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POSITIVE SOLUTIONS FOR CONCAVE-CONVEX ELLIPTIC PROBLEMS INVOLVING $p(x)$-LAPLACIAN

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Abstract. We study the existence and nonexistence of positive solutions of the nonlinear equation

$$-\Delta_{p(x)}u = \lambda k(x)u^q \pm h(x)u^r \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{Q}$$

where $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, is a regular bounded open domain in $\mathbb{R}^N$ and the $p(x)$-Laplacian

$$\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$$

is introduced for a continuous function $p(x) > 1$ defined on Ω. The positive parameter λ induces the bifurcation phenomena. The study of the equation (Q) needs generalized Lebesgue and Sobolev spaces. In this paper, under suitable assumptions, we show that some variational methods still work. We use them to prove the existence of positive solutions to the problem (Q) in $W_0^{1,p(x)}(\Omega)$. When we prove the existence of minimal solution, we use the sub-super solutions method.

Keywords: variable exponent Sobolev space; $p(x)$-Laplace operator; concave-convex nonlinearities; variational method

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be a bounded regular open domain in $\mathbb{R}^N$ and consider the following problem involving the $p(x)$-Laplace operator with the Dirichlet boundary condition:

$$\begin{cases} -\Delta_{p(x)}u = \lambda k(x)u^q + h(x)u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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and

$$
\begin{cases} -\Delta_{p(x)} u = \lambda k(x) u^q - h(x) u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}
$$

where $p(x) : \overline{\Omega} \to \mathbb{R}$ is a continuous function,

$$
\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u),
$$

λ a positive parameter and, h and k are two functions in $L^{\infty}(\Omega)$. This type of partial differential equations with $p(x)$-Laplacian attracted many and many researchers in the last century because of their importance and their applications. They describe various physical phenomena. For example, they model the electro-rheological fluids, which is an important category of non-Newtonian fluids (see [18]). They are used in image restoration (see [6]), elasticity (see [24]) and the process of filtration throughout porous media (see [4]). The perception of such physical models has been made easier by the deployment of many mathematics research such as [1], [2], [5], [7], [10], [15], [20], [21], [22], [23] and the references therein.

The study of the problem (1.1) goes back to 1994 when Ambrosetti et al. in [3] considered the problem

$$
\begin{cases} -\Delta u = \lambda u^q + u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}
$$

for

$$
0 < q < 1 < r < 2^* - 1, \tag{1.4}
$$

where 2^* is the critical Sobolev exponent given by

$$
2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N > 2, \\ \infty & \text{if } N \leqslant 2. \end{cases} \tag{1.5}
$$

After that, the problem (1.3) was generalized by Rădulescu and Repovš in [17] for concave-convex nonlinearities. They studied the solvability of the problem

$$
\begin{cases} -\Delta u = \lambda k(x) u^q \pm h(x) u^r & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}
$$

for q and r satisfying the condition (1.4) and, $h(x)$ and $k(x)$ being positive functions verifying

$$\operatorname*{ess\,inf}_{x\in\Omega} k(x) > 0 \quad \text{and} \quad \operatorname*{ess\,inf}_{x\in\Omega} h(x) > 0. \tag{1.7}$$

Later, Saoudi in [19] considered the problem (1.6) with the p-Laplace operator

$$-\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

for

$$0 \leqslant q < p-1 < r < p^* - 1, \tag{1.8}$$

where

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geqslant N \end{cases}$$

was the corresponding critical Sobolev exponent.

In this paper, we consider (1.1) and (1.2) where $p(x)$ is a function satisfying

$$\min_{x\in\overline{\Omega}} p(x) > 1.$$

In order to introduce our result, let

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x),$$

and through this paper, we suppose that

$$0 \leqslant q < p^- - 1 \leqslant p^+ - 1 < r < p^*(x) - 1, \tag{1.9}$$

where

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{otherwise} \end{cases} \tag{1.10}$$

is the critical Sobolev exponent. We consider the following definition of solutions (weak solution).

Definition 1.1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1) if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x = \lambda\int_\Omega k(x)u^q\varphi\,\mathrm{d}x + \int_\Omega h(x)u^r\varphi\,\mathrm{d}x \tag{1.11}$$

for all $\varphi \in C_{\mathrm{c}}^\infty(\Omega)$.

