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RETRACTS THAT ARE KERNELS OF LOCALLY NILPOTENT DERIVATIONS

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Abstract. Let k be a field of characteristic zero and B a k-domain. Let R be a retract of B being the kernel of a locally nilpotent derivation of B. We show that if $B = R \oplus I$ for some principal ideal I (in particular, if B is a UFD), then $B = R^{[1]}$, i.e., B is a polynomial algebra over R in one variable. It is natural to ask that, if a retract R of a k-UFD B is the kernel of two commuting locally nilpotent derivations of B, then does it follow that $B \cong R^{[2]}$? We give a negative answer to this question. The interest in retracts comes from the fact that they are closely related to Zariski's cancellation problem and the Jacobian conjecture in affine algebraic geometry.

Keywords: retract; locally nilpotent derivation; kernel; Zariski's cancellation problem

MSC 2020: 14R10, 13N15

1. Introduction

Throughout the paper, k stands for a field of characteristic zero and a k-algebra refers to a commutative k-algebra with identity 1. A subalgebra R of a k-algebra S is called a *retract* if there is an idempotent k-algebra endomorphism (called a *retraction*) φ of S such that $\varphi(S) = R$ (for more equivalent conditions, see Definition 2.1 below). In the category of k-algebras, a k-algebra P is a projective object if and only if P is a retract of some polynomial algebra in not necessarily finite number of variables.

The study of retracts of polynomial algebras $k^{[n]} := k[x_1, \ldots, x_n]$ is closely related to some problems in affine algebraic geometry. For example, Shpilrain and Yu in [15] showed that the 2-dimensional Jacobian conjecture is equivalent to the statement that, for each pair of polynomials $f, g \in k[x_1, x_2]$ with $\det J_{x_1,x_2}(f, g)$

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invertible, $k[f]$ is a retract of $k[x_1, x_2]$. In [7], [20], retracts were applied to the automorphic orbit problem for polynomial algebras in two variables. And by the use of retracts, the second author gave in [16] a new method for describing automorphisms of the endomorphism semigroups of free algebras such as polynomial algebras and free Poisson algebras.

Retracts were also involved with Zariski's cancellation problem: if A is a k-algebra such that $A^{[1]} \cong k^{[n+1]}$ then does it follow that $A \cong k^{[n]}$? (Cf. [13], Chapter 6 or [19].) Zariski's cancellation problem has an affirmative answer for $n \leqslant 2$ and is still open for any $n \geqslant 3$. (Gupta in [8] and [10] showed that if $\operatorname{char} k > 0$ then it has a negative answer for all $n \geqslant 3$.) Zariski's cancellation problem has an affirmative answer if the following problem concerning retracts has a positive solution: Is every proper retract of the polynomial algebra $k^{[n]}$ isomorphic to a polynomial algebra over k? (Cf. [3].)

Only a few results concerning retracts have been obtained up to now. Costa in [3] showed that every proper retract of $k^{[2]}$ is of the form $k[p]$ for some $p \in k^{[2]}$, Shpilrain and Yu in [15] showed further that there is an automorphism φ of $k^{[2]}$ such that $\varphi(p) = x_1 + x_2 q$ for some $q \in k^{[2]}$. The authors described in [12] retracts of $k^{[n]}$ induced by retractions with sparse homogeneous parts.

Retracts that are kernels of locally nilpotent derivations were studied by Chakraborty, Dasgupta, Dutta and Gupta in [2], in particular they showed that, for a k-UFD B, if R is a retract of B being the kernel of a locally nilpotent derivation of B, then $B = R^{[1]}$ (see [2]), Corollary 4.3 if B is a domain but not a UFD, it can happen that $B \not\cong R^{[1]}$, whence the relation between R and B was studied given the additional condition that $B = S^{[n]}$ for some Noetherian normal domain S and $S \subseteq R$, see [2], Theorem 4.5.

In this paper, we show that if B is a k-domain and R is a retract of B being the kernel of a locally nilpotent derivation of B such that $B = R \oplus I$ for some principal ideal I, then $B = R^{[1]}$ (see Theorem 2.5), this generalizes Corollary 4.3 of [2] since it is the case if B is a UFD. Note that Theorem 2.5 also follows from the work of Das and Dutta (see [5]) on the codimension-one $\mathbb{A}^1$-fibration with retraction, see Remark 2.6. Our proof is self-contained using the technique of locally nilpotent derivations.

