i}\right),$$

which is not homogeneous and related to variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ and variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$.

In recent years, the study of differential equations and variational problems with variable exponent growth conditions has been an interesting topic resulting from nonlinear electrorheological fluids (see [16]) and elastic mechanics (see [21]).

The present paper deals with a class of nonlinear Dirichlet eigenvalue problems governed by two differentiable operators with variable "rheological" exponent. In such a way, the associated energy is a double phase functional generated by both the gradient and the Laplace operators with variable growth, that formulated in the form of variational integrals governed by these nonhomogenous potentials. We point out the recent works of [2], [14], [20] in the framework of double phase problems with variable growth involving on operator, namely the $p(\cdot)$-Laplacian.

The same problem, for $w \equiv 1$ and $q \equiv p$ was studied by El Khalil et al [5]. The authors established the existence of at least one non-decreasing sequence of positive eigenvalues. The smallest eigenvalue λ_1 of

$$(\Delta^2_{p(\cdot)} - \Delta_p(\cdot), W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega))$$

is positive and admits the following variational characterization:

$$\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}\left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)}\right) \mathrm{d}x : \int_\Omega \frac{1}{p(x)}|u|^{p(x)} \,\mathrm{d}x = 1\right\}. \tag{2}$$

Motivated by the works [1], [10], [12], [13], [15] and through the Ljusternik--Schnireleman principle on C^1-manifolds [17], we prove the existence of at least one non-decreasing sequence of nonnegative eigenvalues $(\lambda_k)_{k\geq 1}$, such that $\lambda_k \nearrow +\infty$.

Throughout this paper, we will work under the following hypotheses on the problem (1) :

(H1) $1 < q(x) < p(x) < \min\left\{\frac{N}{2}, p_2^*(x)\right\}$ for all $x \in \overline{\Omega}$.

(H2) $w \in L^{r(\cdot)}(\Omega)$ with $r(x) > \frac{N}{2}$ and $w(x) > 0$ a.e. $x \in \Omega$.

The rest of this article is organized as follows. In Section 2, we recall some basic facts about the variable exponent Lebesgue and Sobolev spaces. In the third section, we present some important basic lemmas which allow us to prove our main results. In Section 4, we give the main results and their proofs.

2 Preliminaries on variable exponent spaces

To study both $p(\cdot)$-harmonic and $p(\cdot)$-biharmonic problems, we introduce some basic properties of the variable exponent Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. For details, we refer to the book [3] and the references therein. Set

$$C_1^+(\overline{\Omega}) := \{h \in C(\overline{\Omega}) \text{ and } h(x) > 1, \forall x \in \overline{\Omega}\}.$$

Define

$$h^+ := \max_{x\in\overline{\Omega}} h(x), \quad h^- := \min_{x\in\overline{\Omega}} h(x), \quad \text{for any } h \in C_1^+(\overline{\Omega}).$$

To handle better the generalized Lebesgue-Sobolev spaces we define first, for $p(\cdot) \in C_1^+(\overline{\Omega})$, the $p(\cdot)$-modular functional $\rho_{p(\cdot)}$ defined on $L^{p(\cdot)}(\Omega)$ by

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\,\mathrm{d}x.$$

The generalized Lebesgue space is defined as

$$L^{p(\cdot)}(\Omega) = \{u\colon \Omega \to \mathbb{R} \text{ measurable and } \rho_{p(\cdot)}(u) < +\infty\}.$$

We endow it with the Luxemburg norm

$$|u|_{p(\cdot)} = \inf\left\{\mu > 0 : \rho_{p(\cdot)}\left(\frac{u}{\mu}\right) \le 1\right\}.$$

For any $m \in \mathbb{N}^*$, the Sobolev space with variable exponent $W^{m,p(\cdot)}(\Omega)$ is defined by

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \le m\},$$

equipped with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} = \|u\|_{m,p(\cdot)} = \sum_{|\alpha|\le m} |D^\alpha u|_{p(\cdot)}.$$

$W^{m,p(\cdot)}(\Omega)$ has the same topological features as $L^{p(\cdot)}(\Omega)$. For more details, we refer the reader to [8], [7]. We denote by $W_0^{m,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$.

Proposition 1. *[7] The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is separable, uniformly convex, reflexive and its conjugate dual space is $L^{p'(\cdot)}(\Omega)$ where $p'(\cdot)$ is the conjugate function of $p(\cdot)$, i.e.,*

$$p'(x) = \frac{p(x)}{p(x)-1} \quad \text{for all } x \in \Omega.$$

For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, the Hölder's type inequality

$$\left|\int_\Omega u(x)v(x)\,\mathrm{d}x\right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u(x)|_{p(\cdot)}|v(x)|_{p'(\cdot)} \le 2|u(x)|_{p(\cdot)}|v(x)|_{p'(\cdot)} \tag{3}$$

holds true. Moreover, if $p_1, p_2, p_3 \in C_1^+(\overline{\Omega})$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, then for any $u \in L^{p_1(\cdot)}(\Omega)$, $v \in L^{p_2(\cdot)}(\Omega)$ and $w \in L^{p_3(\cdot)}(\Omega)$, the following inequality holds true [6, Proposition 2.5]

$$\Big|\int_\Omega uvw\,\mathrm{d}x\Big| \le \Big(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\Big)|u|_{p_1(\cdot)}|v|_{p_2(\cdot)}|w|_{p_3(\cdot)} \le 3|u|_{p_1(\cdot)}|v|_{p_2(\cdot)}|w|_{p_3(\cdot)}\,. \tag{4}$$