$C_c^\infty(\Omega)$ denotes the space of all C^∞ real functions with the compact support included in Ω. In a similar way, we define the weak solution for the problem (1.2). In the next section, we define the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$ and we give some of its properties that will be used later. Our main results follow.

Theorem 1.1. *Suppose that the positive functions h and k satisfy the condition* (1.7) *and q and r satisfy* (1.9). *Then there exists a positive number $\lambda^\star$ such that:*

(i) *If $0 < \lambda < \lambda^\star$, then the problem* (1.1) *has a positive minimal solution u_λ.*
(ii) *If $\lambda = \lambda^\star$, then the problem* (1.1) *has a positive solution.*
(iii) *If $\lambda > \lambda^\star$, then the problem* (1.1) *does not have a nontrivial solution.*

For the problem (1.2), we will prove the following theorem.

Theorem 1.2. *Assume that the positive functions h and k verify the condition* (1.7) *and q and r satisfy* (1.9). *Then there exists a positive real $\lambda^\star$ such that:*

(i) *If $\lambda > \lambda^\star$, then the problem* (1.2) *has a positive solution.*
(ii) *If $\lambda < \lambda^\star$, then the problem* (1.2) *does not have a nontrivial solution.*

2. Preliminaries

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geqslant 2$. When we have to deal with equations involving $p(x)$-Laplace operator, we can use variable exponent Lebesgue spaces $L^{p(x)}(\Omega)$ introduced by Orlicz, see [16]. Many studies were carried out, see for example [24], [25] and the references therein. For further properties and details on the space $L^{p(x)}(\Omega)$, we can refer to [3], [14]. For the variable exponent Sobolev spaces $W^{1,p(x)}(\Omega)$, we refer to [8], [9], [11], [12], [14].

First, let

$$C_+(\overline{\Omega}) = \{p(x) \in C(\overline{\Omega})\colon\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For $p(x) \in C_+(\overline{\Omega})$, we define the generalized Lebesgue space $L^{p(x)}(\Omega)$ as

$$L^{p(x)}(\Omega) = \Big\{u\colon\ \Omega \to \mathbb{R}\colon\ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\Big\}, \tag{2.1}$$

endowed with the norm (Luxemburg norm)

$$|u|_{p(x)} = \inf\Big\{\lambda > 0\colon\ \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,\mathrm{d}x \leqslant 1\Big\}. \tag{2.2}$$

The space $L^{p(x)}(\Omega)$ is a Banach space and it is reflexive if and only if $1 < p^- \leqslant p^+ < \infty$. When $p^+ < \infty$, $C_c(\Omega)$, the space of continuous real functions with compact support included in Ω is dense in $L^{p(x)}(\Omega)$.

Also, we have an important embedding result. If p_1 and p_2 are two functions in $C_+(\overline{\Omega})$ such that $p_1 \leqslant p_2$ in Ω, then there exists a continuous embedding

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega).$$

Now, for k being a positive integer number and $p(x) \in C_+(\overline{\Omega})$, the generalized Sobolev space is

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega)\colon\ D^\alpha u \in L^{p(x)}(\Omega) \text{ for all } |\alpha| \leqslant k\}, \tag{2.3}$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ is the multi-index, $|\alpha| = \sum_{i=1}^{N} \alpha_i$ is the order of α and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}$$

is the derivative in the distribution sense.

On $W^{k,p(x)}(\Omega)$, we consider the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leqslant k} |D^\alpha u|_{L^{p(x)}(\Omega)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_{\rm c}^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ and we endow it by the norm

$$\|u\|_{p(x)} := |\nabla u|_{p(x)}. \tag{2.4}$$

An important tool when we deal with variable exponent Sobolev spaces is the function $\varrho\colon\ L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\varrho(u) = \int_\Omega |u|^{p(x)}\,{\rm d}x,$$

which possesses the following properties

$$|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \leqslant \varrho(u) \leqslant |u|_{p(x)}^{p^+}, \tag{2.5}$$

$$|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \leqslant \varrho(u) \leqslant |u|_{p(x)}^{p^-}, \tag{2.6}$$

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \varrho(u_n - u) \to 0. \tag{2.7}$$

The following result generalizes the Sobolev embedding theorem.