We consider further that if R is a retract of a k-UFD B being the kernel of two commuting locally nilpotent derivations of B, then does it follow that $B \cong R^{[2]}$? We give a negative answer to this question (see Example 2.13, Proposition 2.14). We also describe retracts of $k^{[n]}$ with the transcendence degree two using Jelonek's embedding theorem for affine spaces, see Theorem 2.10.

2. Retracts that are kernels of locally nilpotent derivations

First, we recall some notions and facts concerning retracts and locally nilpotent derivations, see [3], [6] for details.

Definition 2.1 ([3]). A subalgebra R of a k-algebra S is called a *retract* if it satisfies any of the following equivalent conditions:

(1) there is an idempotent k-algebra endomorphism (called a *retraction*) of S such that $\varphi(S) = R$,

(2) there is a k-algebra homomorphism $\varphi\colon S \to R$ such that $\varphi|_R = \mathrm{id}_R$,

(3) $S = R \oplus I$ for some ideal I of S.

A k-derivation D of a k-algebra S is a k-linear map $D\colon S \to S$ satisfying the Leibnitz rule $D(ab) = D(a)b + aD(b)$ for any $a, b \in S$. We say that D is *locally nilpotent* if for each $u \in S$, there exists some positive integer n_u such that $D^{n_u}(u) = 0$. We write $\ker D$ for the kernel of D, and we denote by $\mathrm{LND}_k(S)$ the set of all locally nilpotent k-derivations of S.

Definition 2.2 ([6], Section 1.1). Let S be a k-algebra, $D \in \mathrm{LND}_k(S)$ and $A = \ker D$. An element $r \in S$ with $Dr \neq 0$ and $D^2 r = 0$ is called a *local slice* of D.

Any nonzero locally nilpotent k-derivation D has a local slice.

Lemma 2.3 ([6], Section 1.4). *Let B be a k-domain, $0 \neq D \in \mathrm{LND}_k(B)$ and $A = \ker D$. Take any local slice r of D. Then $B_{Dr} = A_{Dr}[s]$, where $s = r/Dr$. Moreover, the extension $\widetilde{D}$ of D on B_{Dr} acts as ∂_s on B_{Dr}.*

Given a k-domain B and $0 \neq D \in \mathrm{LND}_k(B)$ for any $b \in B$ put $\deg_D(b) = \min\{n \in \mathbb{N}\colon\ D^{n+1}(b) = 0\}$.

Further, set $\deg_D(0) = -\infty$ by convention. One may see that $\deg_D(b) = 0$ if and only if $b \in \ker D$, and $\deg_D(b) = 1$ if and only if b is a local slice of D. Lemma 2.3 implies that $\deg_D(b)$ equals to the degree of b as a polynomial in s. So $\deg_D$ is a degree function on B.

Lemma 2.4. *Let R be a retract of a k-domain B such that $B = R \oplus (h)$ for some $h \in B$. Then for any integer $m \geqslant 1$,*

$$B = R \oplus Rh \oplus \ldots \oplus Rh^{m-1} \oplus Bh^m.$$

P r o o f. Observe that $B = R \oplus Bh = R \oplus (R \oplus Bh)h = R \oplus Rh \oplus Bh^2$. In this way, we have

$$B = R \oplus Rh \oplus \ldots \oplus Rh^{m-1} \oplus Bh^m$$

for any integer $m \geqslant 1$. □

Theorem 2.5. *Let B be a k-domain and R a retract of B such that $R = \ker D$ for some $0 \neq D \in \mathrm{LND}_k(B)$. If $B = R \oplus I$ for some principal ideal I of B (in particular, if B is a k-UFD), then $B = R^{[1]}$.*

Proof. Let $I = (h)$ and let φ be the projection from B to R regarding to the decomposition $B = R \oplus I$. Then $I = \ker \varphi$ and φ is a retraction such that $\varphi(B) = R$. Take a local slice p of D. Since $\varphi(p) \in R = \ker D$, we have that $p - \varphi(p) \in I$ is also a local slice of D. Replacing p by $p - \varphi(p)$ we may assume that $p \in I$, say $p = hv$ for some $v \in B$. Observe that

$$1 = \deg_D p = \deg_D(hv) = \deg_D(h) + \deg_D(v),$$

where $\deg_D(h) \geqslant 1$ (since $h \notin R = \ker D$). Hence, $\deg_D(h) = 1$, i.e., h is a local slice of D.