We recall also the following proposition, which will be needed later,

Proposition 2. *[4] Let p and q be measurable functions such that $p \in L^\infty(\Omega)$ and $1 < p(x)q(x) < \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(\cdot)}(\Omega)$, $u \neq 0$. Then*

$$\min\big\{|u|^{p^+}_{p(x)q(x)}, |u|^{p^-}_{p(x)q(x)}\big\} \le \big||u|^{p(x)}\big|_{q(x)} \le \max\big\{|u|^{p^+}_{p(x)q(x)}, |u|^{p^-}_{p(x)q(x)}\big\}\,. \tag{5}$$

Note that weak solutions of problem (1) are considered in the generalized Sobolev space

$$X := W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)\,,$$

endowed with the norm

$$\|u\|_{p(\cdot)} = |\Delta u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}\,,$$

is a separable and reflexive Banach space. Moreover, $\|\cdot\|_{p(\cdot)}$ and $|\Delta \cdot|_{p(\cdot)}$ are two equivalent norms of X by [18].

Let

$$\|u\| = \inf\Big\{\alpha > 0 : \int_\Omega \Big[\Big|\frac{\Delta u(x)}{\alpha}\Big|^{p(x)} + \Big|\frac{\nabla u(x)}{\alpha}\Big|^{p(x)}\Big]\,\mathrm{d}x \le 1\Big\}.$$

Then, $\|\cdot\|$ is equivalent to the norms $\|\cdot\|_{p(\cdot)}$ and $|\Delta \cdot|_{p(\cdot)}$ in X.

We consider the functional

$$\Lambda_{p(\cdot)}(u) = \int_\Omega \big(|\Delta u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}\big)\,\mathrm{d}x\,, \quad u \in X$$

and give the following fundamental proposition.

Proposition 3. *For $u \in X$, the following relations hold*

- $\|u\| < 1$ *(respectively =; >)* $\Leftrightarrow \Lambda_{p(\cdot)}(u) < 1$ *(respectively =; >),*
- $\|u\| \le 1 \Rightarrow \|u\|^{p^+} \le \Lambda_{p(\cdot)}(u) \le \|u\|^{p^-}$,
- $\|u\| \ge 1 \Rightarrow \|u\|^{p^-} \le \Lambda_{p(\cdot)}(u) \le \|u\|^{p^+}$, *for all $u_n \in X$ we have*
- $\|u_n\| \to 0 \Leftrightarrow \Lambda_{p(\cdot)}(u_n) \to 0$,
- $\|u_n\| \to \infty \Leftrightarrow \Lambda_{p(\cdot)}(u_n) \to \infty$.

The proof of this proposition is similar to the proof of [7, Theorem 1.3]

Lemma 1. *[7] For all $p, r \in C_1^+(\overline{\Omega})$ such that $r(x) \le p_m^*(x)$ for all $x \in \overline{\Omega}$, then there is a continuous embedding*

$$W^{m,p(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)\,,$$

where

$$p_m^*(x) := \begin{cases} \frac{Np(x)}{N-mp(x)} & \text{if } mp(x) < N, \\ +\infty & \text{if } mp(x) \ge N. \end{cases}$$

If we replace $\le$ with $<$ the embedding is compact.

Remark 1. Regarding hypotheses (H1) and (H2), let

$$r'(x) = \frac{r(x)}{r(x)-1} \quad \text{and} \quad s(x) = \frac{r(x)q(x)}{r(x)-q(x)}\,.$$

Then, $s(x) < p_2^*(x)$ for all $x \in \overline{\Omega}$. Consequently the embedding $X \hookrightarrow L^{s(\cdot)}(\Omega)$ is compact and continuous.

3 Auxiliary results

In this section, we investigate some auxiliary results which allow us to prove our main results. Here and henceforth, we denote by X the generalized Sobolev space $X := W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$ equipped with the norm $\|\cdot\|$, X^* its dual space. For simplicity we write $u_n \rightharpoonup u$ and $u_n \to u$ to denote the weak convergence and strong convergence of sequence u_n in X, respectively.

We are interested in the weak solutions of (1) belonging to the space X in the sense below.

Definition 1. We understand a function $u \in X$ is a weak solution of (1), if for all $v \in X$,

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta v \,\mathrm{d}x + \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \,\mathrm{d}x = \lambda \int_\Omega w(x)|u|^{q(x)-2}uv \,\mathrm{d}x\,. \quad (6)$$

Moreover, if $u \in X \setminus \{0\}$, then we say that λ is the eigenvalue of problem (1) corresponding to the eigenfunction u.