Theorem 2.1 ([14]). *Let $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, be an open regular bounded domain in $\mathbb{R}^N$ and assume that $p, r \in C_+(\overline{\Omega})$ are such that $p(x) \leqslant r(x) \leqslant p^*(x)$ for all $x \in \overline{\Omega}$. Then there exists a continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}$.*

Also, the embedding is compact if $r(x) < p^*(x)$ for a.e. $x \in \Omega$.

3. Proof of Theorem 1.1

In Theorem 1.1 (i), we want to prove the existence of the minimal solution, for this we use the sub-super-solutions method.

Definition 3.1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a *sub-solution* of (1.1) if

$$-\Delta_{p(x)} u \leqslant \lambda k(x) u^q + u^r. \tag{3.1}$$

In the same way, a function $u \in W_0^{k,p(x)}(\Omega)$ is called a *super-solution* of (1.1) if $u > 0$ in Ω and the reverse inequality holds in (3.1).

Let

$$\lambda^* := \sup\{\lambda > 0\colon\ (1.1) \text{ has a solution}\} \tag{3.2}$$

and the energy functional E_λ be given by

$$E_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\,\mathrm{d}x - \frac{\lambda}{q+1}\int_\Omega k(x) u^{q+1}\,\mathrm{d}x - \frac{1}{r+1}\int_\Omega h(x) u^{r+1}\,\mathrm{d}x \tag{3.3}$$

on the Sobolev space $W_0^{1,p(x)}(\Omega)$.

Proof of Theorem 1.1. (i) The proof will be given in several steps.

Step 1. (There exists λ_0 such that for all $0 < \lambda < \lambda_0$ the problem has a solution.) Consider the eigenvalue Dirichlet problem

$$\begin{cases} -\Delta_{p(x)} \tilde{u} = \lambda k(x) \tilde{u}^q & \text{in } \Omega,\\ \tilde{u}|_{\partial\Omega} = 0, \quad \tilde{u} > 0 & \text{in } \Omega. \end{cases} \tag{3.4}$$

For q satisfying (1.9), there exists $\lambda_{0,1}$ such that for all $0 < \lambda < \lambda_{0,1}$, the problem (3.4) has a positive solution $\tilde{u}_\lambda$, see [13]. Let $\underline{u}_\lambda = \varepsilon \tilde{u}_\lambda$, then

$$-\Delta_{p(x)} \underline{u}_\lambda = \lambda k(x) \varepsilon^{p(x)-1} \tilde{u}_\lambda^q.$$

For $\varepsilon \in (0,1)$ sufficiently small and $q \leqslant p^- - 1$, we have

$$\lambda k(x) \varepsilon^{p(x)-1} \tilde{u}_\lambda^q \leqslant \lambda k(x) \varepsilon^q \tilde{u}_\lambda^q \leqslant \lambda k(x) \varepsilon^q \tilde{u}_\lambda^q + h(x) \varepsilon^r \tilde{u}_\lambda^r \tag{3.5}$$

and so $\underline{u}_\lambda$ is a sub-solution of the problem (1.1).

Now, let v be the positive solution to the problem

$$\begin{cases} -\Delta_{p(x)} v = \lambda + 1 & \text{in } \Omega,\\ v|_{\partial\Omega} = 0, \ v > 0 & \text{in } \Omega. \end{cases} \tag{3.6}$$

Let

$$F(u) = \lambda k(x) u^q + h(x) u^r. \tag{3.7}$$

Consider $\overline{u}_\lambda = Tv(x)$, where T is a real number such that

$$-\Delta_{p(x)}\overline{u}_\lambda \geqslant F(TM) \geqslant F(\overline{u}_\lambda), \tag{3.8}$$

where $M = \max\{1, \|v\|_\infty\}$. We have $-\Delta_{p(x)}\overline{u}_\lambda = T^{p(x)-1}(\lambda+1)$ and

$$F(\overline{u}_\lambda) \equiv \lambda k(x)T^q v^q + h(x)T^r v^r \leqslant \lambda c_1 T^q M^q + c_2 T^r M^r \tag{3.9}$$