Now we show that $R[h] = B$. For that purpose, take any $f \in B$. Let $a = D(h)$. Since h is a local slice, we have $R[h]_a = B_a$ due to Lemma 2.3, and thus there exist some positive integer $m(f)$ and some $r_0, r_1, \ldots, r_t \in R$ such that

$$a^{m(f)} f = r_0 + r_1 h + \ldots + r_t h^t. \tag{2.1}$$

By Lemma 2.4, $B = R \oplus Rh \oplus \ldots \oplus Rh^t \oplus Bh^{t+1}$, say

$$f = c_0 + c_1 h + \ldots + c_t h^t + d h^{t+1} \tag{2.2}$$

for some $c_0, c_1, \ldots, c_t \in R$ and some $d \in B$. Combining (2.1) and (2.2), we obtain that

$$a^{m(f)} c_0 + a^{m(f)} c_1 h + \ldots + a^{m(f)} c_t h^t + a^{m(f)} d h^{t+1} = r_0 + r_1 h + \ldots + r_t h^t. \tag{2.3}$$

Since $a, c_i, r_i \in R$ and $B = R \oplus Rh \oplus \ldots \oplus Rh^{t-1} \oplus Bh^{t+1}$, we obtain from (2.3) that $a^{m(f)} d h^{t+1} = 0$ and thus $d = 0$. Then it follows from (2.2) that $f \in R[h]$. Therefore, $R[h] = B$.

Finally, assume that B is a k-UFD. Since R is a retract of B, $B = R \oplus J$ for some ideal J of B. It suffices to show that J is a principal ideal. Similar as above, we may take a local slice $p \in J$. Let $p = p_1 p_2 \ldots p_s$ be the decomposition of p into irreducible elements. Since

$$1 = \deg_D(p) = \deg_D(p_1) + \deg_D(p_2) + \ldots + \deg_D(p_s),$$

there is exact one i such that $\deg_D(p_i) = 1$, say $\deg_D(p_1) = 1$ and $\deg_D(p_2) = \ldots = \deg_D(p_s) = 0$, i.e., p_1 is a local slice and $p_2, \ldots, p_s \in R = \ker D$. Noticing that $B = R \oplus J$, $p = p_1 p_2 \ldots p_s \in J$ and $p_2, \ldots, p_s \in R$, we have $p_1 \in J$.

Let $a_1 = D(p_1)$. Then $B_{a_1} = R[p_1]_{a_1}$ due to Lemma 2.3. For any $u \in J$, there exist some positive integer $m(u)$ and some $r_0, r_1, \dots, r_t \in R$ such that

$$a_1^{m(u)} u = r_0 + r_1 p_1 + \ldots + r_t p_1^t.$$

Since $u, p_1 \in J$ we have $r_0 \in J \cap R$ and thus $r_0 = 0$. It follows that $a_1^{m(u)} u \in (p_1)$. Since B is a UFD and p_1 is irreducible, we have then $u \in (p_1)$ or $a_1^{m(u)} \in (p_1)$. If $a_1^{m(u)} \in (p_1)$, then noticing that $\ker D$ is factorially closed and $a_1 \in \ker D$ we have $p_1 \in \ker D = R$, a contradiction. So $u \in (p_1)$. Therefore, $J = (p_1)$ as desired. $\square$

Remark 2.6. Theorem 2.5 also follows from some results of Das and Dutta in [5], where they investigated a codimension-one $\mathbb{A}^1$-fibration with retraction. More precisely, combining Lemma 3.6 and Remark 3.7 in [5], one has the following conclusion: If R is a retract of a domain B with a retraction $\varphi\colon B \to R$ such that (i) $\ker \varphi = GB$ for some $G \in B$ and (ii) $B \otimes_K R = K^{[1]}$, where K is the fractions field of R, then $B = R[G]$. In Theorem 2.5, the hypothesis $B = R \oplus I$ for some principal ideal I ensures that (i) is satisfied and the hypothesis R is the kernel of some locally nilpotent derivation of B ensures that (ii) is satisfied. Our proof is self-contained using the technique of locally nilpotent derivations.