For any $\lambda > 0$ we define $\mathcal{A}_\lambda \colon X \to \mathbb{R}$ by

$$\mathcal{A}_\lambda(u) = \int_\Omega \Big(\frac{|\Delta u(x)|^{p(x)}}{p(x)} + \frac{|\nabla u(x)|^{p(x)}}{p(x)}\Big)\,\mathrm{d}x - \lambda \int_\Omega \frac{w(x)}{q(x)}|u(x)|^{q(x)}\,\mathrm{d}x\,.$$

Then, $\mathcal{A}_\lambda \in C^1(X, \mathbb{R})$ and

$$\begin{aligned}\langle \mathrm{d}\mathcal{A}_\lambda(u), v\rangle = &\int_\Omega \big(|\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x) + |\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)\big)\,\mathrm{d}x \\ &- \lambda \int_\Omega w(x)|u(x)|^{q(x)-2}u(x)v(x)\,\mathrm{d}x\,,\end{aligned}$$

for all $u, v \in X$. Thus the weak solution of (1) are exactly the critical points of $\mathcal{A}_\lambda$.

We define the first Rayleigh quotient by

$$\lambda_1 := \inf_{u \in X\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}\left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)}\right) \mathrm{d}x}{\int_\Omega \frac{w(x)}{q(x)} |u|^{q(x)} \,\mathrm{d}x} \,. \tag{7}$$

Let us introduce the functionals $\Phi, \Psi \colon X \to \mathbb{R}$ defined by

$$\Phi(u) = \int_\Omega \left(\frac{|\Delta u(x)|^{p(x)}}{p(x)} + \frac{|\nabla u(x)|^{p(x)}}{p(x)} \right) \mathrm{d}x \,,$$

$$\Psi(u) = \int_\Omega \frac{w(x)}{q(x)} |u|^{q(x)} \,\mathrm{d}x \,.$$

Lemma 2. *The functionals Φ and Ψ are even of class C^1 on X, with the Gâteaux derivative given by*

$$\left\langle \mathrm{d}\Phi(u), v \right\rangle = \int_\Omega |\Delta u(x)|^{p(x)-2} \Delta u(x) \Delta v(x) \,\mathrm{d}x + \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) \,\mathrm{d}x \,,$$

and

$$\left\langle \mathrm{d}\Psi(u), v \right\rangle = \int_\Omega w(x) |u(x)|^{q(x)-2} u(x) v(x) \,\mathrm{d}x \,.$$

The proof of Lemma 2 is based on standard arguments, and hence the details are omitted.

Lemma 3. *$\mathcal{V} = \{u \in X;\, \Psi(u) = 1\}$ is a closed C^1-manifold.*

Proof.
- $\mathcal{V} = \Psi^{-1}\{1\}$. Thus $\mathcal{V}$ is closed.
- For all $x \in \overline{\Omega}$, we have $q^- \le q(x) \le q^+$, then for all $u \in \mathcal{V}$

$$\left\langle \mathrm{d}\Psi(u), u \right\rangle = \int_\Omega w(x) |u(x)|^{q(x)} \,\mathrm{d}x \ge q^- > 0 \,.$$

Then, $\mathcal{V}$ is a C^1-manifold of X with codimension one. □

Lemma 4. *[5] We have the following properties:*

i) $\Delta^2_{p(\cdot)} \colon W_0^{2,p(\cdot)}(\Omega) \to W^{-2,p'(\cdot)}(\Omega)$ *is a strictly monotone operator, that is,*

$$\left\langle \Delta^2_{p(\cdot)} u - \Delta^2_{p(\cdot)} v, u - v \right\rangle > 0 \,, \quad \textit{for all } u \ne v \in W_0^{2,p(\cdot)}(\Omega).$$

ii) $\Delta^2_{p(\cdot)} \colon W_0^{2,p(\cdot)}(\Omega) \to W^{-2,p'(\cdot)}(\Omega)$ *is a continuous, bounded homeomorphism.*

iii) $\Delta^2_{p(\cdot)} \colon W_0^{2,p(\cdot)}(\Omega) \to W^{-2,p'(\cdot)}(\Omega)$ *is a mapping of type (S_+), that is, if*

$$u_n \rightharpoonup u \quad \textit{in } W_0^{2,p(\cdot)}(\Omega)$$

and

$$\limsup_{n\to\infty} \left\langle \Delta^2_{p(\cdot)} u_n, u_n - u \right\rangle \le 0 \,,$$

then $u_n \to u$ in $W_0^{2,p(\cdot)}(\Omega)$.

Lemma 5. *[9] We have the following properties:*

i) $-\Delta_{p(\cdot)}\colon W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ *is a homeomorphism.*

ii) $-\Delta_{p(\cdot)}\colon W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ *is a strictly monotone operator.*

iii) $-\Delta_{p(\cdot)}\colon W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p'(\cdot)}(\Omega)$ *is a mapping of type* (S_+).