with $c_1 = \|k\|_{L^\infty}$ and $c_2 = \|h\|_{L^\infty}$. So, it is enough to look for T such that

$$\begin{aligned} &\lambda+1 \geqslant \lambda c_1 T^{q+1-p^-}M^q + c_2 T^{r+1-p^-}M^r \quad \text{when } T \geqslant 1,\\ &\lambda+1 \geqslant \lambda c_1 T^{q+1-p^+}M^q + c_2 T^{r+1-p^+}M^r \quad \text{when } 1 > T, \end{aligned} \tag{3.10}$$

because

$$\begin{aligned} &T^{p(x)-1}(\lambda+1) \geqslant T^{p^- -1}(\lambda+1) \quad \text{when } T \geqslant 1,\\ &T^{p(x)-1}(\lambda+1) \geqslant T^{p^+ -1}(\lambda+1) \quad \text{when } 1 > T. \end{aligned}$$

Let

$$\varphi(T) = \begin{cases} \lambda A T^{q+1-p^-} + BT^{r+1-p^-} & \text{when } T \geqslant 1,\\ \lambda A T^{q+1-p^+} + BT^{r+1-p^+} & \text{when } 1 > T, \end{cases} \tag{3.11}$$

where $A = c_1 M^q$, $B = c_2 M^r$ and $\varphi(T)$ is a continuous function.

$$\lim_{T\to 0^+}\varphi(T) = \lim_{T\to\infty}\varphi(T) = \infty, \tag{3.12}$$

since $q+1-p^- < 0 < r+1-p^-$ and $q+1-p^+ < 0 < r+1-p^+$, so φ reaches its a minimum in $[0,\infty)$. For λ small enough, an elementary computation points out that the function φ reaches its minimum at $T_0 = C\lambda^{1/(r-q)}$ where C is a positive constant and

$$C = (AB^{-1}(r-p^+ +1)(p^+ - q - 1)^{-1})^{1/(r-q)}.$$

Then, there exists $\lambda_{0,2}$ such that for $0 < \lambda < \lambda_{0,2}$,

$$\overline{u}_\lambda(x) = T_0 v$$

is a super-solution of problem (1.1). Let $\lambda_0 = \inf\{\lambda_{0,1}, \lambda_{0,2}\}$. Consider $\lambda \in (0, \lambda_0)$. In order to prove that $\underline{u}_\lambda \leqslant \overline{u}_\lambda$, we use the fact that $\tilde{u}$ is bounded and for ε sufficiently small, we have

$$-\Delta_{p(x)}\underline{u}_\lambda \leqslant \lambda k(x)\varepsilon^{p(x)-1}\tilde{u}^q \leqslant \lambda k(x)\overline{u}_\lambda^q \leqslant -\Delta_{p(x)}\overline{u}_\lambda. \tag{3.13}$$

By the weak comparison principle, we conclude that $\underline{u}_\lambda \leqslant \overline{u}_\lambda$ and then (1.1) has a solution u between the sub-solution and the super-solution. So, the set $\Lambda = \{\lambda > 0\colon$ (1.1) has a weak solution$\}$ is not empty and $\lambda^\star := \sup\{\lambda > 0\colon$ (1.1) has a weak solution$\}$ exists in $(0,\infty)$.

Step 2. (Λ is an interval containing $(0,\lambda^\star)$.) Let $\lambda \in \Lambda$ and λ' be such that $0 < \lambda' < \lambda$. Since $\lambda \in \Lambda$, the problem (1.1) has a positive solution for this value λ which is a super-solution for the equation (1.1) when λ is replaced by λ'. However, $\underline{u}_\nu = \varepsilon\tilde{u}_\nu$ is a sub-solution for some $0 < \nu < \lambda_{0,1}$ for ε small enough. As in Step 1, by the inequality (3.13) and the comparison principle, we have $\underline{u}_\nu \leqslant \overline{u}_\lambda$ and so the problem (1.1), when λ is replaced by λ', has a positive solution and then $\lambda' \in \Lambda$. So Γ is an interval containing $(0,\lambda^\star)$.