Corollary 2.7. *Let R be a retract of $k^{[3]} = k[x, y, z]$ which is the kernel of some nonzero locally nilpotent derivation of $k^{[3]}$. Then there is a coordinate system f, g, h of $k^{[3]}$ such that $R = k[f, g]$.*

P r o o f. Due to Theorem 2.5, $R[h] = k^{[3]}$ for some $h \in k^{[3]}$. By Miyanishi's theorem (cf. [6], Theorem 5.1), the kernel of any locally nilpotent derivation of $k^{[3]}$ is isomorphic to $k^{[2]}$. So $R = k[f, g]$ for some $f, g \in k^{[3]}$. Thus $k[f, g, h] = k[x, y, z]$, i.e., f, g, h is a coordinate system of $k^{[3]} = k[x, y, z]$. $\square$

Remark 2.8. There exists a retract of $k^{[3]} = k[x, y, z]$ with transcendence degree two which is not the kernel of any locally nilpotent derivation of $k^{[3]}$, for example the retract $R = k[x + xz, y]$ defined by the retraction φ of $k^{[3]}$, $\varphi(x) = x + xz$, $\varphi(y) = y$, $\varphi(z) = 0$. In fact, if $R = \ker D$ for some $0 \neq D \in \mathrm{LND}_k(k^{[3]})$, then $R[h] = k^{[3]}$ for some $h \in k^{[3]}$ due to Theorem 2.5, and thus $x + xz$ is a coordinate of $k^{[3]}$, a contradiction.

It was shown in [2] (and independently in [14]) that every retract R of $k^{[n]}$ with transcendence degree two is isomorphic to $k^{[2]}$. We give below an explicit description for such retracts using Jelonek's embedding theorem for affine spaces.

An embedding $\alpha\colon \mathbb{A}_k^r \to \mathbb{A}_k^n$ is called *rectifiable* if there exists some $\varphi \in \operatorname{Aut}(\mathbb{A}_k^n)$ such that $\alpha = \varphi \circ j$, where $j\colon \mathbb{A}_k^r \to \mathbb{A}_k^n$, $(x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$ is the standard embedding. The well-known Abhyanker-Moh-Suzuki theorem (see [1], [17]) says that every embedding of $\mathbb{A}_k^1$ to $\mathbb{A}_k^2$ is rectifiable. And Craighero in [4] showed that, when $n \geqslant 4$, every embedding of $\mathbb{A}_k^1$ to $\mathbb{A}_k^n$ is rectifiable. A more general result is as follows.

Lemma 2.9 (Jelonek [11]). *If $n > 2r + 1$, then every embedding of $\mathbb{A}_k^r$ to $\mathbb{A}_k^n$ is rectifiable.*

Theorem 2.10. *Let $n > 5$ and let R be a retract of $B = k^{[n]} = k[x_1, x_2, \ldots, x_n]$ with transcendence degree two. Then there exists a $\psi \in \operatorname{Aut}_k(k^{[n]})$ such that $\psi(R) = k[x_1 + h_1, x_2 + h_2]$, where h_1, h_2 belong to the ideal $(x_3, \ldots, x_n)$ of B.*

Proof. By [2], [14], $R \cong k^{[2]}$, say $R = k[f_1, f_2]$ for some $f_1, f_2 \in B$. Let $\widetilde{\varphi}\colon B \to B$, $\widetilde{\varphi}(x_i) = \varphi_i(f_1, f_2)$, $1 \leqslant i \leqslant n$, be a retraction such that $\widetilde{\varphi}(B) = R$. Then

$$\varrho\colon \mathbb{A}_k^2 \to \mathbb{A}_k^n, \quad (x_1, x_2) \mapsto (\varphi_1(x_1, x_2), \ldots, \varphi_n(x_1, x_2))$$