Lemma 6. *The functional* $\mathrm{d}\Psi$ *is completely continuous, namely,* $u_n \rightharpoonup u$ *in* X *implies* $\mathrm{d}\Psi(u_n) \to \mathrm{d}\Psi(u)$ *in* X^*.

Proof. Let $u_n \rightharpoonup u$ in X. For any $v \in X$, by Hölder's type inequality (4) and due to the fact that $X \hookrightarrow L^{s(\cdot)}(\Omega)$ is continuous, it follows that

$$\begin{aligned}\left|\left\langle \mathrm{d}\Psi(u_n) - \mathrm{d}\Psi(u), v\right\rangle\right| &= \left|\int_\Omega w(x)\big(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\big)v\,\mathrm{d}x\right| \\ &\le 3\left|w(x)\right|_{r(\cdot)}\left||u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right|_{\frac{q(\cdot)}{q(\cdot)-1}}|v|_{s(\cdot)} \\ &\le 3C\left|w(x)\right|_{r(\cdot)}\left||u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right|_{\frac{q(\cdot)}{q(\cdot)-1}}\|v\|, \quad C > 1.\end{aligned}$$

On the other hand, using the compact embedding of X into $L^{q(\cdot)}(\Omega)$, we have $u_n \to u$ in $L^{q(\cdot)}(\Omega)$. Due the fact that the map

$$L^{q(\cdot)}(\Omega) \ni u \mapsto |u|^{q(x)-2}u \in L^{\frac{q(\cdot)}{q(\cdot)-1}}(\Omega),$$

is continuous, we get

$$|u_n|^{q(x)-2}u_n \to |u|^{q(x)-2}u \quad \text{in } L^{\frac{q(\cdot)}{q(\cdot)-1}}(\Omega).$$

That is,

$$\mathrm{d}\Psi(u_n) \to \mathrm{d}\Psi(u) \quad \text{in } L^{\frac{q(\cdot)}{q(\cdot)-1}}(\Omega).$$

Recall that the embedding

$$L^{\frac{q(\cdot)}{q(\cdot)-1}}(\Omega) \hookrightarrow X^*,$$

is compact. Thus

$$\mathrm{d}\Psi(u_n) \to \mathrm{d}\Psi(u) \quad \text{in } X^*.$$

□

Let us conclude this section with show that the functional Φ satisfies the Palais--Smale condition (in short the (PS) condition) on $\mathcal{V}$.

Proposition 4. *The functional* Φ *satisfies the (PS) condition on* $\mathcal{V}$. *Namely, we will prove that for sequence* $\{u_n\} \subset \mathcal{V}$ *satisfying*

$$|\Phi(u_n)| \le d \quad \text{for some } d > 0 \text{ and all } n \ge 1. \tag{PS1}$$

$$\mathrm{d}\Phi(u_n) \to 0 \quad \text{in } X^*, \text{ as } n \to \infty. \tag{PS2}$$

has a convergent subsequence in X.

Proof. Let $\{u_n\}_{n\geq 1}$ be a sequence of Palais-Smale of Φ in X. Since

$$\begin{aligned}\int_\Omega \frac{1}{p(x)}\left(|\Delta u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)\mathrm{d}x &\geq \frac{1}{p^+}\int_\Omega\left(|\Delta u_n|^{p(x)}+|\nabla u_n|^{p(x)}\right)\mathrm{d}x\\ &= \frac{1}{p^+}\Lambda_{p(\cdot)}(u_n)\,,\end{aligned}$$

this fact, combined with Equation (PS1), implies that

$$\Lambda_{p(\cdot)}(u_n)\leq p^+ d\,.$$

That is $\Lambda_{p(\cdot)}(u_n)$ is bounded in $\mathbb{R}$. Thus, without loss of generality, we can assume that

$$u_n \rightharpoonup u \quad \text{for some functions } u\in X \text{ and } \Lambda_{p(\cdot)}(u_n)\to \ell\,.$$

For the rest we distinguish two cases:

Case 1. If $\ell = 0$, then $u_n \to 0$ in X and the proof is finished.

Case 2. If $\ell \neq 0$, then we argue as follows.

From Equation (PS2), $\mathrm{d}\Phi(u_n)\to 0$. i.e.,

$$\eta_n = \mathrm{d}\Phi(u_n) - \delta_n \mathrm{d}\Psi(u_n) \to 0 \quad \text{as } n\to+\infty\,, \tag{8}$$

where

$$\delta_n = \frac{\langle \mathrm{d}\Phi(u_n), u_n\rangle}{\langle \mathrm{d}\Psi(u_n), u_n\rangle}\,.$$

The idea is to prove that

$$\limsup_{n\to\infty}\langle \Delta^2_{p(\cdot)}u_n - \Delta_{p(\cdot)}u_n, u_n - u\rangle \leq 0\,.$$