Step 3. (For $\lambda \in (0,\lambda^\star)$, the equation (1.1) has a minimal solution.) Since $\lambda \in (0,\lambda^\star)$, then the problem (1.1) has a positive solution U. U is a super-solution to (1.1). Consider $\underline{u}_\nu = \varepsilon\tilde{u}_\nu$ for some $0 < \nu < \lambda$ and ε small enough such that $\underline{u}_\nu$ is a sub-solution for the problem (1.1) and $\underline{u}_\nu \leqslant U$.

Let $u_0 = \underline{u}_\nu$ and consider the monotone iterative scheme

$$\tag{3.14} \begin{cases} -\Delta_{p(x)} u_n = \lambda k(x) u_{n-1}^q + h(x) u_{n-1}^r & \text{in } \Omega,\\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

We obtain from the weak comparison principle that (u_n) is a nondecreasing sequence and $u_0 \leqslant u_1 \leqslant \ldots \leqslant U$. So the sequence (u_n) is uniformly bounded in $W_0^{1,p(x)}(\Omega)$ and converges to a solution u_λ of (1.1). We remark that the construction of the sequence (u_n) does not depend on the super-solution U and so any solution v of the equation of (1.1) can be considered as a super-solution. We have $u_0 \leqslant u_\lambda \leqslant v$ and then u_λ is the minimal positive solution.

(ii) *Step 1.* ($\lambda^\star$ is a finite real.) Let $\lambda \in (0,\lambda^*)$ and u_λ be a minimal solution of the problem (1.1). Put

$$\Omega_1 = \{x \in \Omega\colon\ u_\lambda(x) \geqslant 1\}, \quad \Omega_2 = \{x \in \Omega\colon\ u_\lambda(x) < 1\}.$$

By density, we can take $\varphi = u_\lambda^{-q}$ as a test function in the weak formulation of (1.1) and we obtain

$$\tag{3.15} \begin{aligned} \int_\Omega |\nabla u_\lambda|^{p(x)-2}\nabla u_\lambda\cdot\nabla u_\lambda^{-q}\,\mathrm{d}x &= \lambda\int_\Omega k(x)u_\lambda^{q-q}\,\mathrm{d}x + \int_\Omega h(x)u_\lambda^{r-q}\,\mathrm{d}x\\ &\geqslant \lambda\int_\Omega k(x)u_\lambda^{q-q}\,\mathrm{d}x \geqslant \lambda C\int_\Omega u_\lambda^{q-q}\,\mathrm{d}x\\ &= \lambda C|\Omega|_{N-1}. \end{aligned}$$

Using Poincaré inequality and by the Young inequality with $\gamma \in (0,1)$, the first term on the left-hand side of (3.15) may be estimated as

$$\begin{aligned}
&\int_\Omega |\nabla u_\lambda|^{p(x)-2}\nabla u_\lambda \cdot \nabla u_\lambda^{-q}\,\mathrm{d}x \\
&\qquad = q\int_\Omega |\nabla u_\lambda|^{p(x)} u_\lambda^{-q-1}\,\mathrm{d}x \\
&\qquad \leqslant q\biggl(\int_\Omega (\gamma^{-1/(p(x)-1)}|\nabla u_\lambda|^{p(x)})^{(p(x)-1)/p(x)}(\gamma u_\lambda^{-q-1})^{1/p(x)}\,\mathrm{d}x\biggr) \\
&\qquad \leqslant q\biggl(\int_\Omega \frac{p(x)-1}{p(x)}\gamma^{-1/(p(x)-1)}|\nabla u_\lambda|^{p(x)}\,\mathrm{d}x + \int_\Omega \frac{\gamma}{p(x)} u_\lambda^{-q-1}\,\mathrm{d}x\biggr) \\
&\qquad \leqslant C_1\int_\Omega |\nabla u_\lambda|^{p(x)}\,\mathrm{d}x + C_2\int_\Omega u_\lambda^{-q-1}\,\mathrm{d}x.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_\Omega |\nabla u_\lambda|^{p(x)-2}\nabla u_\lambda \cdot \nabla u_\lambda^{-q}\,\mathrm{d}x &\leqslant C_1\int_{\Omega_1} |\nabla u_\lambda|^{p(x)}\,\mathrm{d}x + C_2\int_{\Omega_1} u_\lambda^{-q-1}\,\mathrm{d}x \\
&\quad + C_1\int_{\Omega_2} |\nabla u_\lambda|^{p(x)}\,\mathrm{d}x + C_2\int_{\Omega_2} u_\lambda^{-q-1}\,\mathrm{d}x \\
&= \mathbf{I_{1,2}} + \mathbf{II_{3,4}}.
\end{aligned}$$