is an embedding. Let $j\colon \mathbb{A}_k^2 \to \mathbb{A}_k^n$, $(x_1, x_2) \mapsto (x_1, x_2, 0, \ldots, 0)$, be the standard embedding. Since $n > 5$, by Lemma 2.9, there exists a $\psi \in \operatorname{Aut}(\mathbb{A}_k^n)$ such that $\psi j = \varrho$. Thus for $j = 1, 2$, $f_i \psi j = f_i \varrho$, i.e., $f_i \psi(x_1, x_2, 0, \ldots, 0) = f_i(\varphi_1(x_1, x_2), \ldots, \varphi_n(x_1, x_2))$. So

$$f_i \psi(f_1, f_2, 0, \ldots, 0) = f_i(\varphi_1(f_1, f_2), \ldots, \varphi_n(f_1, f_2)) = f_i,$$

where the last equality is due to $\widetilde{\varphi}|_R = \mathrm{id}|_R$. So

$$f_i \psi(x_1, x_2, 0, \ldots, 0) = x_i,$$

which implies that $f_i \psi = x_i + h_i$, where $h_i \in (x_3, \ldots, x_n)$, $i = 1, 2$. Therefore,

$$\widetilde{\psi}(R) = k[\widetilde{\psi}(f_1), \widetilde{\psi}(f_2)] = k[x_1 + h_1, x_2 + h_2],$$

where $\widetilde{\psi}$ is the automorphism of $k^{[n]}$ corresponding to $\psi \in \operatorname{Aut}(\mathbb{A}_k^n)$. $\square$

Finally, we consider retract being the kernel of two commuting locally nilpotent derivations. It is natural to state the following problem.

Problem 2.11. *Let R be a retract of a k-UFD B such that R is the kernel of two B-linearly independent commuting locally nilpotent derivations of B. Does it follow that $B \cong R^{[2]}$?*

The condition of B-linear independence is necessary, for otherwise the kernels of the two derivations are the same, whence $B \cong R^{[1]}$ due to Theorem 2.5.

Proposition 2.12. *Let B be a k-UFD and R a retract of B such that $R = \ker D_1 \cap \ker D_2$ for two B-linearly independent commuting derivations $D_1, D_2 \in \mathrm{LND}_k(B)$. Then*

(1) $\ker D_1 = R[h_1]$ *and* $\ker D_2 = R[h_2]$ *for some* $h_1, h_2 \in B$,

(2) $R_w[h_1, h_2] = B_w$ *for some* $w \in R$.

Proof. Noticing that B is a UFD and D_1 is a locally nilpotent derivation of B, we have that $B_1 := \ker D_1$ is a UFD because $\ker D_1$ is factorially closed. Since D_1 and D_2 commute, D_2 restricts to $B_1 = \ker D_1$. Noticing that R is a retract of B, it is easy to verify that R is also a retract of B_1. Due to Theorem 2.5, $B_1 = R[h_1]$ for some $h_1 \in B_1$ and $D_2|_{B_1} = w\partial_{h_1}$ for some $w \in R$.

Similarly, $B_2 := \ker D_2 = R[h_2]$ for some $h_2 \in B_2$. Since $D_2(h_1) = w \in R$, h_1 is a local slice of D_2. Therefore, by Lemma 2.3 we have

$$B_w = (\ker D_2)_w[h_1] = R_w[h_2, h_1].$$

□

The following example gives a negative answer to Problem 2.11.

Example 2.13. Let $B = k[x, y, u, v]/(x^a + y^b + u^c v)$, where $a, b, c \geqslant 2$ are integers and $\gcd(a, b) = 1$. Then B is a UFD and there are two commuting locally nilpotent derivations D_1 and D_2 on B:

$$\begin{aligned} &D_1(x) = D_1(u) = 0, \quad D_1(v) = by^{b-1}, \quad D_1(y) = -u^c, \\ &D_2(y) = D_2(u) = 0, \quad D_2(v) = ax^{a-1}, \quad D_2(x) = -u^c. \end{aligned}$$

One may verify that $D_1D_2 = D_2D_1 = 0$ and $R := \ker D_1 \cap \ker D_2 = k[u]$. By Proposition 2.14 below, $B \ncong k^{[3]}$ and thus $B \ncong R^{[2]}$.