Indeed, notice that

$$\langle \Delta^2_{p(\cdot)}u_n - \Delta_{p(\cdot)}u_n, u_n - u\rangle = \Lambda_{p(\cdot)}(u_n) - \langle \Delta^2_{p(\cdot)}u_n - \Delta_{p(\cdot)}u_n, u\rangle\,.$$

Applying η_n of (8) to u, we deduce that

$$\theta_n = \langle \Delta^2_{p(\cdot)}u_n - \Delta_{p(\cdot)}u_n, u\rangle - \delta_n\langle \mathrm{d}\Psi(u_n), u\rangle \to 0 \quad \text{as } n\to\infty.$$

Therefore

$$\langle \Delta^2_{p(\cdot)}u_n - \Delta_{p(\cdot)}u_n, u_n - u\rangle = \Lambda_{p(\cdot)}(u_n) - \theta_n - \frac{\langle \mathrm{d}\Phi(u_n), u_n\rangle}{\langle \mathrm{d}\Psi(u_n), u_n\rangle}\langle \mathrm{d}\Psi(u_n), u\rangle\,.$$

That is,

$$\langle \Delta^2_{p(\cdot)}u_n - \Delta_{p(\cdot)}u_n, u_n - u\rangle = \frac{\Lambda_{p(\cdot)}(u_n)}{\langle \mathrm{d}\Psi(u_n), u_n\rangle}\left(\langle \mathrm{d}\Psi(u_n), u_n\rangle - \langle \mathrm{d}\Psi(u_n), u\rangle\right) - \theta_n\,.$$

On the other hand, from Lemma 6, $\mathrm{d}\Psi$ is also completely continuous. So

$$\mathrm{d}\Psi(u_n) \to \mathrm{d}\Psi(u) \quad \text{and} \quad \big\langle \mathrm{d}\Psi(u_n), u_n \big\rangle \to \big\langle \mathrm{d}\Psi(u), u \big\rangle .$$

Then

$$\begin{aligned}&\big|\big\langle \mathrm{d}\Psi(u_n), u_n \big\rangle - \big\langle \mathrm{d}\Psi(u_n), u \big\rangle\big| \\ &\qquad \le \big|\big\langle \mathrm{d}\Psi(u_n), u_n \big\rangle - \big\langle \mathrm{d}\Psi(u), u \big\rangle\big| + \big|\big\langle \mathrm{d}\Psi(u_n), u \big\rangle - \big\langle \mathrm{d}\Psi(u), u \big\rangle\big| .\end{aligned}$$

It follows that

$$\begin{aligned}&\big|\big\langle \mathrm{d}\Psi(u_n), u_n \big\rangle - \big\langle \mathrm{d}\Psi(u_n), u \big\rangle\big| \\ &\qquad \le \big|\big\langle \mathrm{d}\Psi(u_n), u_n \big\rangle - \big\langle \mathrm{d}\Psi(u), u \big\rangle\big| + \|\mathrm{d}\Psi(u_n) - \mathrm{d}\Psi(u)\|_* \|u\| ,\end{aligned}$$

where $\|\cdot\|_*$ is the dual norm associated to the norm $\|\cdot\|$. This implies that

$$\big\langle \mathrm{d}\Psi(u_n), u_n \big\rangle - \big\langle \Psi(u_n), u \big\rangle \to 0 \quad \text{as } n \to \infty.$$

Combining with the above equalities, we obtain

$$\begin{aligned}&\limsup_{n\to+\infty} \big\langle \Delta^2_{p(\cdot)} u_n - \Delta_{p(\cdot)} u_n, u_n - u \big\rangle \\ &\qquad \le \frac{\ell}{\big\langle \mathrm{d}\Psi(u), u \big\rangle} \limsup_{n\to\infty} \Big(\big\langle \mathrm{d}\Psi(u_n), u_n \big\rangle - \big\langle \mathrm{d}\Psi(u_n), u \big\rangle \Big) .\end{aligned}$$

We deduce

$$\limsup_{n\to\infty} \big\langle \Delta^2_{p(\cdot)} u_n - \Delta_{p(\cdot)} u_n, u_n - u \big\rangle \le 0 .$$

By ii) of Lemma 4 and ii) of Lemma 5, $u_n \to u$ in X. Hence, the proof of the proposition is completed. □

4 Main results

In this section, we show that the problem (1) has at least one non-decreasing sequence of positive eigenvalues by using the results of Ljusternik-Schnireleman principle on C^1-manifolds [17]. In other words, we use a local minimization for the corresponding energy functional.

Let

$$\Sigma_j = \big\{ H \subset \mathcal{V} : H \text{ is compact, } H = -H \text{ and } \gamma(H) \ge j \big\} ,$$

where $\gamma(H) = j$ is the Krasnoselskii genus of the set H, i.e.,

$$\gamma(H) = \inf\big\{ j : \text{there exists an odd continuous map from } H \text{ to } \mathbb{R}^j \setminus \{0\} \big\}.$$

The main result of this paper is given by the following theorem.