On one hand, by using (2.5) we get

$$\mathbf{I}_{1,2} \leqslant C_1\|u_\lambda\|_{p(x)}^{p^+} + C_2\int_{\Omega_1} u_\lambda^{p(x)}\,\mathrm{d}x \leqslant C_1\|u_\lambda\|_{p(x)}^{p^+} + C_2\|u_\lambda\|_{p(x)}^{p^+}.$$

On the other hand, using (2.6)we obtain

$$\begin{aligned}
\mathbf{II}_{3,4} &\leqslant C_1\|u_\lambda\|_{p(x)}^{p^-} + C_2\int_{\Omega_2} u_\lambda^{-r-1}\,\mathrm{d}x && (-r-1 < -q-1) \\
&\leqslant C_1\|u_\lambda\|_{p(x)}^{p^-} + C_2\biggl(\int_{\Omega_2} u_\lambda^{p(x)}\,\mathrm{d}x\biggr)^{(-r-1)/p(x)} && (-r-1 < p(x)) \\
&\leqslant C_1\|u_\lambda\|_{p(x)}^{p^-} + C_2\|u_\lambda\|_{p(x)}^{-r-1},
\end{aligned}$$

where C_1, C_2 are positive constants. So, we have

$$\int_\Omega |\nabla u_\lambda|^{p(x)-2}\nabla u_\lambda \cdot \nabla u_\lambda^{-q}\,\mathrm{d}x \leqslant c(\|u_\lambda\|_{p(x)}^{p^+} + \|u_\lambda\|_{p(x)}^{p^-} + \|u_\lambda\|_{p(x)}^{-r-1})$$

for some constant $c > 0$. From the inequality (3.15), we deduce that $\lambda^\star < \infty$.

Step 2. (The problem (1.1) has a solution when $\lambda = \lambda^\star$.) Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be a sequence such that $\lambda_k \uparrow \lambda^\star$ as $k \to \infty$. Since $\lambda_k \in (0, \lambda^\star)$, there exists $u_k = u_{\lambda_k}$, a weak positive solution to (1.1) for $\lambda = \lambda_k$, and $u_k \geqslant \underline{u}_{\lambda_k} \geqslant 0$. We claim that the sequence (u_k) is bounded in $W_0^{1,p(x)}(\Omega)$. Indeed, if it is not bounded, then $\|u\|_{p(x)} \to \infty$ up to a subsequence. So, we get

$$E_\lambda(u) \geqslant \frac{1}{p^+}\|u\|_{p(x)}^{p^-} - C_1\|u\|_{q+1}^{q+1} - C_2\|u\|_{r+1}^{r+1},$$

where C_1 and C_2 are positive constants. By using (1.9) and since the embeddings $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{r+1}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ are continuous, for $\|u\|_{p(x)} > 1$ we get

$$E_\lambda(u) \geqslant \frac{1}{p^+}\|u\|_{p(x)}^{p^-} - K_1\|u\|_{p(x)}^{q+1} - K_2\|u\|_{p(x)}^{r+1}, \tag{3.16}$$

where K_1, K_2 are positive constants. So, if $\|u\|_{p(x)} \to \infty$, then $E_\lambda(u) \to \infty$. Thus, up to a subsequence,

$$E_{\lambda_k}(u_k) \to \infty \quad \text{as } k \to \infty. \tag{3.17}$$

On the other hand, u_k is a solution for the problem (1.1) with $\lambda = \lambda_k$. We know that the equation (1.11) remains true for $u \in W_0^{1,p(x)}(\Omega)$ by density. Taking u_k as a test function in (1.11), we obtain

$$\int_\Omega |\nabla u_k|^{p(x)}\,\mathrm{d}x = \lambda_k \int_\Omega k(x)u_k^{q+1}\,\mathrm{d}x + \int_\Omega h(x)u_k^{r+1}\,\mathrm{d}x. \tag{3.18}$$