The Makar-Limanov invariant and Derksen invariant are powerful tools for distinguishing an $\mathbb{A}^n_k$-like affine variety from $\mathbb{A}^n_k$. The Derksen invariant $\mathrm{DK}(S)$ of a k-algebra S is the subalgebra of S generated by all kernels of locally nilpotent derivations on S, cf. [6], Chapter 9. Note that for $S = k^{[n]}$, $\mathrm{DK}(S) = S$ if $n > 1$.

Proposition 2.14. *Let B be as in Example* 2.13. *Then* $\mathrm{DK}(B) = k[x, y, u] \neq B$ *and thus* $B \ncong k^{[3]}$.

Proof. It suffices to show that $\mathrm{DK}(B) = k[x, y, u]$. Consider the $\mathbb{Z}^2$ grading $\mathfrak{g}$ on B in the lexicographic order such that x, y, u, v are homogeneous with degrees

$$\deg_{\mathfrak{g}} u = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \deg_{\mathfrak{g}} v = \begin{pmatrix} c \\ -ab \end{pmatrix}, \quad \deg_{\mathfrak{g}} x = \begin{pmatrix} 0 \\ -b \end{pmatrix}, \quad \deg_{\mathfrak{g}} y = \begin{pmatrix} 0 \\ -a \end{pmatrix}.$$

Let D_1 and D_2 be as in Example 2.13. Then D_1 and D_2 are both $\mathfrak{g}$-homogeneous, and $\ker D_1 = k[u, x]$, $\ker D_2 = k[u, y]$. So $\mathrm{DK}(B) \supseteq k[x, y, u]$.

To show that $\mathrm{DK}(B) \subseteq k[x,y,u]$, take any nonzero $D \in \mathrm{LND}_k(B)$ and any $f \in \ker D$. Denote by $\overline{f}$ and $\overline{D}$ the highest homogeneous parts of f and D, respectively, regarding the grading $\mathfrak{g}$. Then $\overline{D}$ is a $\mathfrak{g}$-homogeneous locally nilpotent derivation of B, and $\overline{f} \in \ker \overline{D}$. By [6], Lemma 9.8, the kernel of a nonzero $\mathfrak{g}$-homogeneous locally nilpotent derivation of B is $k[u,x]$ or $k[u,y]$. So $\overline{f} \in k[u,x]$ or $\overline{f} \in k[u,y]$, which implies that $\deg_{\mathfrak{g}} \overline{f} < \binom{0}{0}$ and thus $\deg_{\mathfrak{g}} f < \binom{0}{0}$. If $i_2 > 0$ is such that

$$\deg_{\mathfrak{g}} u^{i_1} v^{i_2} x^{i_3} y^{i_4} = \begin{pmatrix} -i_1 + i_2 c \\ -(i_2 ab + i_3 b + i_4 a) \end{pmatrix} < \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then $i_1 > i_2 c$, and by the relation $x^a + y^b + u^c v = 0$, we have $u^{i_1} v^{i_2} \in k[x,y,u]$. It follows that $f \in k[x,y,u]$ since $\deg_{\mathfrak{g}} f < \binom{0}{0}$. Therefore, $\mathrm{DK}(B) \subseteq k[x,y,u]$, as desired. □

Remark 2.15. Proposition 2.14 can follow from some general deep results in the literature. The conclusion $B \ncong k^{[3]}$ follows from the equivalence of (iv) and (ix) in [9], Theorem 3.11. The description $\mathrm{DK}(B) = k[x,y,u]$ follows from [9], Proposition 3.7. (Precisely, Proposition 3.7 of [9] says that if $\mathrm{DK}(B) \neq k[x,y,u]$, then there exist $z, t \in k[x,y]$ such that $k[z,t] = k[x,y]$ and $x^a + y^b = f(z) + g(z)t$ for some $f(z), g(z) \in k[z]$. Let d_1 and d_2 be the t-degrees of x and y, respectively. Since $k[z,t] = k[x,y]$, Jung's theorem (cf. [18], Section 5.1) ensures that $d_1 \mid d_2$ or $d_2 \mid d_1$. The equality $x^a + y^b = f(z) + g(z)t$ implies that $ad_1 = bd_2$, contradicts the condition $\gcd(a,b) = 1$. Hence, $\mathrm{DK}(B) = k[x,y,u]$.) Our proof of Proposition 2.14 is simple and self-contained.

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