Theorem 1. *For any integer $j \in \mathbb{N}^*$,*

$$\lambda_j := \inf_{H\in\Sigma_j} \max_{u\in H} \Phi(u) ,$$

is a critical value of Φ *restricted on* $\mathcal{V}$*. More precisely, there exists* $u_j \in H$ *such that*

$$\lambda_j = \Phi(u_j) = \sup_{u \in H} \Phi(u)\,,$$

and u_j *is a solution of* (6) *associated to the positive eigenvalue* λ_j*. Moreover,*

$$\lambda_j \to \infty\,, \quad \text{as } j \to \infty.$$

We proceed by two lemmas.

Lemma 7. *For any* $j \in \mathbb{N}^*$, $\Sigma_j \neq \emptyset$.

Proof. Since X is separable. Therefore, for any $j \in \mathbb{N}^*$, there exists $(\varphi_i)_{i\geq 1}$ linearly dense in X such that

$$\begin{cases} \operatorname{supp}(\varphi_i) \cap \operatorname{supp}(\varphi_j) = \emptyset & \text{if } i \neq j, \\ \operatorname{meas}(\operatorname{supp}(\varphi_i)) > 0 & \text{for } i \in \{1, 2, \dots, j\}. \end{cases}$$

Thanks to hypotheses (H2) we can choose φ_i such that $\int_\Omega \frac{w(x)}{q(x)} |\varphi_i|^{q(x)}\,\mathrm{d}x = 1$.

Let $X_j = \operatorname{Span}\{\varphi_1, \varphi_2, \dots, \varphi_j\}$ be the vector subspace of X generated by j vectors $\{\varphi_1, \varphi_2, \dots, \varphi_j\}$. Then, it is clear that

- $\dim X_j = j$.
- $\displaystyle\int_\Omega \frac{w(x)}{q(x)} |u(x)|^{q(x)}\,\mathrm{d}x > 0$ for all $u \in X_j \setminus \{0\}$.

Note that $X_j \subset L^{q(\cdot)}(\Omega)$ because $X_j \subset X \subset L^{q(\cdot)}(\Omega)$. Thus the norm $\|\cdot\|$ and $|\cdot|_{q(\cdot)}$ are equivalent on X_j because X_j is a finite dimensional space. Consequently the map

$$u \mapsto |u| := \inf\left\{\alpha > 0 : \int_\Omega \left|\frac{u(x)}{\alpha}\right|^{q(x)} \mathrm{d}x \leq 1\right\},$$

defines a norm on X_j. Denote $S := \{u \in X_j : |u| = 1\}$ the unit sphere of X_j.

Let us introduce the functional

$$\begin{aligned} g\colon \mathbb{R}^+ \times X_j &\longrightarrow \mathbb{R} \\ (s, u) &\mapsto \Psi(su)\,. \end{aligned}$$

On one hand, it is clear that

- $g(0, u) = 0$.
- $g(s, u)$ is non decreasing with respect to s.

Moreover, for $s > 1$ we have

$$g(s, u) \geq s^{q^-}\Psi(u)\,,$$

so that $\lim_{s\to+\infty} g(s,u) = +\infty$. Therefore, for every $u \in S_1$ fixed, there is a unique value $s = s(u) > 0$ such that $g(s(u),u) = 1$.

On the other hand, since

$$\frac{\partial g}{\partial s}(s(u),u) = \int_\Omega (s(u))^{q(x)-1} w(x)|u|^{q(x)}\,\mathrm{d}x \geq \frac{q^-}{s(u)} g(s(u),u) = \frac{q^-}{s(u)} > 0\,.$$

The implicit function theorem implies that the map $u \mapsto s(u)$ is continuous and even by uniqueness.

Now, take the compact $H_j := \mathcal{V} \cap X_j$. Since the map $h\colon S \to H_j$ defined by $h(u) = s(u)\cdot u$ is continuous and odd, it follows by the property of genus that $\gamma(H_j) = j$. Therefore $H_j \in \Sigma_j$. □

Lemma 8. *$\lambda_j \to \infty$ as $j \to \infty$.*

Proof. Let $(e_n, e_k^*)_{n,k}$ be a bi-orthogonal system such that $e_n \in X$ and $e_k^* \in X^*$, the $(e_n)_n$ are linearly dense in X and the $(e_k^*)_k$ are total for the dual X^*.

For $j \in \mathbb{N}^*$, set

$$X_j = \mathrm{Span}\{e_1,\dots,e_j\} \quad \text{and} \quad X_j^\perp = \mathrm{Span}\{e_{j+1}, e_{j+2},\dots\}.$$

By property of genus, we have for any $H \in \Gamma_j$, it is $H \cap X_{j-1}^\perp \neq \emptyset$.