And so,

$$E_{\lambda_k}(u_k) \leqslant \frac{1}{p^-}\int_\Omega |\nabla u_k|^{p(x)}\,\mathrm{d}x - \left(\lambda_k \int_\Omega k(x)u_k^{q+1}\,\mathrm{d}x + \int_\Omega h(x)u_k^{r+1}\,\mathrm{d}x\right) \leqslant 0,$$

which contradicts (3.17). Then the sequence (u_k) is bounded in $W_0^{1,p(x)}(\Omega)$. By the generalized compactness Sobolev embedding, there exists an element $u \in W_0^{1,p(x)}(\Omega)$ such that

$$u_k \rightharpoonup u \quad \text{in } W_0^{1,p(x)}(\Omega) \text{ (weakly)} \quad \text{as } k \to \infty, \tag{3.19}$$

$$u_k \to u \quad \text{in } L^q(\Omega) \quad \text{as } k \to \infty, \tag{3.20}$$

$$u_k \to u \quad \text{a.e. in } \Omega \quad \text{as } k \to \infty. \tag{3.21}$$

We get

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi\,\mathrm{d}x = \lambda^* \int_\Omega k(x)u^q\varphi\,\mathrm{d}x + \int_\Omega h(x)u^r\varphi\,\mathrm{d}x \tag{3.20}$$

for any $\varphi \in W_0^{1,p(x)}(\Omega)$. If u is not nonnegative, we can take $|u|$ and by the comparison principle, the solution is positive. So for $\lambda = \lambda^\star$, the problem (1.1) has a positive solution.

(iii) By the definition of λ^*, there is no solution for the problem (1.1) for all $\lambda > \lambda^*$ and this accomplishes the proof of Theorem 1.1. $\square$

4. Proof of Theorem 1.2

Recall that the norm in $L^s(\Omega)$, when $1 \leqslant s < \infty$, is given by

$$\|u\|_s := \left(\int_\Omega |u|^s \, \mathrm{d}s\right)^{1/s}.$$

Consider $J_\lambda \colon W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, \mathrm{d}x - \frac{\lambda}{q+1} \int_\Omega k(x) u^{q+1} \, \mathrm{d}x + \frac{1}{r+1} \int_\Omega h(x) u^{r+1} \, \mathrm{d}x. \tag{4.1}$$

Let

$$\Lambda := \{\lambda > 0 \colon \text{(1.2) has a nontrivial solution}\}.$$

The proof of Theorem 1.2 will be also given in several steps.

(i) *Step 1.* (Λ is a nonempty set.) Let $\lambda > 0$. For $u \in W_0^{1,p(x)}(\Omega)$ and for $\|u\|_{p(x)} > 1$, we have

$$J_\lambda(u) \geqslant \frac{1}{p^+} \|u\|_{p(x)}^{p^-} - C_1 \|u\|_{q+1}^{q+1} + C_2 \|u\|_{r+1}^{r+1},$$

where C_1 and C_2 are positive constants. From (1.9) and continuous embedding theorems, we have

$$J_\lambda(u) \geqslant \frac{1}{p^+} \|u\|_{p(x)}^{p^-} - K \|u\|_{p(x)}^{q+1}, \tag{4.2}$$

where K is a positive constant. Then $J_\lambda(u) \to \infty$ as $\|u\|_{p(x)} \to \infty$ and the functional J is coercive. The lower semi-continuity of the norms induces the lower semi-continuity of the functional J.

Let (u_n) be a minimizing sequence of J_λ in $W_0^{1,p(x)}(\Omega)$, which is bounded in $W_0^{1,p(x)}(\Omega)$, since J is coercive. With no loss of generality, we can suppose that (u_n) is nonnegative, converges pointwise and converges weakly to u in $W_0^{1,p(x)}(\Omega)$.

Furthermore, using the boundedness of (u_n) in $W_0^{1,p(x)}(\Omega)$ and the weak lower semi-continuity of the norm $\|\cdot\|_{p(x)}$, we obtain

$$J_\lambda(u) \leqslant \liminf_{n\to\infty} J_\lambda(u_n).$$

Consequently, u is a global minimizer of J_λ in $W_0^{1,p(x)}(\Omega)$ and u is nonnegative.

Step 2. (The function u introduced in Step 1 is a nontrivial solution of (1.2) for some $\lambda > 0$.) We have $J_\lambda(0) = 0$ and we claim that $J(u) < 0$. To prove the claim, consider

$$\mathcal{M} = \left\{ w \in W_0^{1,p(x)}(\Omega)\colon \ \frac{1}{q+1}\int_\Omega k(x)|w|^{q+1}\,\mathrm{d}x = 1 \right\} \tag{4.3}$$

and let I be the functional defined on $\mathcal{M}$ by

$$I(w) = \int_\Omega \frac{1}{p(x)}|\nabla w|^p(x)\,\mathrm{d}x + \frac{1}{r+1}\int_\Omega h(x)|w|^{r+1}\,\mathrm{d}x. \tag{4.4}$$

Since $\mathcal{M}$ is a weakly closed subset of $W_0^{1,p(x)}(\Omega)$, we have a minimizing sequence (v_n) of I in $\mathcal{M}$. By the continuous embedding theorem and the condition (1.9), the functional I is coercive. So the sequence (v_n) is bounded in $W_0^{1,p(x)}(\Omega)$. Since $W_0^{1,p(x)}(\Omega)$ is reflexive, up to a subsequence, (v_n) converges weakly to some $v \in W_0^{1,p(x)}(\Omega)$. But $\mathcal{M}$ is weakly closed in $W_0^{1,p(x)}(\Omega)$, therefore $v \in \mathcal{M}$, and by the weak lower semi-continuity of J, we have $I(v) \leqslant \liminf\limits_{n\to\infty} J(v_n)$ and $I(v) = \min\{I(w)\colon\ w \in \mathcal{M}\}$. That is

$$\frac{1}{q+1}\int_\Omega k(x)|v|^{q+1}\,\mathrm{d}x = 1 \tag{4.5}$$

and

$$I(v) = \int_\Omega \frac{1}{p}|\nabla v|^p\,\mathrm{d}x + \frac{1}{r+1}\int_\Omega h(x)|v|^{r+1}\,\mathrm{d}x. \tag{4.6}$$

Let $c := I(v)$. Then $J_\lambda(v) = c - \lambda < 0$ for any $\lambda > c$. So, there exists $v \in W_0^{1,p(x)}(\Omega)$ such that $J_\lambda(v) < 0$. Thus $J_\lambda(u) < 0$ and this completes the proof of Step 2.

Step 3. (There exists a positive real number $\lambda^\star$ such that the problem (1.2) has a positive solution for all $\lambda > \lambda^\star$.) In the beginning, let $\lambda^\star := \inf\Lambda$. As a consequence of Step 1, $\lambda^\star \in (0,\infty)$. Let $\lambda > \lambda^\star$. λ is not a lower bound of Λ and so there exists $\mu \in \Lambda$ such that $\lambda^\star \leqslant \mu \leqslant \lambda$. Since $\mu \in \Lambda$, the functional J_μ has a nontrivial critical point $u_\mu \in W_0^{1,p(x)}(\Omega)$ and the function u_μ is a sub-solution of the problem (1.2) (i.e. $(1.2)_\lambda$).

Let us consider the minimization problem

$$\inf\{J_\lambda(w)\colon\ w \in W_0^{1,p(x)}(\Omega) \text{ and } u_\mu \leqslant w\}. \tag{4.7}$$

As in Step 2, there exists a nonnegative function u_λ, the critical point of the functional J_λ such that $u_\lambda \geqslant u_\mu$. Since u_μ is nontrivial, the function u_λ is a nonnegative, nontrivial solution of the problem (1.2). By the maximum principle (see [10]), u_λ is positive.

(ii) This is due to the definition of $\lambda^\star$. □

5. Conclusion

Problems with the Laplace operator need many methods to be solved and this depends on the nonlinearities of the problem. Since the p-Laplace operator is nonlinear, when we deal with problems involving this operator, the study of the solvability becomes more difficult.

Here, with the $p(x)$-Laplace operator, it is not easy to investigate the existence and the nonexistence of the positive solution and to perform one step, we can need more than one method. As perspectives, we can study the problems (1.1) and (1.2) when the growths of the solutions are variables.

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