We claim that

$$t_j = \inf_{H\in\Sigma_j} \sup_{u\in H\cap X_{j-1}^\perp} \Phi(u) \to \infty \quad \text{as } j \to \infty.$$

Indeed, if not, for large j there exists $u_j \in X_{j-1}^\perp$ with

$$\int_\Omega \frac{w(x)}{q(x)} |u_j(x)|^{q(x)}\,\mathrm{d}x = 1$$

such that

$$t_j \leq \Phi(u_j) \leq M, \quad \text{for some } M > 0 \text{ independent of } j.$$

Thus

$$\|u_j\| \leq (p^+ M)^{\frac{1}{p^-}}\,.$$

This implies that $(u_j)_j$ is bounded in X. For a subsequence of $\{u_j\}$ if necessary, we can assume that $\{u_j\}$ converges weakly in X and strongly in $L^{p(\cdot)}(\Omega)$.

By our choice of $X_{j-1}^\perp$, we have $u_j \rightharpoonup 0$ in X because $\langle e_k^*, e_n\rangle = 0$, for any $n > k$. This contradicts the fact that $\int_\Omega \frac{w(x)}{q(x)}|u_j(x)|^{q(x)}\,\mathrm{d}x = 1$ for all j.

Indeed, from Lemma 6, $\mathrm{d}\Psi$ is completely continuous, so $\langle \mathrm{d}\Psi(u_j), u_j\rangle \to 0$.

On the other hand, since $\int_\Omega \frac{w(x)}{q(x)}|u_j(x)|^{q(x)}\,\mathrm{d}x = 1$, and

$$\langle \mathrm{d}\Psi(u_j), u_j\rangle = \int_\Omega w(x)|u_j(x)|^{q(x)}\mathrm{d}x \geq q^- \int_\Omega \frac{w(x)}{q(x)}|u_j(x)|^{q(x)}\mathrm{d}x \geq q_2^- \geq 1\,,$$

since $\langle \mathrm{d}\Psi(u_j), u_j\rangle \geq 1$, for all j, $\langle \mathrm{d}\Psi(u_j), u_j\rangle \to l \geq 1$. Therefore $l \neq 0$.

Since $\lambda_j \geq t_j$, we get $\lambda_j \to \infty$ as $j \to \infty$, this complete the proof. □

Proof of Theorem 1. Applying lemmas 7, 8 and Ljusternik-Schnirelemann theory to the problem (1), we have for each $j \in \mathbb{N}^*$, λ_j is a critical value of Φ on C^1-manifold $\mathcal{V}$, such that

$$\lambda_j \to \infty\,, \quad \text{as } j \to \infty.$$

□

Corollary 1. *The following hold:*

i) $\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}\left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)}\right) \mathrm{d}x : u \in X \textit{ and } \int_\Omega \frac{w(x)}{q(x)}|u|^{q(x)}\, \mathrm{d}x = 1\right\}$.

ii) $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to +\infty$.

iii) $\lambda_1 = \inf \Lambda$ *(i.e.,* λ_1 *is the smallest eigenvalue in the spectrum* Λ *of* (1)*).*

Proof. i) For $u \in \mathcal{V}$, set $H_1 = \{u, -u\}$. It is clear that $\gamma(K_1) = 1$, Φ is even and

$$\Phi(u) = \max_{H_1} \Phi \geq \inf_{H\in\Sigma_1} \max_{u\in H} \Phi(u)\,.$$

Thus

$$\inf_{u\in\mathcal{V}} \Phi(u) \geq \inf_{H\in\Sigma_1} \max_{u\in H} \Phi(u) = \lambda_1\,.$$

On the other hand, for all $H \in \Sigma_1$ and $u \in H$, we have

$$\sup_{u\in H} \Phi \geq \Phi(u) \geq \inf_{u\in\mathcal{V}} \Phi(u)\,.$$

It follows that

$$\inf_{H\in\Sigma_1} \max_{H} \Phi = \lambda_1 \geq \inf_{u\in\mathcal{V}} \Phi(u)\,.$$

Then

$$\lambda_1 = \inf\left\{\int_\Omega \frac{1}{p(x)}\left(|\Delta u|^{p(x)} + |\nabla u|^{p(x)}\right) \mathrm{d}x : u \in X \text{ and } \int_\Omega \frac{w(x)}{q(x)}|u|^{q(x)}\, \mathrm{d}x = 1\right\}.$$

ii) For all $i \geq j$, we have $\Sigma_i \subset \Sigma_j$ and in view of the definition of $\lambda_i, i \in \mathbb{N}^*$, we get $\lambda_i \geq \lambda_j$. As regards $\lambda_n \to \infty$, this has been proved in Theorem 1.

iii) Let $\lambda \in \Lambda$. Thus there exists u_λ an eigenfunction of λ such that

$$\int_\Omega \frac{w(x)}{q(x)}|u_\lambda|^{q(x)}\, \mathrm{d}x = 1\,.$$

Therefore

$$\Delta^2_{p(x)} u_\lambda - \Delta_{p(x)} u_\lambda = \lambda w(x)|u_\lambda|^{q(x)-2}u_\lambda \quad \text{in } \Omega.$$

Then

$$\int_\Omega \frac{1}{p(x)}\left(|\Delta u_\lambda|^{p(x)} + |\nabla u_\lambda|^{p(x)}\right) \mathrm{d}x = \lambda \int_\Omega \frac{w(x)}{q(x)}|u_\lambda|^{q(x)}\, \mathrm{d}x\,.$$

In view of the characterization of λ_1 in (7), we conclude that

$$\lambda = \frac{\int_\Omega \frac{1}{p(x)}\left(|\Delta u_\lambda|^{p(x)} + |\nabla u_\lambda|^{p(x)}\right) \mathrm{d}x}{\int_\Omega \frac{w(x)}{q(x)}|u_\lambda|^{q(x)}\, \mathrm{d}x} = \int_\Omega \frac{1}{p(x)}\left(|\Delta u_\lambda|^{p(x)} + |\nabla u_\lambda|^{p(x)}\right) \mathrm{d}x \geq \lambda_1\,.$$

This implies that $\lambda_1 = \inf \Lambda$. □

References

[1] T.A. Bui: $W^{1,p(\cdot)}$ estimate for renormalized solutions of quasilinear equations with measure data and Reifenberg domains. *Advances in Nonlinear Analysis* 7 (4) (2018) 517–533.

[2] M. Cencelj, V.D. Rădulescu, D.D. Repovš: Double phase problems with variable growth. *Nonlinear Analysis* 177 (2018) 270–287.

[3] L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka: *Lebesgue and Sobolev spaces with variable exponents.* Lecture Notes in Mathematics, Springer (2011).

[4] D. Edmunds, J. Rákosník: Sobolev embeddings with variable exponent. *Studia Mathematica* 143 (3) (2000) 267–293.

[5] A. El Khalil, M.D. Morchid Alaoui, A. Touzani: On the Spectrum of problems involving both $p(x)$-Laplacian and $P(x)$-Biharmonic. *Advances in Science, Technology and Engineering Systems Journal* 2 (5) (2017) 134–140.

[6] X. Fan, X. Han: Existence and multiplicity of solutions for $p(x)$-Laplacian equations in Dirichlet problem in $\mathbb{R}^N$. *Nonlinear Analysis: Theory, Methods & Applications* 59 (1–2) (2004) 173–188.

[7] X.L. Fan, X. Fan: A Knobloch-type result for $p(t)$-Laplacian systems. *Journal of mathematical analysis and applications* 282 (2) (2003) 453–464.

[8] X.L. Fan, X. Fan: A Knobloch-type result for $p(t)$-Laplacian systems. *Journal of mathematical analysis and applications* 282 (2) (2003) 453–464.

[9] X.L. Fan, Q.H. Zhang: Existence of solutions for $p(x)$-Laplacian Dirichlet problem. *Nonlinear Analysis: Theory, Methods & Applications* 52 (8) (2003) 1843–1852.

[10] K. Kefi, V.D. Rădulescu: On a $p(x)$-biharmonic problem with singular weights. *Zeitschrift für angewandte Mathematik und Physik* 68 (80) (2017) 1–13.

[11] A. Scapellato: Regularity of solutions to elliptic equations on Herz spaces with variable exponents. *Boundary Value Problems* 2019 (1) (2019) 1–9.

[12] M. Mihăilescu, V. Rădulescu: A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 462 (2073) (2006) 2625–2641.

[13] V.D. Rădulescu: Nonlinear elliptic equations with variable exponent: old and new. *Nonlinear Analysis: Theory, Methods & Applications* 121 (2015) 336–369.

[14] V.D. Rădulescu: Isotropic and anisotropic double-phase problems: old and new. *Opuscula Mathematica* 39 (2) (2019) 259–279.

[15] V.D. Rădulescu, D.D. Repovš: *Partial differential equations with variable exponents: variational methods and qualitative analysis.* Monographs and Research Notes in Mathematics, CRC press (2015).

[16] M. Růžička: *Electrorheological fluids: modeling and mathematical theory*. Lecture Notes in Mathematics, 1748, Springer Science & Business Media (2000).

[17] A. Szulkin: Ljusternik-Schnirelmann theory on C^1-manifolds. *Annales de l'Institut Henri Poincare (C) Non Linear Analysis* 5 (2) (1988) 119–139.

[18] A. Zang, Y. Fu: Interpolation inequalities for derivatives in variable exponent Lebesgue-Sobolev spaces. *Nonlinear Analysis: Theory, Methods & Applications* 69 (10) (2008) 3629–3636.

[19] E. Zeidler: *Nonlinear Functional Analysis and Its Applications: II/B: Nonlinear Monotone Operators*. Springer (1990). Translated from the German by the author and Leo F. Boron.

[20] Q. Zhang, V.D. Rădulescu: Double phase anisotropic variational problems and combined effects of reaction and absorption terms. *Journal de Mathématiques Pures et Appliquées* 118 (2018) 159–203.

[21] V.V. Zhikov: Averaging of functionals of the calculus of variations and elasticity theory (in Russian). *Izv. Akad. Nauk SSSR Ser. Mat.* 50 (4) (1986) 675–710.

Received: 11 August 2019
Accepted for publication: 23 September 2019
Communicated by: Diana Barseghyan